Progress and Challenges in the Relativistic Two-Body Problem

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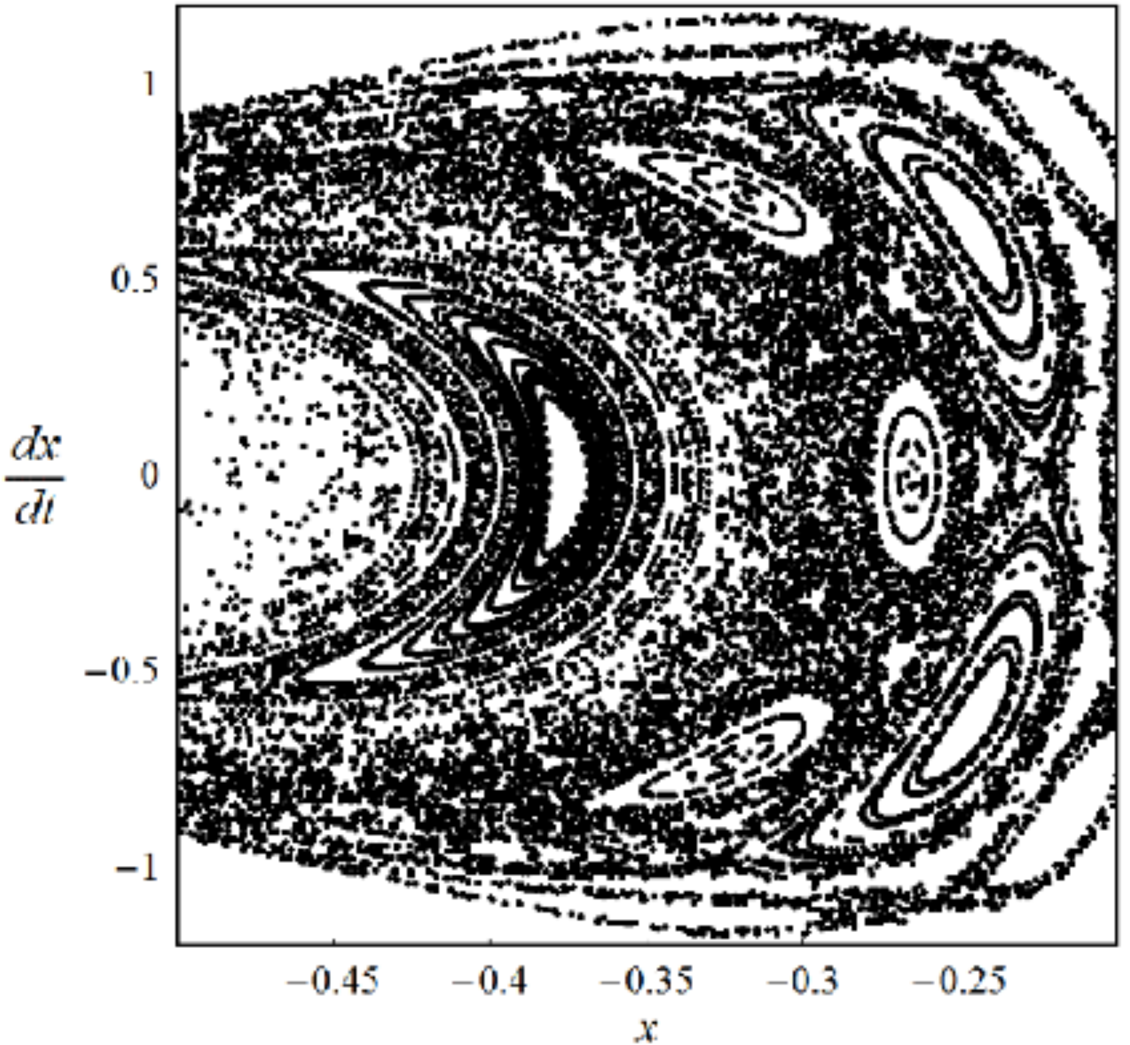
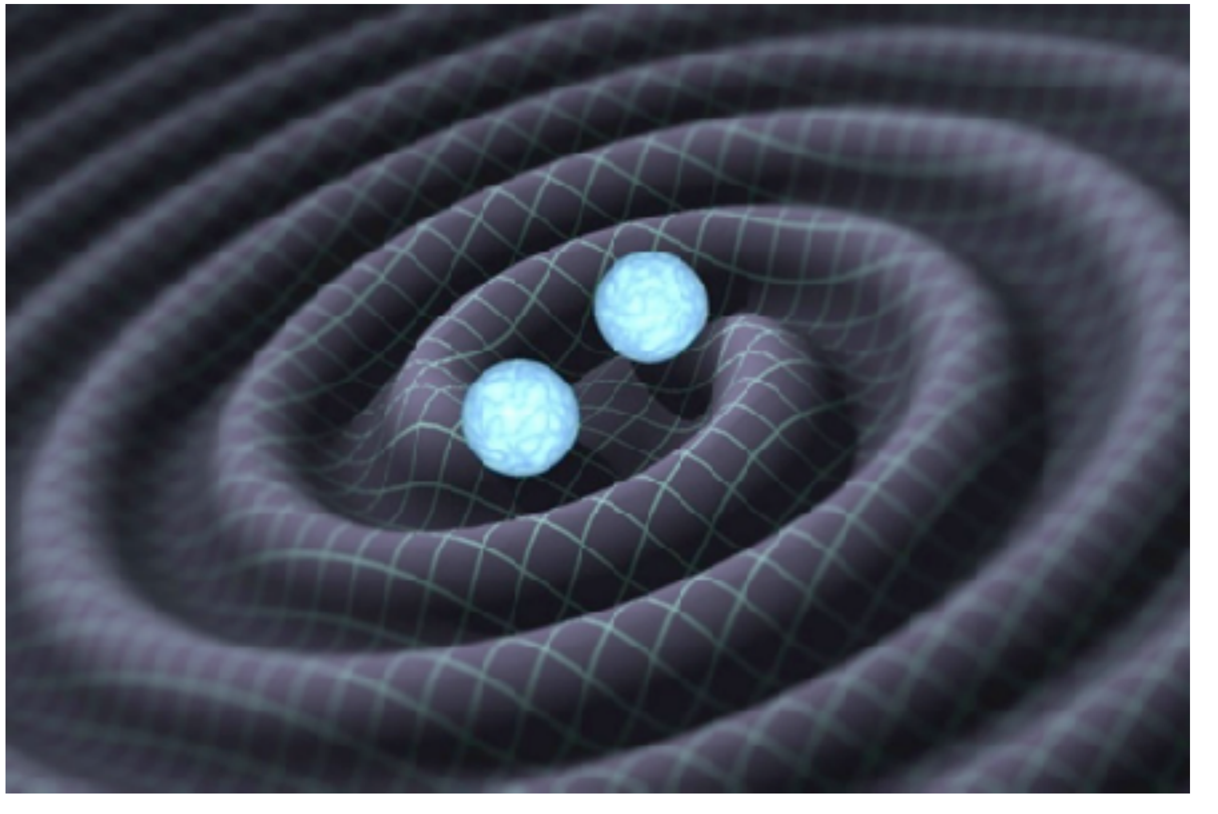
Henri Poincaré

«Il n'y a pas de problèmes résolus,
il y a seulement des problèmes
plus ou moins résolus »

«There are no solved problems,
there are only
more or less solved problems »





Tools used for the 2-body pb

Post-Newtonian (PN) approximation (expansion in 1/c)

Post-Minkowskian (PM) approximation (expansion in G)

Multipolar post-Minkowskian (MPM) approximation theory to the GW emission of binary systems

Matched Asymptotic Expansions useful both for the motion of strongly self-gravitating bodies, and for the nearzone-wavezone matching

Gravitational Self-Force (SF): expansion in m1/m2

Effective One-Body (EOB) Approach

Numerical Relativity (NR)

Effective Field Theory (EFT)

Quantum scattering amplitude —> classical PM approximation theory aided by Double-Copy, « Feynman-integral Calculus », Experimental Mathematic

Tutti Frutti method

The GR two-body problem (1)

1912-1916: Einstein introduced both the **PM, nonlinearity expansion**:

$$g_{\mu\nu} = \eta_{\mu\nu} + G h_{1\mu\nu} + G^2 h_{2\mu\nu} + \cdots$$

and the **PN expansion**: v/c <<1, T^ij << T^0i << T^00; hence h_0i << h_00,…

Droste 1912-1916 develops the PN expansion, using

$$\frac{1}{c}\frac{\partial}{\partial t} \ll \frac{\partial}{\partial x}$$

$$\Box = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \Delta + O\left(\frac{1}{c^2}\right)$$

Droste-Lorentz 1917, Einstein-Infeld-Hoffmann 1938:
1PN-accurate dynamics (and Lagrangian) of 2-body systems

$$\mathrm{L}_{\mathrm{PN}}(\boldsymbol{r}(t), \boldsymbol{r}'(t), \boldsymbol{v}(t), \boldsymbol{v}'(t)) = \mathrm{L}_{\mathrm{N}} + \frac{1}{c^2}\mathrm{L}_2 \qquad (2.1\,a)$$

with

$$\mathrm{L}_{\mathrm{N}} = \frac{1}{2}mv^2 + \frac{1}{2}m'v'^2 + \frac{\mathrm{G}mm'}{\mathrm{R}} \qquad (2.1\,b)$$

$$\mathrm{L}_2 = \frac{1}{8}mv^4 + \frac{1}{8}m'v'^4 + \frac{\mathrm{G}mm'}{2\mathrm{R}}\left[3v^2 + 3v'^2 - 7(vv') - (\mathrm{N}v)(\mathrm{N}v') - \mathrm{G}\frac{m+m'}{\mathrm{R}}\right] \qquad (2.1\,c)$$

The GR two-body problem (2)

Higher PN approximations ca.1970
(Chandrasekhar-Nutku'69, Chandraskhar-Esposito'70,
Burke'69-70,Thorne'69, Ohta-Okamura-Kimura-Hiida'73)

IR difficulties at 2PN (v^4/c^4) and 2.5PN (v^5/c^5):
incomplete and inconclusive results at 2PN and 2.5PN

Root of IR difficulties: general retarded wave

$$\Box\phi = 0$$

$$\phi(t, \mathbf{x}) = \sum_\ell \partial_x^\ell \left(\frac{S(t - \frac{r}{c})}{r} \right)$$

$$= \sum_\ell \partial_x^\ell \left(\frac{S(t))}{r} - \frac{1}{c}\dot{S}(t) + \frac{1}{2c^2}\ddot{S}(t)r - \frac{1}{6c^3}\dddot{S}(t)r^2 \cdots \right)$$

Burke (69-70) suggested to use **Matched Asymptotic Expansions** to have a well-defined matching between nearzone and wavezone gravitationa fields, and to derive the Radiation-Reaction force acting on the system. However, his implementation was flawed (see Blanchet-TD'84)

The GR two-body problem: PM comes back

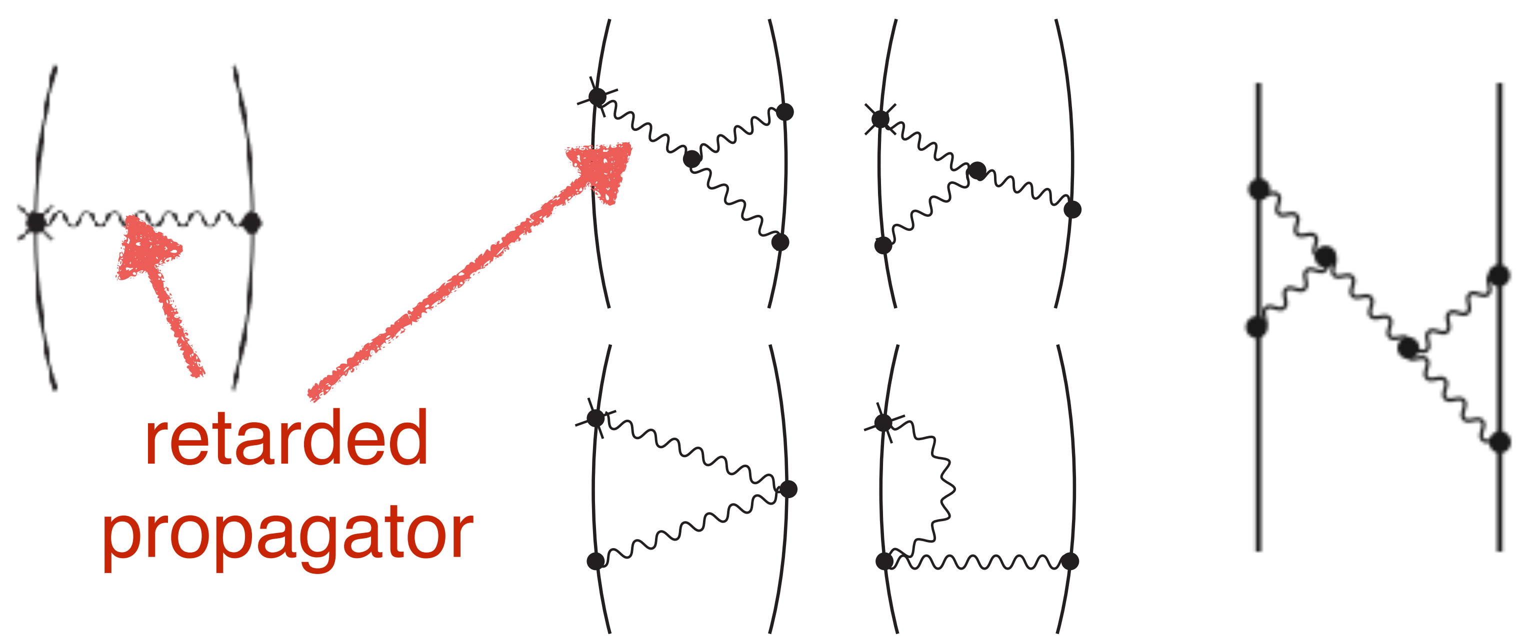
September 1974: Discovery binary pulsar PSR1913+16 (Hulse-Taylor’75)
An observational handle on gravitational radiation-reaction (Wagoner’75)

December 1978: 9th Texas Symposium (Munich):
J. H. Taylor announces that the orbital-period of PSR1913+16 decreases as:
dP_b/dt = (1.33 +/- 0.25) [dP_b/dt]_Quadrupole Formula

Unsatisfactory aspects of the then-existing « derivations » of the dynamics of binary systems in GR (emphasized by J. Ehlers and others):

Divergences appear in the 2.5PN expansion (Chandrasekhar-Esposito’70)
Incomplete treatment of nonlinear effects in the NZ-WZ matching (Burke-Thorne’69)
Inapplicability of weak-field PN to compact objects
No explicit (correct) derivation of the (conservative) 2PN eqs dynamics
Lack of clear proof of a **balance** between system’s mechanical energy loss and GW flux

Motivates a PM-based approach to 2-body dynamics including radiation-reaction
(Rosenblum’78,Westpfahl’79, Bel-TD-Deruelle-Ibanez-Martin’81)



The 2-body pb at G^3 and 1/c^5

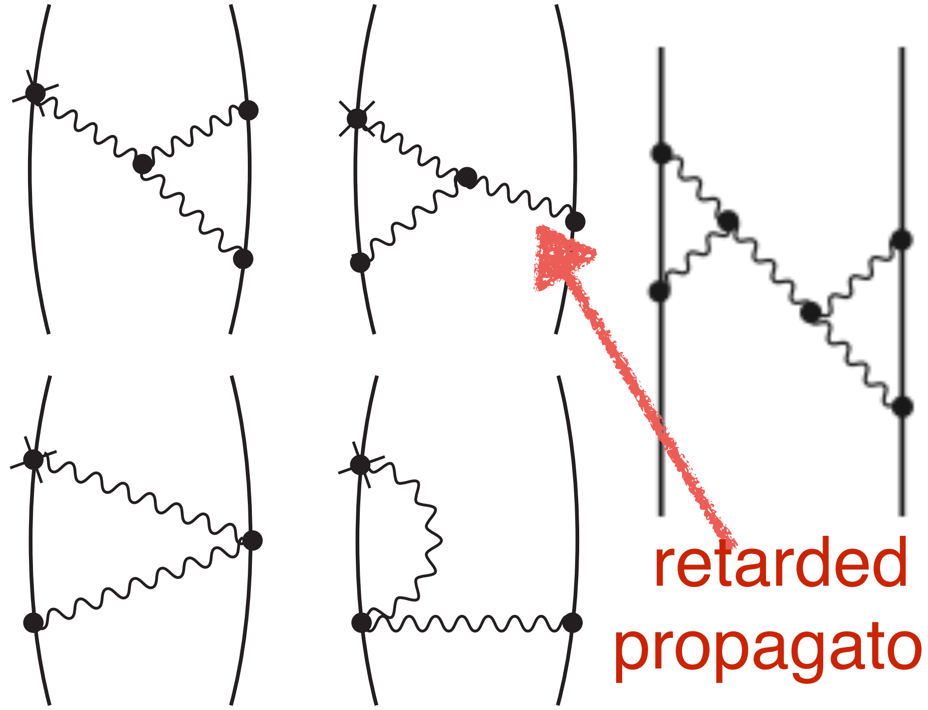
(TD-Deruelle'81, TD'82, using Bel et al.'81)

Use of **PM approximation**: G^2 + part of G^3

Eqs of motion (because non conservative)

Followed by PN expansion of PM for **separating Conservative and Radiation-Reaction Effects**

(—> direct proof of balance of E and L)



$$a^i = A_0^i(z-z') + c^{-2}A_2^i(z-z',v,v') + c^{-4}A_4^i(z-z',v,v',S,S') + c^{-5}A_5^i(z-z',v-v') + O(c^{-6}),$$

dissipative

conservative 2PN

$$L(z,v,a) = L_0(z,v) + c^{-2}L_2(z,v) + c^{-4}L_4(z,v,a).$$

acceleration-dependent Lagrangian

$$L_4(z,v,a) = M_4 + N_4 ,$$

$$\begin{aligned} M_4(z,v,a) = {} & \Sigma\left(\tfrac{1}{16}mv^6\right) + \\ & + \Sigma\, Gmm'R^{-1}\left(\tfrac{7}{8}v^4 + \tfrac{15}{16}v^2v'^2 - 2v^2(vv') + \tfrac{1}{8}(vv')^2 - \tfrac{7}{8}(Nv)^2v'^2 + \right. \\ & \left. + \tfrac{3}{4}(Nv)(Nv')(vv') + \tfrac{3}{16}(Nv)^2(Nv')^2\right) + \\ & + \Sigma\, G^2m^2m'R^{-2}\left(\tfrac{1}{4}v^2 + \tfrac{7}{4}v'^2 - \tfrac{7}{4}(vv') + \tfrac{7}{2}(Nv)^2 + \tfrac{1}{2}(Nv')^2 - \tfrac{7}{2}(Nv)(Nv')\right) + \\ & + \Sigma\, Gmm'\left((Na)\left(\tfrac{7}{8}v'^2 - \tfrac{1}{8}(Nv')^2\right) - \tfrac{7}{4}(v'a)(Nv')\right), \end{aligned}$$

$$N_4(z) = \frac{G^3mm'}{R^3}\left(\frac{1}{2}m^2 + \frac{1}{2}m'^2 + \frac{19}{4}mm'\right).$$

$$A_0^i = -Gm'R^{-2}N^i , \qquad (7)$$

$$A_2^i = Gm'R^{-2}\left(N^i\left(-v^2 - 2v'^2 + 4(vv') + \tfrac{3}{2}(Nv')^2 + 5(Gm/R) + 4(Gm'/R)\right) + (v^i - v'^i)\left(4(Nv) - 3(Nv')\right)\right), \qquad (8)$$

$$A_4^i = B_4^i + C_4^i , \qquad (9)$$

with :

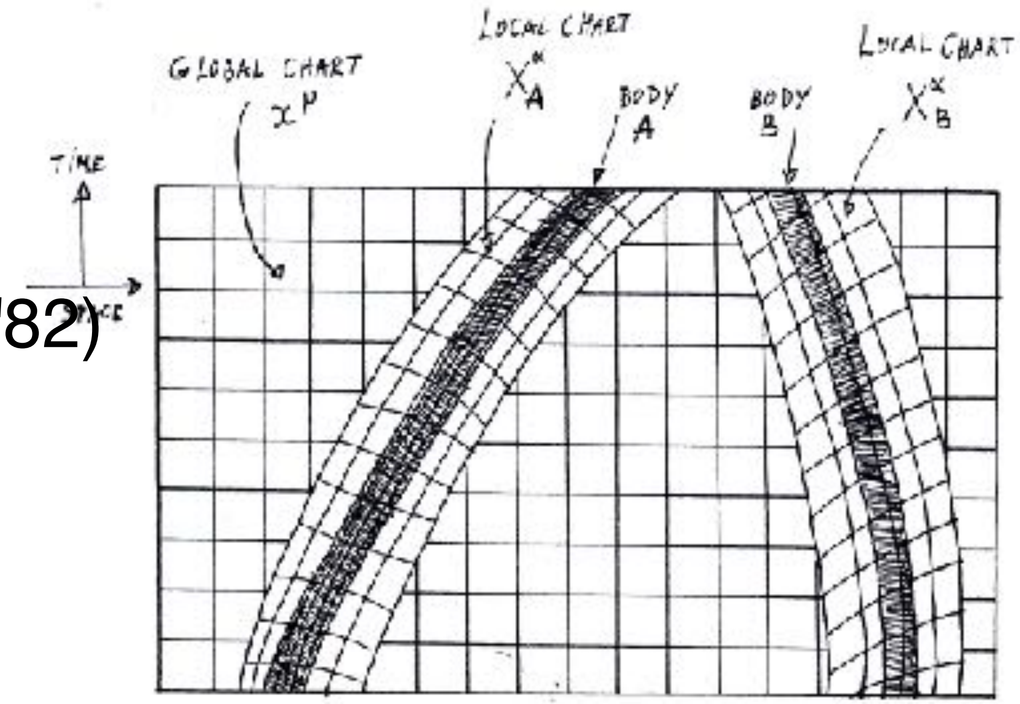
$$\begin{aligned} B_4^i = {} & Gm'R^{-2}\Big(N^i\Big(-2v'^4 + 4v'^2(vv') - 2(vv')^2 + \tfrac{3}{2}v^2(Nv')^2 + \tfrac{9}{2}v'^2(Nv')^2 - 6(vv')(Nv')^2 - \tfrac{15}{8}(Nv')^4 + \\ & + (Gm/R)\left(-\tfrac{15}{4}v^2 + \tfrac{5}{4}v'^2 - \tfrac{5}{2}(vv') + \tfrac{39}{2}(Nv)^2 - 39(Nv)(Nv') + \tfrac{17}{2}(Nv')^2\right) + \\ & + (Gm'/R)\left(4v'^2 - 8(vv') + 2(Nv)^2 - 4(Nv)(Nv') - 6(Nv')^2\right)\Big) + \\ & + (v^i - v'^i)\Big(v^2(Nv') + 4v'^2(Nv) - 5v'^2(Nv') - 4(vv')(Nv) + \\ & + 4(vv')(Nv') - 6(Nv)(Nv')^2 + \tfrac{9}{2}(Nv')^3 + \\ & + (Gm/R)\left(-\tfrac{63}{4}(Nv) + \tfrac{55}{4}(Nv')\right) + (Gm'/R)\left(-2(Nv) - 2(Nv')\right)\Big)\Big), \end{aligned} \qquad (10)$$

$$C_4^i = G^3m'R^{-4}N^i\left(-\tfrac{57}{4}m^2 - 9m'^2 - \tfrac{69}{2}mm'\right), \qquad (11)$$

$$\begin{aligned} A_5^i = {} & \tfrac{4}{5}G^2mm'R^{-3}\Big(v^i\left(-V^2 + 2(Gm/R) - 8(Gm'/R)\right) + \\ & + N^i(NV)\left(3V^2 - 6(Gm/R) + \tfrac{52}{3}(Gm'/R)\right)\Big). \end{aligned} \qquad (12)$$

dissipative G^2/c^5 + G^3/c^5

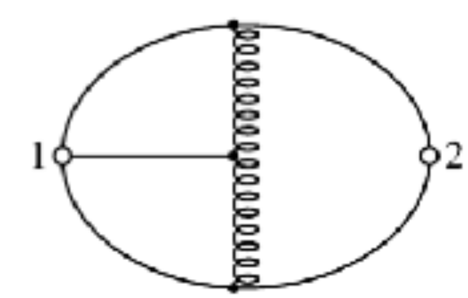
Subtleties in the skeletonized description of strong self-gravity bodies



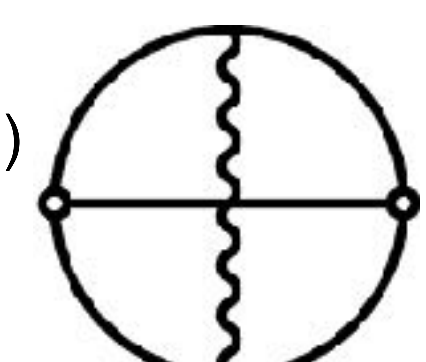
Matched Asymptotic Expansions
for compact bodies (EIH'38, Manasse'63, Demianski-Grischchuk'74, D'Eath'75,Kates'80,TD'82)
Skeletonization (Mathisson'36, Infeld '54)

$$T^{\mu\nu} \rightarrow \int ds\, u^\mu u^\nu \delta(x - z(s))$$

—> **UV divergences: need regularization**
(analytic or dim.reg TD'82)
Introduction of **Love number k of compact bodies**
Finite-size effects can only start at 5 loop (5PN)



Proof that **k_BH=0** (in D=4 TD'82, but not in D \neq 4 Kol-Smolkin'12)
—> **Effacing Property**
direct proof of physical UV finiteness at 3PN= 3 loops
(TD-Jaranowski-Schaefer'01, Blanchet-TD-Esposito-Farese'04),
and 4PN=4 loops (TD-Jaranowski-Schaefer'14,Jaranowski-Schaefer'15,…)



Recent explicit 5PN computation (Bluemlein et al'21)
show the absence of physical UV divergences at 5 PN
But there appear IR divergences at 4PN (4 loop) linked to non-locality (Blanchet-Damour '88).

The GR two-body problem:
PN comes back but helped by MPM

The PM-based derivation of the 2.5PN (G^3/c^5) eom [and parallel work using Hamiltonian approach (Schaefer'85)] built confidence in energy balance, and showed the technical difficulty of computing PM eom at and beyond G^3.
The perspective of detecting GWs from compact binaries gave a strong motivation for improving both analytical methods of **GW generation**, and the analytical accuracy of the **2-body eom** [Cutler et al'93]

Development of the **Multipolar Post-Minkowskian (MPM)** [Blanchet-TD-Iyer]
Effort to **push PN calculations beyond 2.5PN** (Jaranowski-Schaefer'98,Blanchet-Faye'00)
However, it becomes crucial to complete the Near-Zone-only PN approximation, with Wave-Zone information coming from MPM

Perturbative Theory of the Generation of Gravitational Radiation

Einstein '16, '18 (+ Landau-Lifshitz 41, and Fock '55) : h_+, h_x and quadrupole formula

Relativistic, multipolar extensions of LO quadrupole radiation :

Sachs-Bergmann '58, Sachs '61, Mathews '62, Peters-Mathews '63, Pirani '64

Campbell-Morgan '71,

Campbell et al '75,

nonlinear effects:

Bonnor-Rotenberg '66,

Epstein-Wagoner-Will '75-76

Thorne '80, .., Will et al 00

MPM Formalism:

Blanchet-Damour '86,

Damour-Iyer '91,

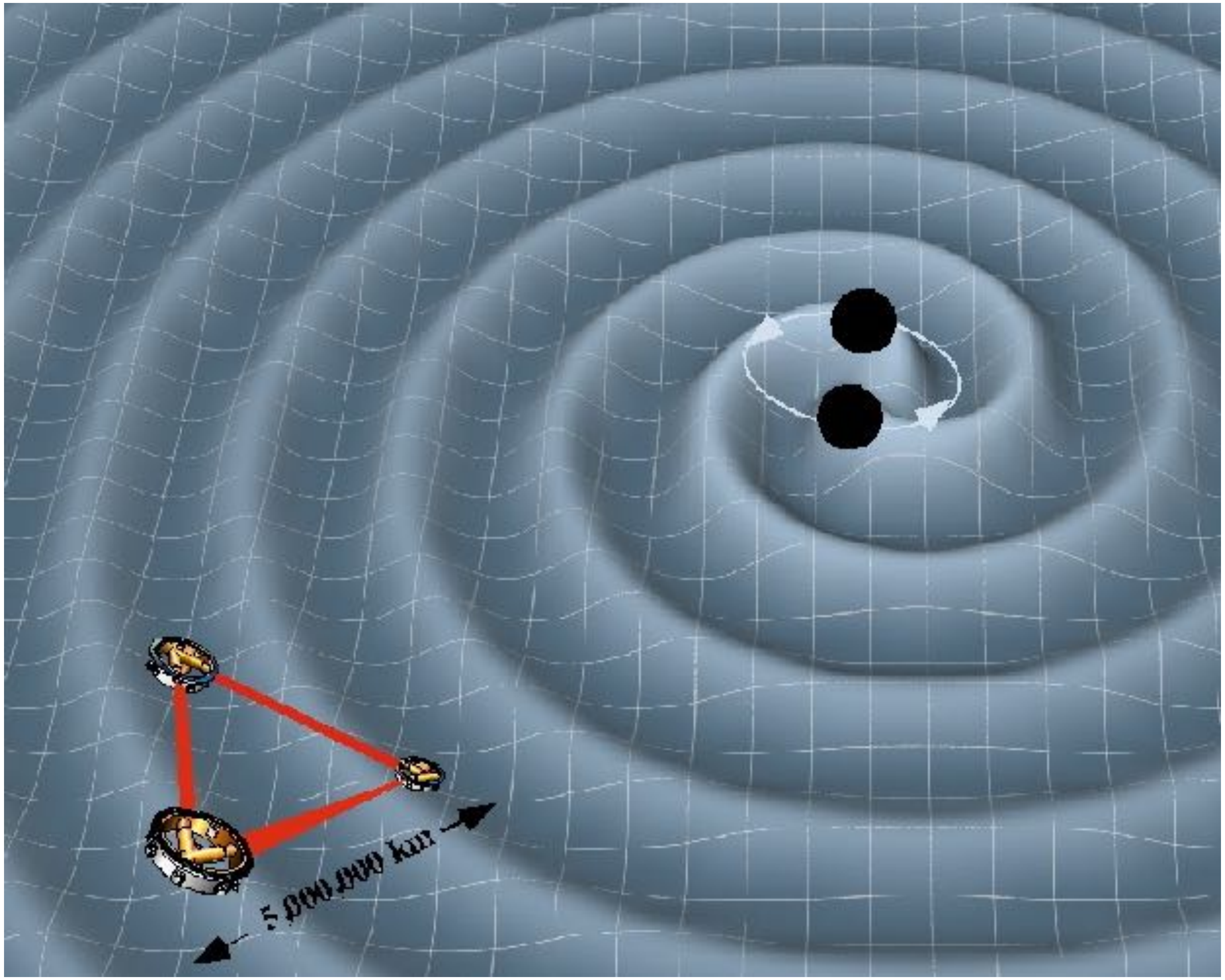
Blanchet '95 '98

Combines multipole exp. ,

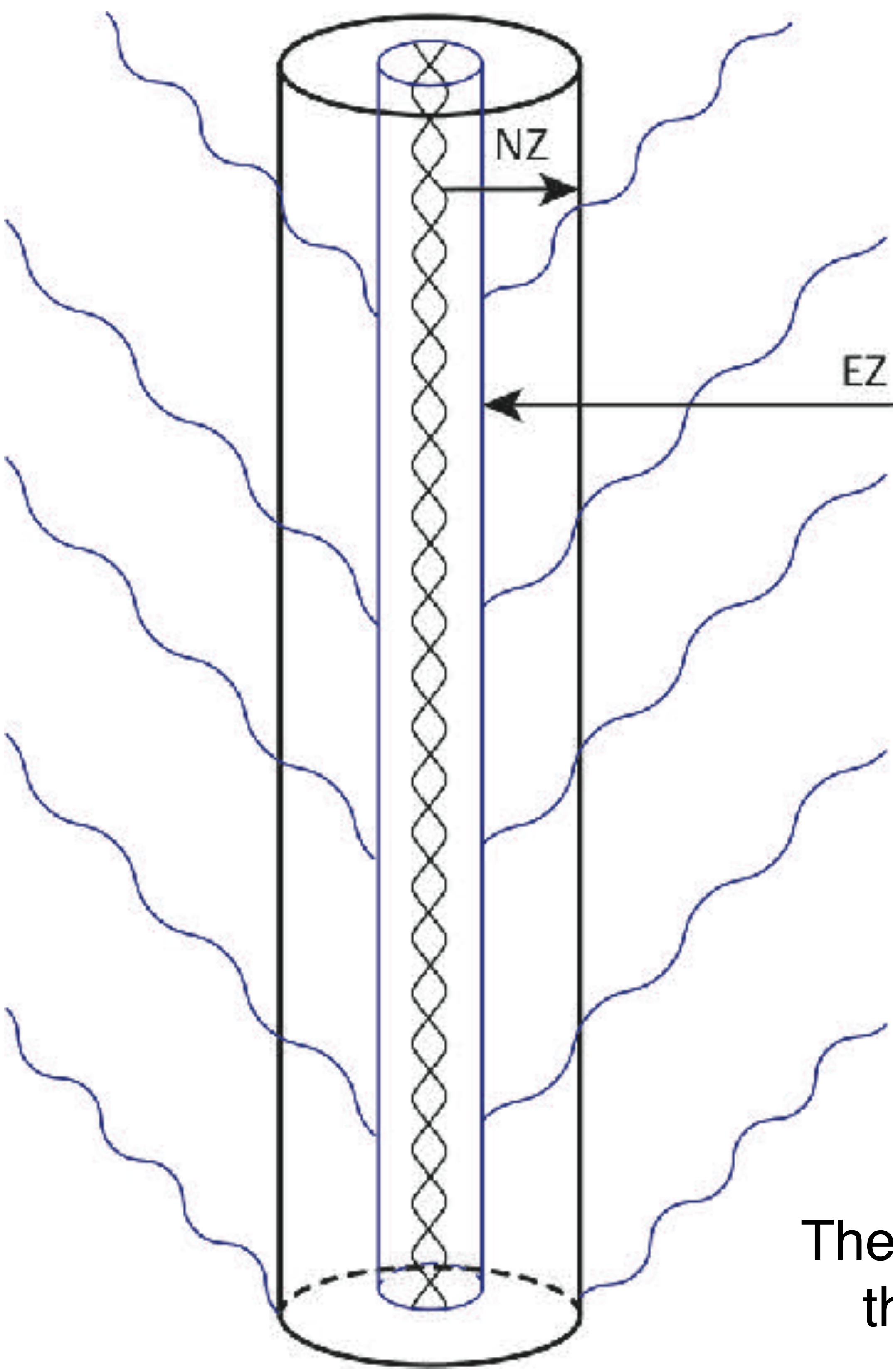
Post Minkowkian exp.,

analytic continuation,

and PN matching



MULTIPOLAR POST-MINKOWSKIAN FORMALISM (BLANCHET-DAMOUR-IYER)



Decomposition of space-time in various overlapping regions:

1. **near-zone: r << lambda : PN**
2. **exterior zone: r >> r_source: MPM**
3. **far wave-zone: Bondi-type expansion**

then matching between the zones

in exterior zone, iterative solution of Einstein's vacuum field equations by means of a double expansion in non-linearity and in multipoles, with crucial use of analytic continuation (complex B) for dealing with formal UV divergences at r=0

$$g = \eta + G h_1 + G^2 h_2 + G^3 h_3 + \ldots,$$
$$\Box h_1 = 0,$$
$$\Box h_2 = \partial\partial h_1 h_1,$$
$$\Box h_3 = \partial\partial h_1 h_1 h_1 + \partial\partial h_1 h_2,$$
$$h_1 = \sum_\ell \partial_{i_1 i_2 \ldots i_\ell}\left(\frac{M_{i_1 i_2 \ldots i_\ell}(t-r/c)}{r}\right) + \partial\partial\ldots\partial\left(\frac{\epsilon_{j_1 j_2 k} S_{k j_3 \ldots j_\ell}(t-r/c)}{r}\right),$$
$$h_2 = FP_B \Box^{-1}_{\rm ret}\left(\left(\frac{r}{r_0}\right)^B \partial\partial h_1 h_1\right) + \ldots,$$
$$h_3 = FP_B \Box^{-1}_{\rm ret} \ldots$$

The PN-matched MPM formalism has allowed to compute the GW emission to very high accuracy (Blanchet et al)

Link radiative multipoles <-> source variables

(Blanchet-Damour '89'92, Damour-Iyer'91, Blanchet '95…)

$$\mathrm{U}_{ij}(U) = \mathrm{M}^{(2)}_{ij}(U) + \frac{2G\mathrm{M}}{c^3}\int_0^{+\infty} d\tau\, \mathrm{M}^{(4)}_{ij}(U-\tau)\left[\ln\left(\frac{c\tau}{2r_0}\right) + \frac{11}{12}\right]$$

tail

$$+\frac{G}{c^5}\left\{-\frac{2}{7}\int_0^{+\infty} d\tau\, \mathrm{M}^{(3)}_{a\langle i}(U-\tau)\mathrm{M}^{(3)}_{j\rangle a}(U-\tau)\right.$$

memory

$$\left. - \frac{2}{7}\mathrm{M}^{(3)}_{a\langle i}\mathrm{M}^{(2)}_{j\rangle a} - \frac{5}{7}\mathrm{M}^{(4)}_{a\langle i}\mathrm{M}^{(1)}_{j\rangle a} + \frac{1}{7}\mathrm{M}^{(5)}_{a\langle i}\mathrm{M}_{j\rangle a} + \frac{1}{3}\varepsilon_{ab\langle i}\mathrm{M}^{(4)}_{j\rangle a}\mathrm{S}_b\right\}$$

instant.

$$+\frac{2G^2\mathrm{M}^2}{c^6}\int_0^{+\infty} d\tau\, \mathrm{M}^{(5)}_{ij}(U-\tau)\left[\ln^2\left(\frac{c\tau}{2r_0}\right) + \frac{57}{70}\ln\left(\frac{c\tau}{2r_0}\right) + \frac{124627}{44100}\right]$$

tail-of-tail

$$+\mathcal{O}\left(\frac{1}{c^7}\right).$$

$$\mathrm{M}_{ij} = \mathrm{I}_{ij} - \frac{4G}{c^5}\left[\mathrm{W}^{(2)}\mathrm{I}_{ij} - \mathrm{W}^{(1)}\mathrm{I}^{(1)}_{ij}\right] + \mathcal{O}\left(\frac{1}{c^7}\right)$$

$$\Sigma = \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2},$$

$$\Sigma_i = \frac{\overline{\tau}^{0i}}{c},$$

$$\Sigma_{ij} = \overline{\tau}^{ij}$$

$$\mathrm{I}_L(u) = \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz\left\{\delta_l\hat{x}_L\Sigma - \frac{4(2l+1)}{c^2(l+1)(2l+3)}\delta_{l+1}\hat{x}_{iL}\Sigma^{(1)}_i\right.$$

$$\left.+\frac{2(2l+1)}{c^4(l+1)(l+2)(2l+5)}\delta_{l+2}\hat{x}_{ijL}\Sigma^{(2)}_{ij}\right\}(\mathbf{x}, u + z|\mathbf{x}|/c), \qquad (85)$$

$$\mathrm{J}_L(u) = \mathcal{FP}\int d^3\mathbf{x}\int_{-1}^{1} dz\, \varepsilon_{ab\langle i_l}\left\{\delta_l\hat{x}_{L-1\rangle a}\Sigma_b - \frac{2l+1}{c^2(l+2)(2l+3)}\delta_{l+1}\hat{x}_{L-1\rangle ac}\Sigma^{(1)}_{bc}\right\}(\mathbf{x}, u + z|\mathbf{x}|/c).$$

Explicit Source Quadrupole Moment at 3.5 PN for a binary system

(Blanchet-Damour-Esposito-Farese-Iyer’05; Blanchet et al;Faye-Marsat-Blanchet-Iyer’12)

$$\mathrm{I}_{ij} = \mu\left(A\,x_{\langle ij\rangle} + B\,\frac{r^2}{c^2}v^{\langle ij\rangle} + \frac{48}{7}\,\frac{G^2m^2\nu}{c^5 r}\,C\,x_{\langle i}v_{j\rangle}\right) + \mathcal{O}\left(\frac{1}{c^8}\right)$$

$$\begin{aligned}
A &= 1 + \gamma\left(-\frac{1}{42} - \frac{13}{14}\nu\right) + \gamma^2\left(-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2\right) \\
&\quad + \gamma^3\left(\frac{395899}{13200} - \frac{428}{105}\ln\left(\frac{r_{12}}{r_0}\right) + \left[\frac{3304319}{166320} - \frac{44}{3}\ln\left(\frac{r_{12}}{r'_0}\right)\right]\nu + \frac{162539}{16632}\nu^2 + \frac{2351}{33264}\nu^3\right) \\
B &= \frac{11}{21} - \frac{11}{7}\nu + \gamma\left(\frac{1607}{378} - \frac{1681}{378}\nu + \frac{229}{378}\nu^2\right) \\
&\quad + \gamma^2\left(-\frac{357761}{19800} + \frac{428}{105}\ln\left(\frac{r_{12}}{r_0}\right) - \frac{92339}{5544}\nu + \frac{35759}{924}\nu^2 + \frac{457}{5544}\nu^3\right), \\
C &= 1 + \gamma\left(-\frac{256}{135} - \frac{1532}{405}\nu\right).
\end{aligned}$$

Challenge: derive the quadrupole moment at **4PN** (mixture of UV and IR subtleties; incomplete result: Marchand et al’20)

Perturbative computation of GW flux from binary system

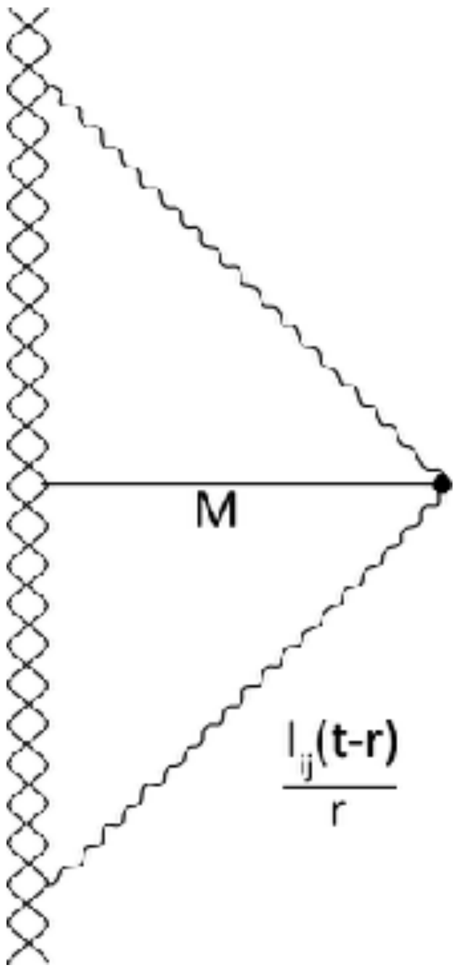
- lowest order : Einstein 1918 Peters-Mathews 63
- 1 + (v^2/c^2) : Wagoner-Will 76
- … + (v^3/c^3) : Blanchet-Damour 92, Wiseman 93
- … + (v^4/c^4) : Blanchet-Damour-Iyer Will-Wiseman 95
- … + (v^5/c^5) : Blanchet 96
- … + (v^6/c^6) : Blanchet-Damour-Esposito-Farèse-Iyer 2004
- … + (v^7/c^7) : Blanchet

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$

$$x = \left(\frac{v}{c}\right)^2 = \left(\frac{G(m_1+m_2)\Omega}{c^3}\right)^{\frac{2}{3}} = \left(\frac{\pi G(m_1+m_2)f}{c^3}\right)^{\frac{2}{3}}$$

$$\begin{aligned}
\mathcal{F} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ & 1 + \left(-\frac{1247}{336} - \frac{35}{12}\nu\right)x + 4\pi x^{3/2} \\
& + \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right)x^2 + \left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2} \\
& + \Bigg[\frac{6643739519}{69854400} + \frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_E - \frac{856}{105}\ln(16\,x) \\
& \quad + \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3\Bigg]x^3 \\
& + \left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\}.
\end{aligned}$$

Nonlocality in time: Tail-transported hereditary effects (Blanchet-Damour '88)



Hereditary (time-dissymetric) modification of the quadrupolar radiation-damping force, signalling a **breakdown of a basic tenet of PN expansion at the 4PN level**: (v/c)^8 fractional

$$g^{\rm in}_{00}(\mathbf{x},t)=-1+\frac{1}{c^2}\left[2\int\frac{d^3\mathbf{y}\,\rho(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|}\right]+\frac{1}{c^4}\left[\partial_t^2 X-2U^2+4\int\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\rho\left[\mathbf{v}^2+U+\frac{\Pi}{2}+\frac{3p}{2\rho}\right]\right]$$

$$+\frac{1}{c^6}{}_6\hat{\Phi}_{00}+\frac{1}{c^7}\left[-\tfrac{2}{5}x_{ab}\,{}^{(5)}I_{ab}(t)\right]+\frac{1}{c^8}{}_8\hat{\Phi}_{00}+\frac{1}{c^9}{}_9\hat{\Phi}_{00}$$

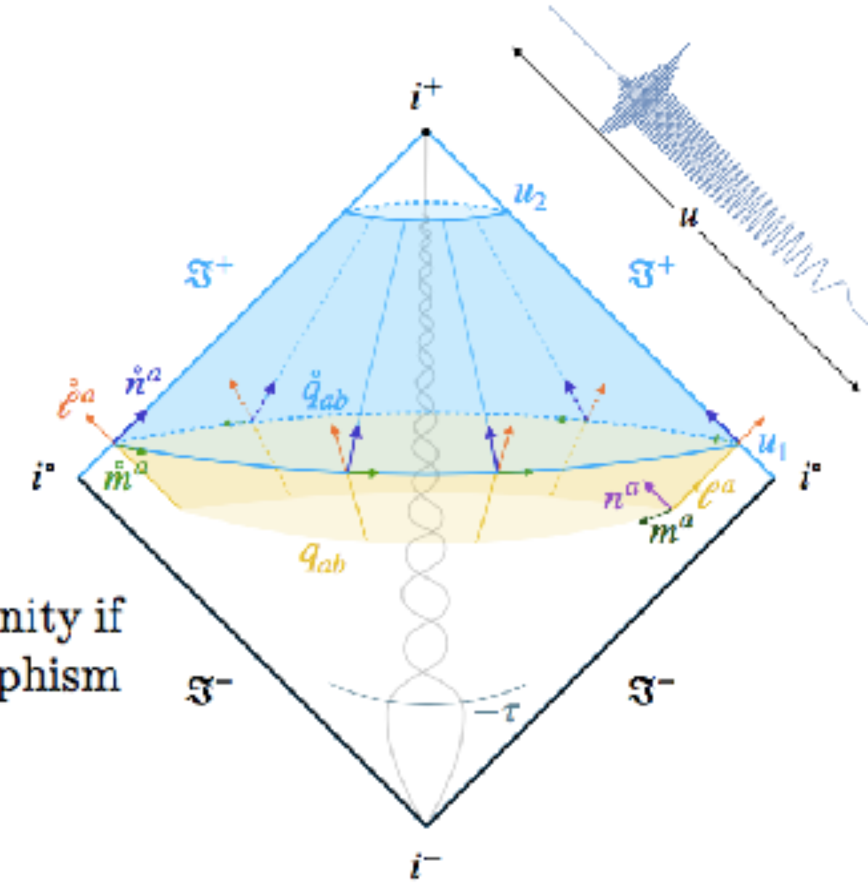
$$+\frac{1}{c^{10}}\left[-\tfrac{8}{5}x_{ab}I(t)\int_0^{+\infty}dv\,\ln\left[\frac{v}{2P}\right]{}^{(7)}I_{ab}(t-v)+{}_{10}\hat{\Phi}_{00}\right]+\cdots\,.$$

generates a **time-symmetric nonlocal-in-time 4PN-level action** (Damour-Jaranowski-Schaefer'14) which was uniquely matched to the local-zone metric via the Regge-Wheeler-Zerilli-Mano-Suzuki-Takasugi- based work of Bini-Damour'13, using the 1st law of binary dynamics (LeTiec-Blanchet-Whiting'12)

$$H^{\rm nonloc}_{\rm 4PN}(t)=-\frac{1}{5}\frac{G^2M}{c^8}I^{(3)}_{ij}(t)\times\mathrm{Pf}_{2r_{12}/c}\int_{-\infty}^{+\infty}\frac{dv}{|v|}I^{(3)}_{ij}(t+v),$$

Challenges in asymptotic spacetime structure

Elegant Penrose conformal reformulation
of Bondi-Sachs asymptotic



Ashtekar-DeLorenzo-Khera'19

Definition 1: A space-time $(\hat{M}, \hat{g}_{ab})$ will be said to be *asymptotically flat* at null infinity if there exists a manifold M with boundary $\mathcal{I}$ equipped with a metric g_{ab} and a diffeomorphism from $\hat{M}$ onto $M \setminus \mathcal{I}$ (with which we identify $\hat{M}$ and $M \setminus \mathcal{I}$) such that:

i) there exists a smooth function Ω on M with $g_{ab} = \Omega^2 \hat{g}_{ab}$ on $\hat{M}$; $\Omega = 0$ on $\mathcal{I}$; and $n_a := \nabla_a \Omega$ is nowhere vanishing on $\mathcal{I}$;

ii) $\mathcal{I}$ is topologically $\mathbb{S}^2 \times \mathbb{R}$; and,

iii) $\hat{g}_{ab}$ satisfies Einstein's equations $\hat{R}_{ab} - \frac{1}{2}\hat{R}\hat{g}_{ab} = 8\pi G\, \hat{T}_{ab}$, where $\Omega^{-2}\hat{T}_{ab}$ has a smooth limit to $\mathcal{I}$.

This definition implies the peeling of the Newman-Penrose scalars Psi4, Psi3,Psi2,Psi1,Psi0

$$\Psi_4 \sim \frac{1}{r}, \Psi_3 \sim \frac{1}{r^2}, \Psi_2 \sim \frac{1}{r^3}, \Psi_1 \sim \frac{1}{r^4}, \Psi_0 \sim \frac{1}{r^5},$$

Violations of peeling for scattering problems

Violations at I^- even in linearized theory (Bardeen-Press,Schmidt-Stewart,Walker-Will,Porrill-Stewart

Violations at I^+ when taking into account tails (TD'86)

$$\Psi_0 = -\frac{6MA_{ij}m^im^j}{r^4} + o\left(\frac{1}{r^4}\right) . \qquad (4.42)$$

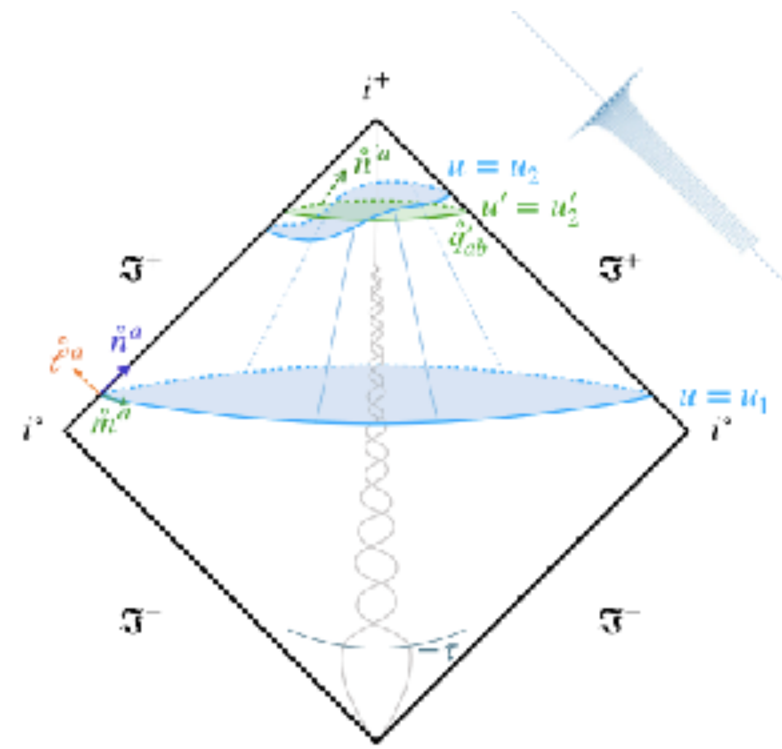
Therefore, if A_{ij} is non-zero, the peeling property does *not* hold on $\mathcal{J}^+$. This means that the conformal metric $\tilde{g}_{\alpha\beta}$, eqn(4.2), *cannot* be C^3 on $\mathcal{J}^+$ (for any choice of the conformal factor Ω). This means also that the Weyl tensor in the conformal space-time, $\tilde{C}^{\alpha}{}_{.\beta\gamma\delta}$ does *not* tend to zero on $\mathcal{J}^+$ (and that $\tilde{K}^{\alpha}{}_{.\beta\gamma\delta} = \Omega^{-1}\tilde{C}^{\alpha}{}_{.\beta\gamma\delta}$ is unbounded near $\mathcal{J}^+$).

Christodoulou-Klainerman' theorem —> stronger generic violation of peeling

BMS « symmetry », definition of angular momentum

General issue:
lack of connection with material source

PM and MPM perturbation theory are useful for providing connection with source. They have their limitations but suggest that that the only **global classical symmetry** is the **Poincaré group**

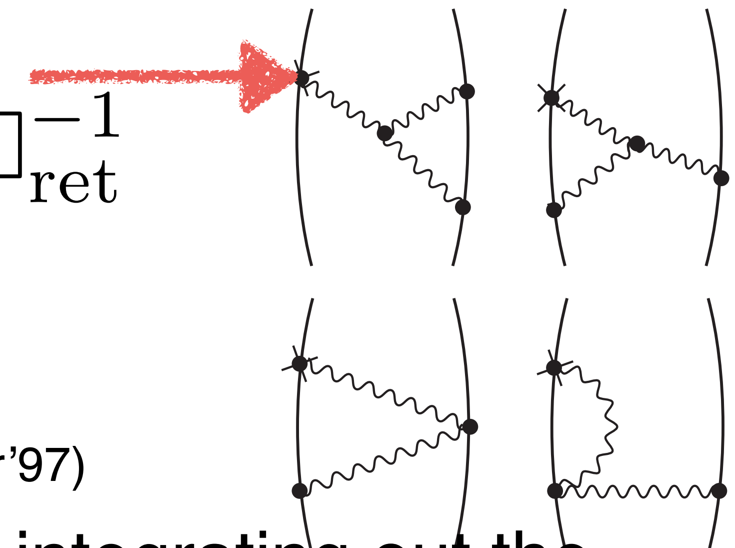


Ashtekar-DeLorenzo-Khera'20

Separating Conservative and Radiation-Reaction Effects

Within the PM approach: one used a PN-expansion of the PM dynamics to separate conservative and radiation-reaction effects;

$$G_{\rm ret} = \Box^{-1}_{\rm ret}$$



Within the ADM approach (Schaefer'85, Jaranowski-Schaefer'97)

Hamiltonian for matter + radiative dof obtained by integrating out the potential-mode-interactions by solving the constraints in a Coulomb-like gauge

$$g^{-1/2}\left[gR + \frac{1}{2}(g_{ij}\pi^{ij})^2 - \pi_{ij}\pi^{ij}\right] = \sum_a (g^{ij}p_{ai}p_{aj} + m_a^2)^{1/2}\delta_a \qquad -2\pi^{ij}{}_{|j} = \sum_a g^{ij}p_{aj}\delta_a$$

Within the Fokker-Wheeler-Feynman approach one uses a time-symmetric Green's function G_sym to define the conservative dynamics (including soft-graviton interactions)

Within the EOB approach one uses balance (modulo Schott terms) between mechanical E-J and GW fluxes to determine the radiation-reaction force

Within the EFT approach (Goldberger-Rothstein'06) one first integrates out the potential gravitons, before taking into account soft-graviton effects

Within the Tutti-Frutti approach (Bini-TD-Geralico'20) one adds nonlocal soft-graviton conservative interactions and uses SF to determine H_loc

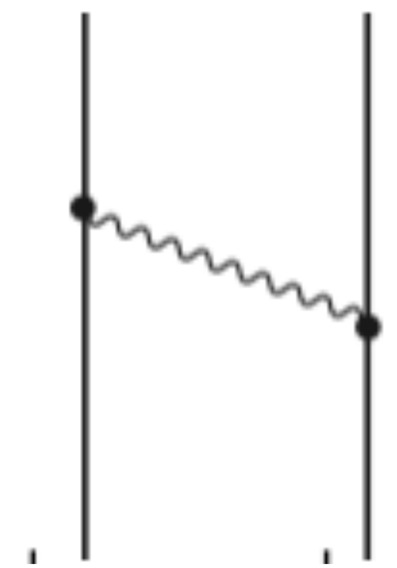
Reduced Worldline Action in Electrodynamics (Fokker 1929)

$$S_{\rm tot}[x_a^\mu, A_\mu] = -\sum_a \int m_a ds_a + \sum_a \int e_a dx_a^\mu A_\mu(x_a) - \int d^D x \frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} + S_{\rm gf}$$

« Integrate out » the field A_mu in the total (particle+field) action

$$S_{\rm eff}^{\rm class}[x_a(s_a)] = -\sum_a m_a \int {\rm d}s_a + \frac{1}{2}\sum_{a,b} e_a e_b \iint {\rm d}x_a^\mu \, {\rm d}x_{b\mu} \, \delta\left((x_a - x_b)^2\right).$$

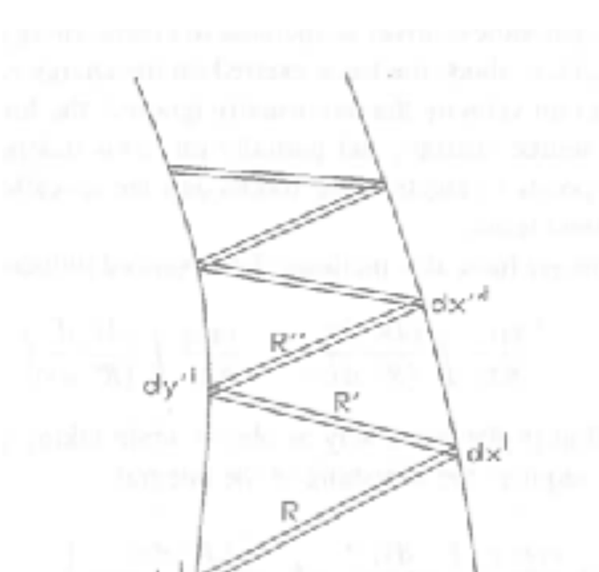
One-photon-exchange diagram

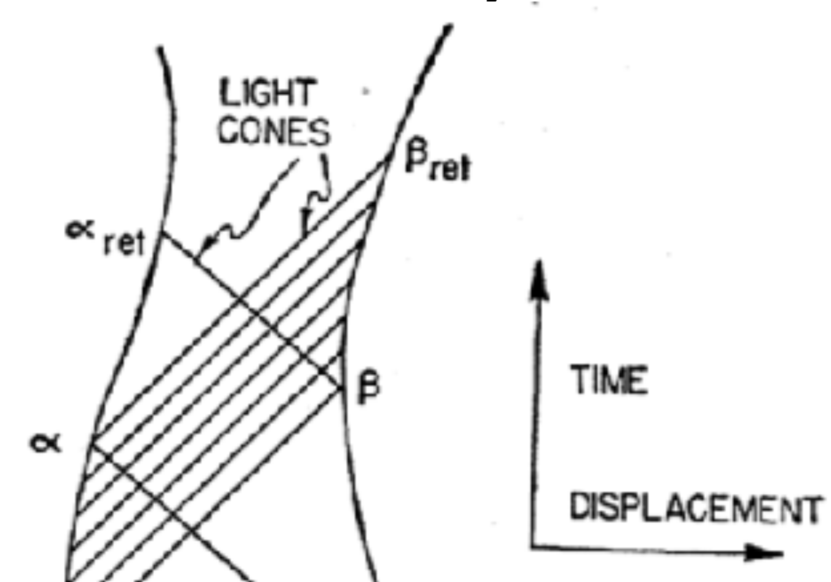


time-symmetric Green function G.

$$G(x) = \delta(-\eta_{\mu\nu}x^\mu x^\nu) = \frac{1}{2r}\left(\delta(t-r) + \delta(t+r)\right) \; ; \; \Box G(x) = -4\pi\delta^4(x)$$

The effective action S_eff(x_a) was heavily used in the (second) Wheeler-Feynman paper (1949) together with similar diagrams to those used by Fokker





FWF Reduced Action in Gravity and its PM Diagrammatic Expansion

$$S^{\rm class}_{\rm eff}[x_a(s_a)] = [S_{\rm pm} + S_{\rm EH} + S_{\rm gf}]_{g_{\mu\nu}(x)\to g^{\rm gf}_{\mu\nu}[x_a(s_a)]}$$

PN: Infeld-Plebanski '60

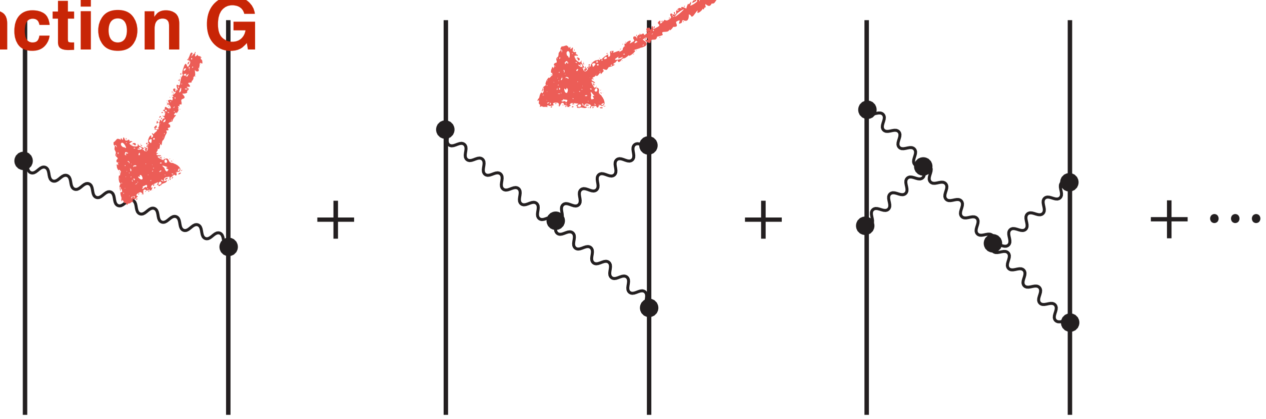
PM:TD-Esposito-Farese '96

Needs gauge-fixed* action and time-symmetric Green function G.
*E.g. Arnowitt-Deser-Misner Hamiltonian formalism or harmonic coordinates.
Perturbatively solving (in dimension D=4 + eps) Einstein's equations
to get the equations of motion and the action for the conservative dynamics

$$g = \eta + h$$
$$S(h,T) = \int \left(\frac{1}{2} h \Box h + \partial\partial hhh + \ldots + (h + hh + \ldots)T\right)$$
$$\Box h = -T + \ldots \to h = GT + \ldots$$
$$S_{\rm red}(T) = \frac{1}{2} TGT + V_3(GT,GT,GT) + \ldots$$

$$\int ds_1 \int ds_2 \int ds_2' (m_1 u_1 u_1)(m_2 u_2 u_2)(m_2 u_2' u_2')$$
$$\times \int d^4x\, \partial_x \partial_x\, G(x_1 - x) G(x - x_2) G(x - x_2')$$

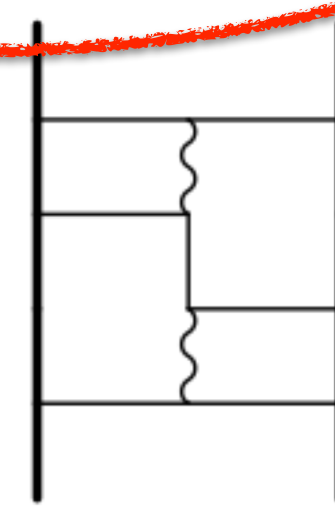
time-symmetric Green's function G



O(G)= 1PM= Newtonian + (v/c)^n corrections

O(G^2)=2PM =1 loop -> 1PN

O(G^3)=3PM =2 loop-> 2PN



O(G^5)=5PM =4 loop-> 4PN

A tale of many Green's functions

$$G_{\rm ret}(x) = \frac{\delta(t - r/c)}{r} \qquad G_{\rm ret} = {\rm P}\frac{1}{k^2} + i\pi {\rm sign}(k^0)\delta(k^2)$$

$$G_{\rm sym}(x) = \frac{\delta(t - r/c) + \delta(t + r/c)}{2r} \qquad G_{\rm sym} = {\rm P}\frac{1}{k^2}$$

$$G^{\rm PN}_{\rm sym}(x) = \frac{\delta(t)}{r} + \frac{r}{2c^2}\ddot{\delta}(t) + \cdots \qquad G^{\rm PN}_{\rm sym} = \frac{1}{\mathbf{k^2}} + \frac{\omega^2}{c^2\mathbf{k^4}} + \cdots$$

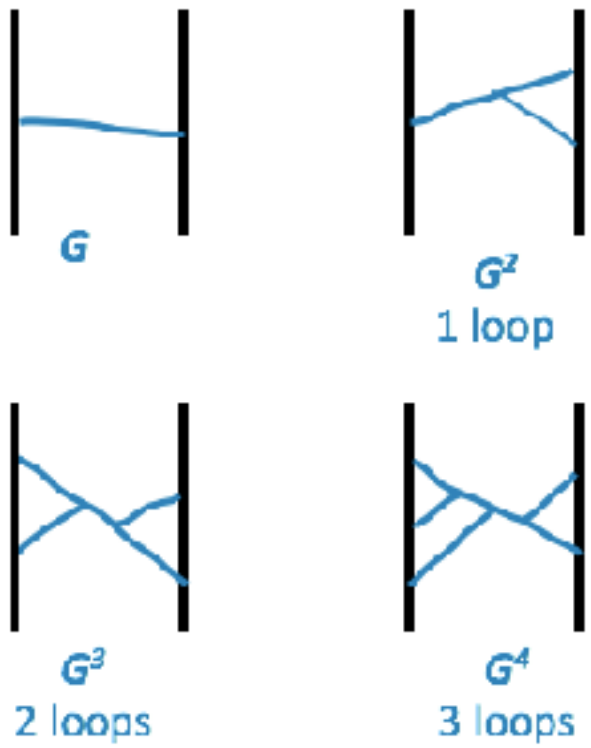
$$G_{\rm F}(x) = \frac{i}{\pi(t^2 - r^2 + i0)} \qquad G_{\rm F} = {\rm P}\frac{1}{k^2} + i\pi\delta(k^2)$$

+ issues of: <in,out>; <in,in>, FWF, Schwinger-Keldysh,…

Effective One-Body (EOB) approach: H + Rad-Reac Force

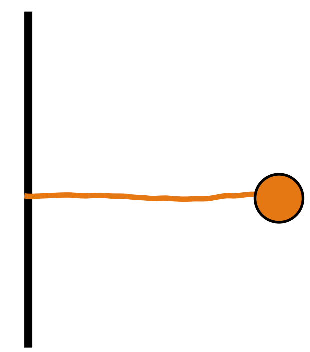
Historically rooted in QM: Brezin-Itzykson-ZinnJustin'70
eikonal scattering amplitude+ Wheeler's:`Think quantum mechanically'

Real 2-body system
(in the c.o.m. frame)



An effective particle of mass mu in some effective metric

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$



$$g^{\mu\nu}_{\rm eff}(X)$$

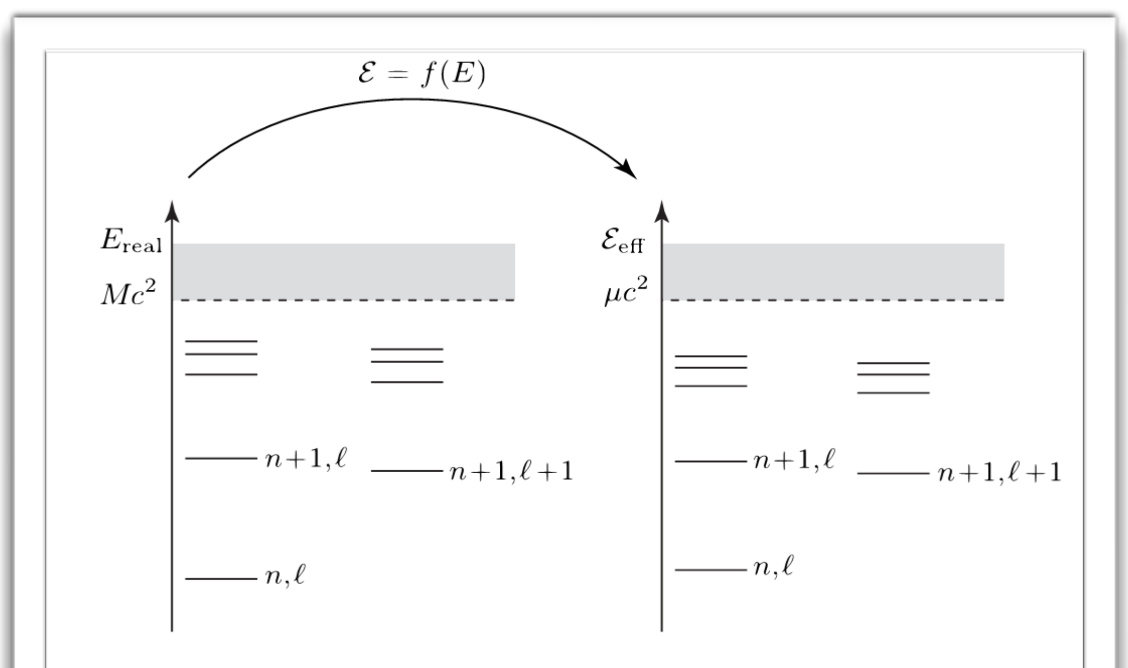
1:1 map

mass-shell constraint

$$0 = g^{\mu\nu}_{\rm eff}(X) P_\mu P_\nu + \mu^2 + Q(X, P)$$

Level correspondence
in the semi-classical limit:
Bohr-Sommerfeld ->
identification of
quantized action variables

$$J = \ell\hbar = \frac{1}{2\pi}\oint p_\varphi d\varphi$$
$$N = n\hbar = I_r + J$$
$$I_r = \frac{1}{2\pi}\oint p_r dr$$



Crucial energy map

$$\mathcal{E} = f(E)$$

2-body Taylor-expanded 3PN Hamiltonian (TD-Jaranowski-Schaefer'01)

$$H_{\rm N}(\mathbf{x}_a,\mathbf{p}_a)=\frac{\mathbf{p}_1^2}{2m_1}-\frac{1}{2}\frac{Gm_1m_2}{r_{12}}+(1\leftrightarrow 2)$$

$$c^2H_{\rm 1PN}(\mathbf{x}_a,\mathbf{p}_a)=-\frac{1}{8}\frac{(\mathbf{p}_1^2)^2}{m_1^3}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(-12\frac{\mathbf{p}_1^2}{m_1^2}+14\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$+\frac{1}{4}\frac{Gm_1m_2}{r_{12}}\frac{G(m_1+m_2)}{r_{12}}+(1\leftrightarrow 2),$$

$$c^4H_{\rm 2PN}(\mathbf{x}_a,\mathbf{p}_a)=\frac{1}{16}\frac{(\mathbf{p}_1^2)^3}{m_1^5}+\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\left(5\frac{(\mathbf{p}_1^2)^2}{m_1^4}-\frac{11}{2}\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}-\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}+5\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right.$$
$$\left.-6\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}-\frac{3}{2}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)$$
$$+\frac{1}{4}\frac{G^2m_1m_2}{r_{12}^2}\left(m_2\left(10\frac{\mathbf{p}_1^2}{m_1^2}+19\frac{\mathbf{p}_2^2}{m_2^2}\right)-\frac{1}{2}(m_1+m_2)\frac{27(\mathbf{p}_1\cdot\mathbf{p}_2)+6(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$-\frac{1}{8}\frac{Gm_1m_2}{r_{12}}\frac{G^2(m_1^2+5m_1m_2+m_2^2)}{r_{12}^2}+(1\leftrightarrow 2),$$

$$c^6H_{\rm 3PN}(\mathbf{x}_a,\mathbf{p}_a)=-\frac{5}{128}\frac{(\mathbf{p}_1^2)^4}{m_1^7}+\frac{1}{32}\frac{Gm_1m_2}{r_{12}}\left(-14\frac{(\mathbf{p}_1^2)^3}{m_1^6}+4\frac{((\mathbf{p}_1\cdot\mathbf{p}_2)^2+4\mathbf{p}_1^2\mathbf{p}_2^2)\mathbf{p}_1^2}{m_1^4m_2^2}+6\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^4m_2^2}\right.$$
$$-10\frac{(\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2-\mathbf{p}_2^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2)\mathbf{p}_1^2}{m_1^4m_2^2}+24\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^4m_2^2}$$
$$+2\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}+\frac{(7\mathbf{p}_1^2\mathbf{p}_2^2-10(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2^3}$$
$$+\frac{(\mathbf{p}_1^2\mathbf{p}_2^2-2(\mathbf{p}_1\cdot\mathbf{p}_2)^2)(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2^3}+15\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^3m_2^3}$$
$$\left.-18\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}+5\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)^3}{m_1^3m_2^3}\right)+\frac{G^2m_1m_2}{r_{12}^2}\left(\frac{1}{16}(m_1-27m_2)\frac{(\mathbf{p}_1^2)^2}{m_1^4}\right.$$
$$-\frac{115}{16}m_1\frac{\mathbf{p}_1^2(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1^3m_2}+\frac{1}{48}m_2\frac{25(\mathbf{p}_1\cdot\mathbf{p}_2)^2+371\mathbf{p}_1^2\mathbf{p}_2^2}{m_1^2m_2^2}+\frac{17}{16}\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^3}+\frac{5}{12}\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^4}{m_1^3}$$
$$-\frac{1}{8}m_1\frac{(15\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)+11(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1))(\mathbf{n}_{12}\cdot\mathbf{p}_1)}{m_1^3m_2}-\frac{3}{2}m_1\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^3(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^3m_2}$$
$$+\frac{125}{12}m_2\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1^2m_2^2}+\frac{10}{3}m_2\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}$$
$$\left.-\frac{1}{48}(220m_1+193m_2)\frac{\mathbf{p}_1^2(\mathbf{n}_{12}\cdot\mathbf{p}_2)^2}{m_1^2m_2^2}\right)+\frac{G^3m_1m_2}{r_{12}^3}\left(-\frac{1}{48}\left(425m_1^2+\left(473-\frac{3}{4}\pi^2\right)m_1m_2-150m_2^2\right)\frac{\mathbf{p}_1^2}{m_1^2}\right.$$
$$+\frac{1}{16}\left(77(m_1^2+m_2^2)+\left(143-\frac{1}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{p}_1\cdot\mathbf{p}_2)}{m_1m_2}+\frac{1}{16}\left(20m_1^2-\left(43+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)^2}{m_1^2}$$
$$\left.+\frac{1}{16}\left(21(m_1^2+m_2^2)+\left(119+\frac{3}{4}\pi^2\right)m_1m_2\right)\frac{(\mathbf{n}_{12}\cdot\mathbf{p}_1)(\mathbf{n}_{12}\cdot\mathbf{p}_2)}{m_1m_2}\right)$$
$$+\frac{1}{8}\frac{G^4m_1m_2^3}{r_{12}^4}\left(\left(\frac{227}{3}-\frac{21}{4}\pi^2\right)m_1+m_2\right)+(1\leftrightarrow 2).$$

Explicit 3PN EOB dynamics (Damour-Jaranowski-Schaefer '01)

A **simple**, but crucial transformation between the real energy and the effective one:

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1+m_2)}.$$

A **simple post-geodesic** effective mass-shell:

$$g_{\rm eff}^{\mu\nu} P'_\mu P'_\nu + \mu^2 c^2 + Q(P'_\mu) = 0\,,$$

$$ds^2_{\rm eff} = -A(R;\nu)dt^2 + B(R;\nu)dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$M = m_1 + m_2\,, \qquad \mu = \frac{m_1 m_2}{m_1+m_2}\,, \qquad \nu = \frac{m_1 m_2}{(m_1+m_2)^2} = \frac{\mu}{M}$$

$$u \equiv \frac{GM}{R\,c^2}$$

$$A^{\rm 3PN}(u) = 1 - 2u + 2\nu\, u^3 + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\nu u^4,$$

$$\overline{D}^{\rm 3PN}(u) = 1 + 6\nu\, u^2 + (52\nu - 6\nu^2)\, u^3,$$

$$\widehat{Q}^{\rm 3PN} \equiv \frac{Q}{\mu^2 c^2} = (8\nu - 6\nu^2)\, u^2\, \frac{p_r^4}{c^4}.$$

A(u) is linear in nu because of striking cancellations

The first EOB vs NR comparisons

Buonanno-Cook-Pretorius 2007

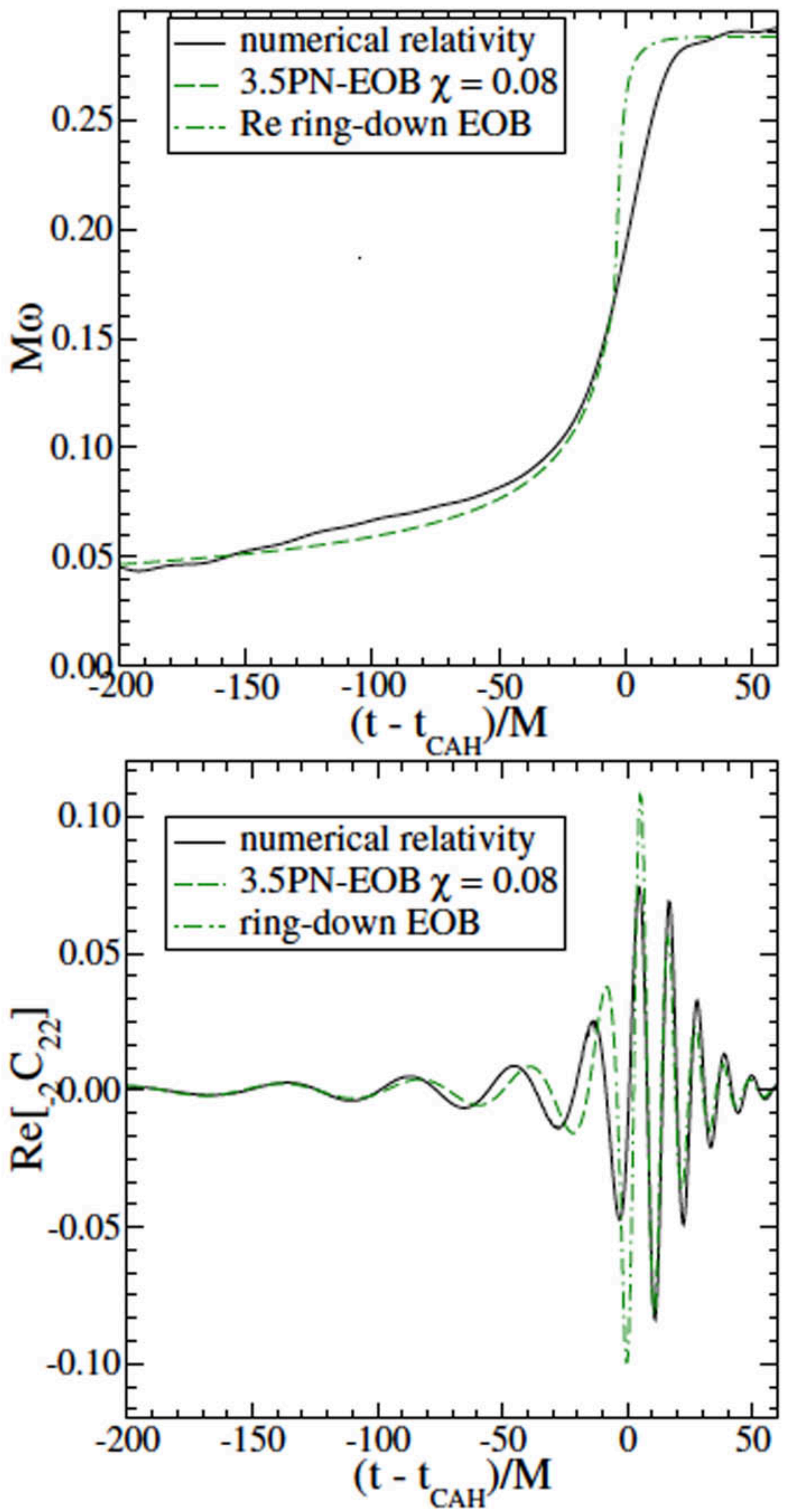
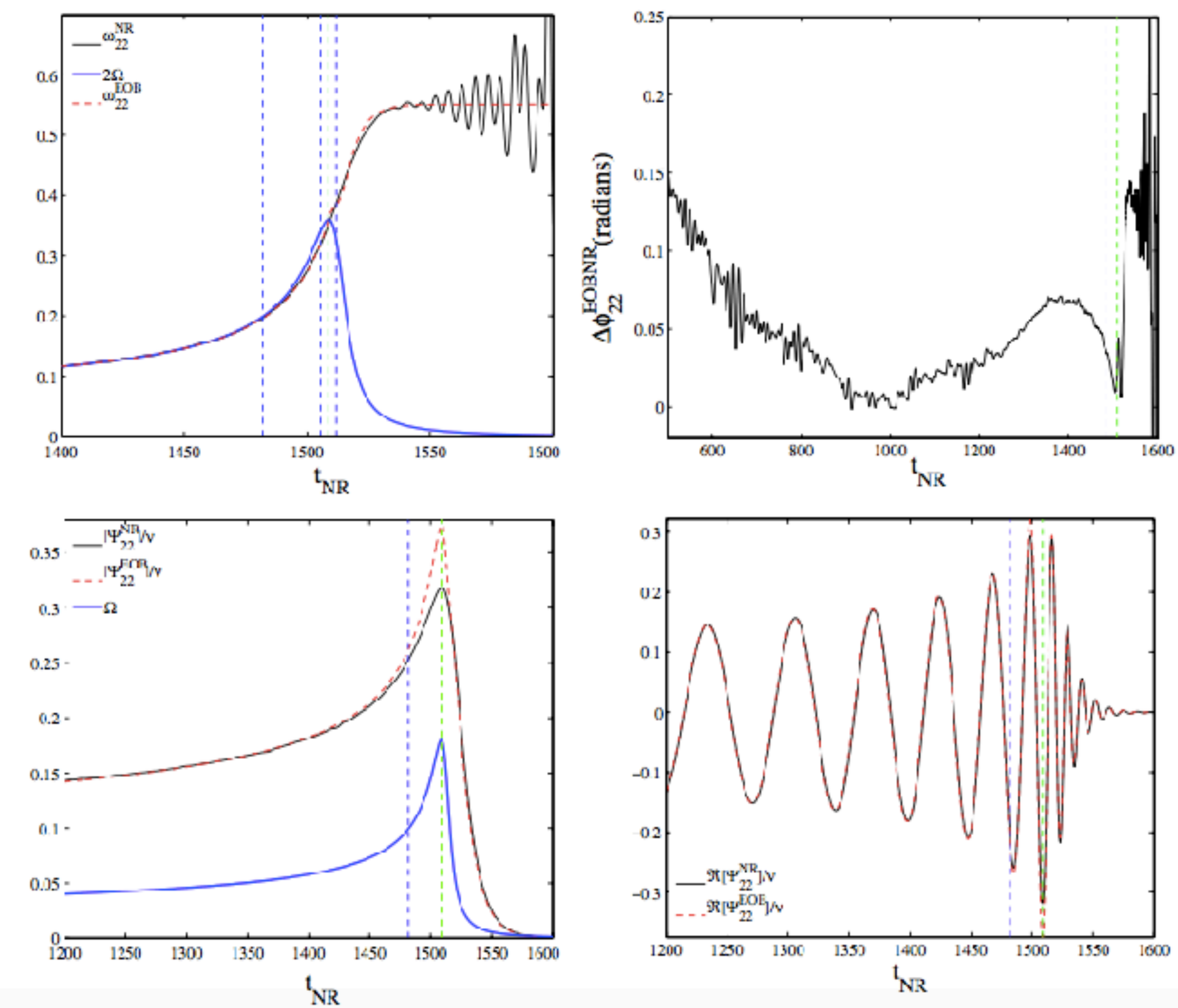


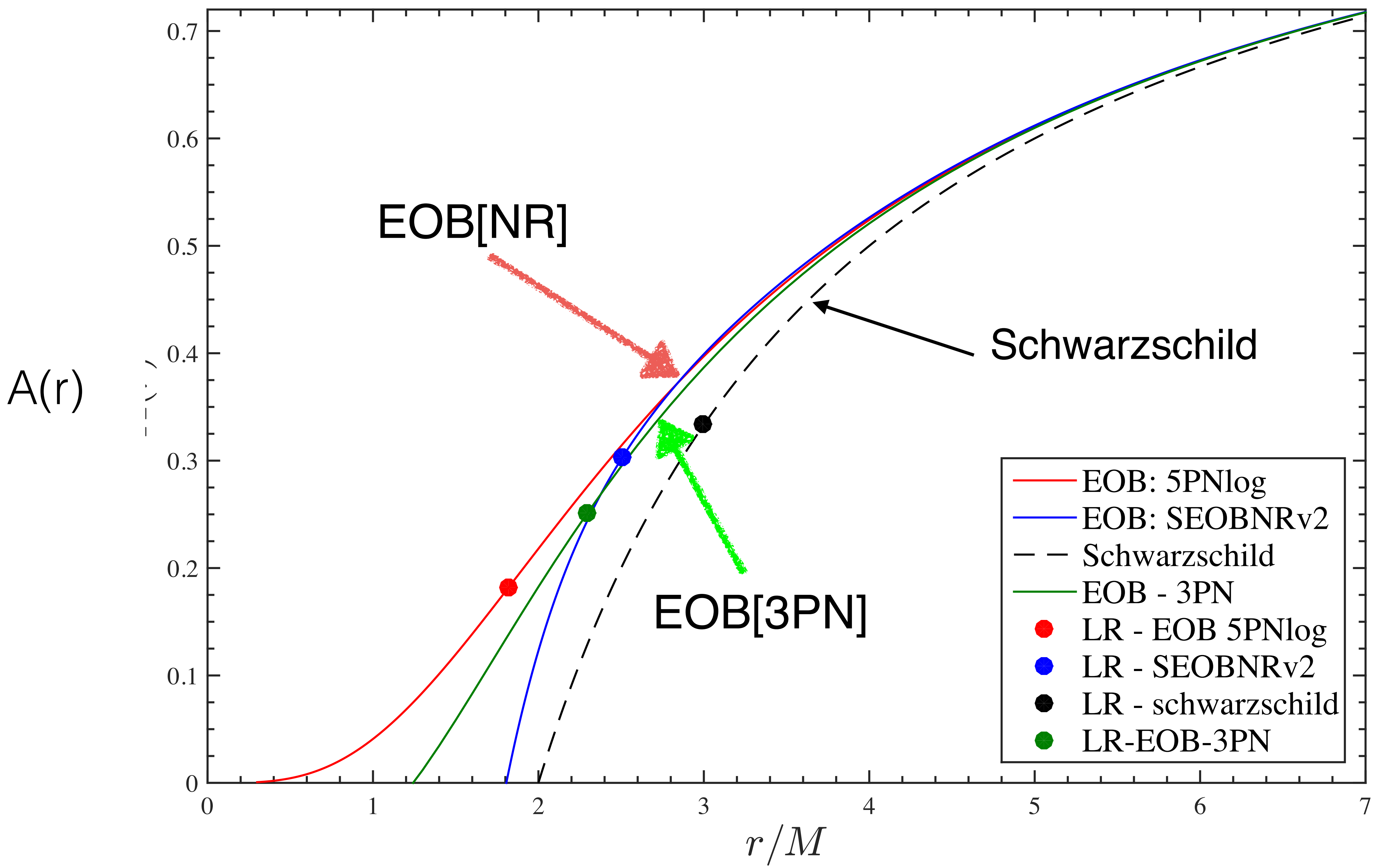
FIG. 21 (color online). We compare the NR and EOB frequency and Re$[_{-2}C_{22}]$ waveforms throughout the entire inspiral−merger−ring-down evolution. The data refers to the $d = 16$ run.

DAMOUR, NAGAR, DORBAND, POLLNEY, AND REZZOLLA — PHYSICAL REVIEW D 77, 084017 (2008)



MAIN RADIAL EOB POTENTIAL A(R)

m1=m2 case

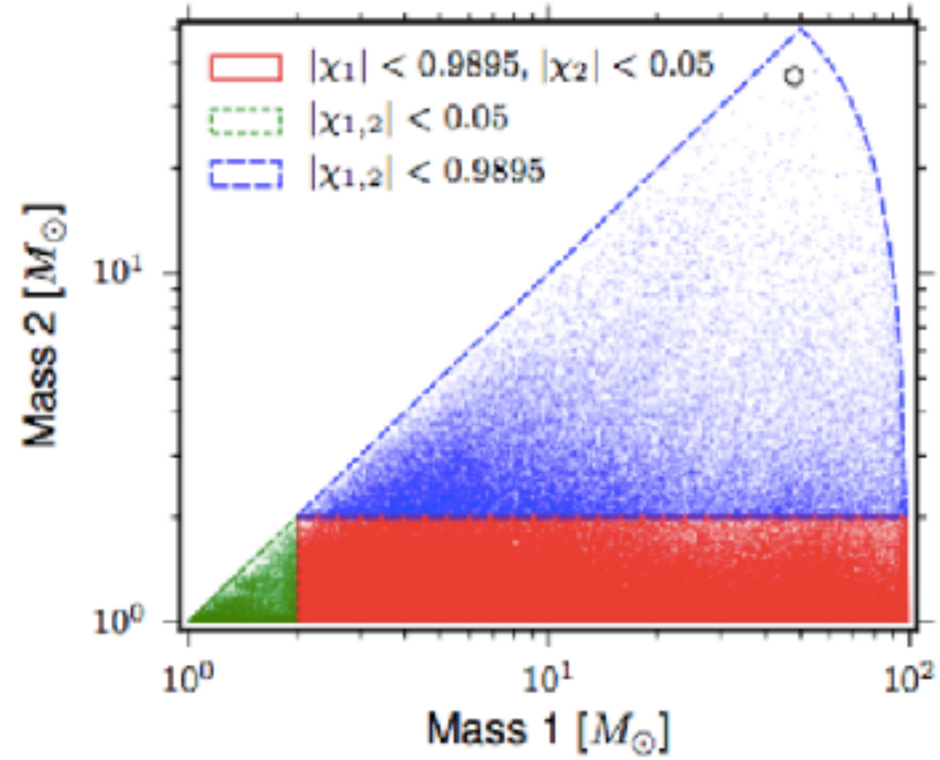


Tutti-Frutti strategy combining PN,PM,MPM,SF,EFT within EOB

(Bini-TD-Geralico'19)

LIGO's bank of EOB search templates

(Taracchini et al.'14, Bohé et al.'17,Ossokine et al.'20, Nagar et al.'20)



SF $m_1 \ll m_2$

NR

MPM

PN $v \ll c$

$R \gg GM/c^2$

PM

$R \gg GM/c^2$

Classical Scattering

EOB

LISA's templates via EOB[SF] ?

EFT

$v \sim c$

$R \sim GM/c^2$

Ongoing Fruitful Dialogue and Information Exchange

QFT perturbation theory

STRING perturbation theory

Quantum Scattering Amplitudes

EOB= Effective-One-Body

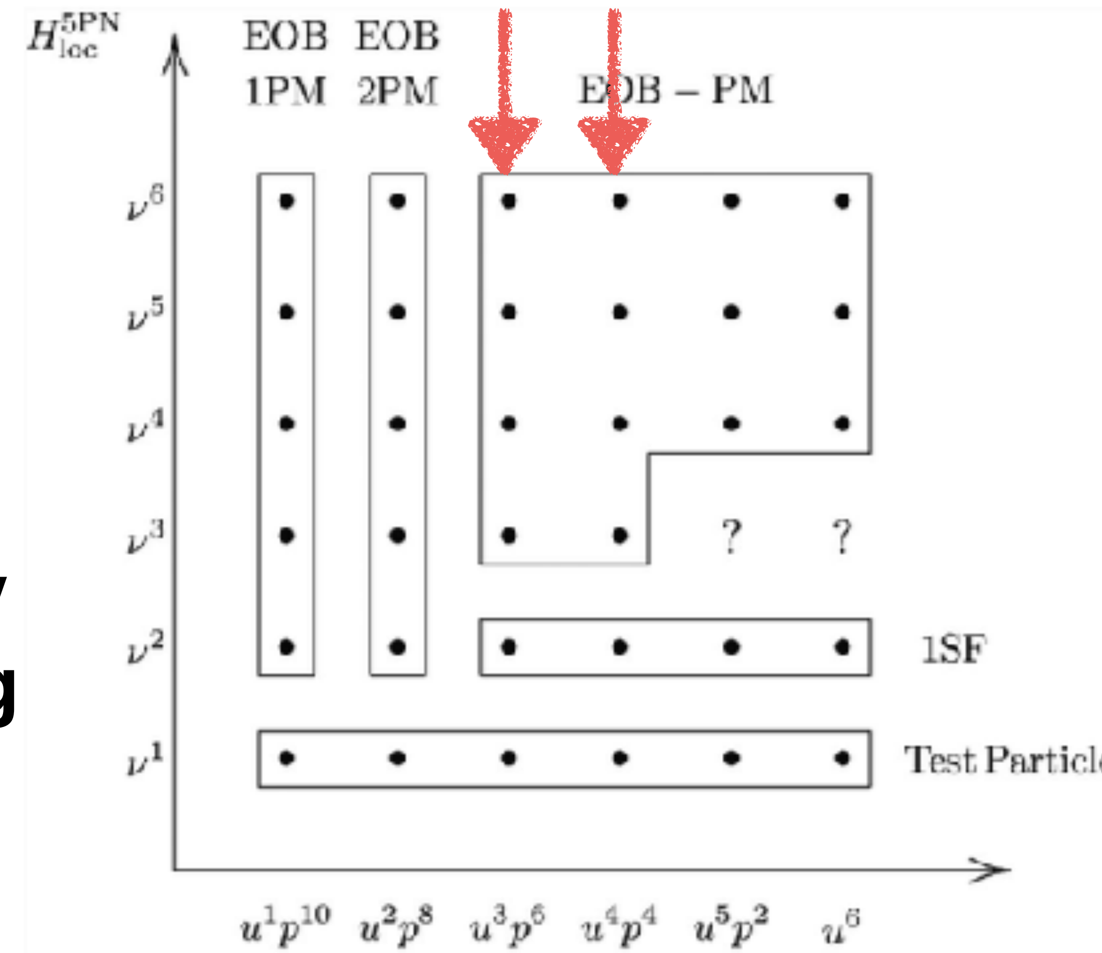
Buonanno-Damour 1999, 2000;
Damour-Jaranowski-Schaefer 2000;
Damour 2001

Novel Approach to Binary Dynamics: Application to the Fifth Post-Newtonian Level

Donato Bini[1,2] Thibault Damour[3] and Andrea Geralico[1]

Tutti Frutti: combine several efficient, complementary tools:

PN
PM
MPM
EOB
SF
Delaunay averaging



Step 1: Use MPM + EFT to separate off the nonlocal part

Step 2: Compute z_1^SF to e^6

Step 3: Use 1st law to transform z_1^SF into a pr^6 EOB Hamilt.

Step 4: Determine H^loc_1SF by subtracting the averaged H^nonloc

Step 5: Use EOB-PM theory to determine most of the nonlinear in nu dependence

$$S^{\leq n\mathrm{PN}}_{\mathrm{tot}}[x_1(s_1), x_2(s_2)] = S^{\leq n\mathrm{PN}}_{\mathrm{loc}}[x_1(s_1), x_2(s_2)] + S^{\leq n\mathrm{PN}}_{\mathrm{nonloc}}[x_1(s_1), x_2(s_2)]$$

$$\delta z_1^{e^6} = \nu\left[\frac{1}{4}u_p^3 + \left(-\frac{53}{12} - \frac{41}{128}\pi^2\right)u_p^4 + C_5 u_p^5 + C_6 u_p^6 + o(u_p^6)\right] + O(\nu^2),$$

$$Q = q_4(u;\nu)p_r^4 + q_6(u;\nu)p_r^6 + q_8(u;\nu)p_r^8 + \cdots$$

$$q_6(u;\nu) = \nu q_{62}^{\nu^1} u^2 + \nu q_{63}^{\nu^1} u^3 + O(u^{7/2}) + O(\nu^2)$$

SIXTH POST-NEWTONIAN LOCAL-IN-TIME DYNAMIC

(Bini-TD-Geralico'20)

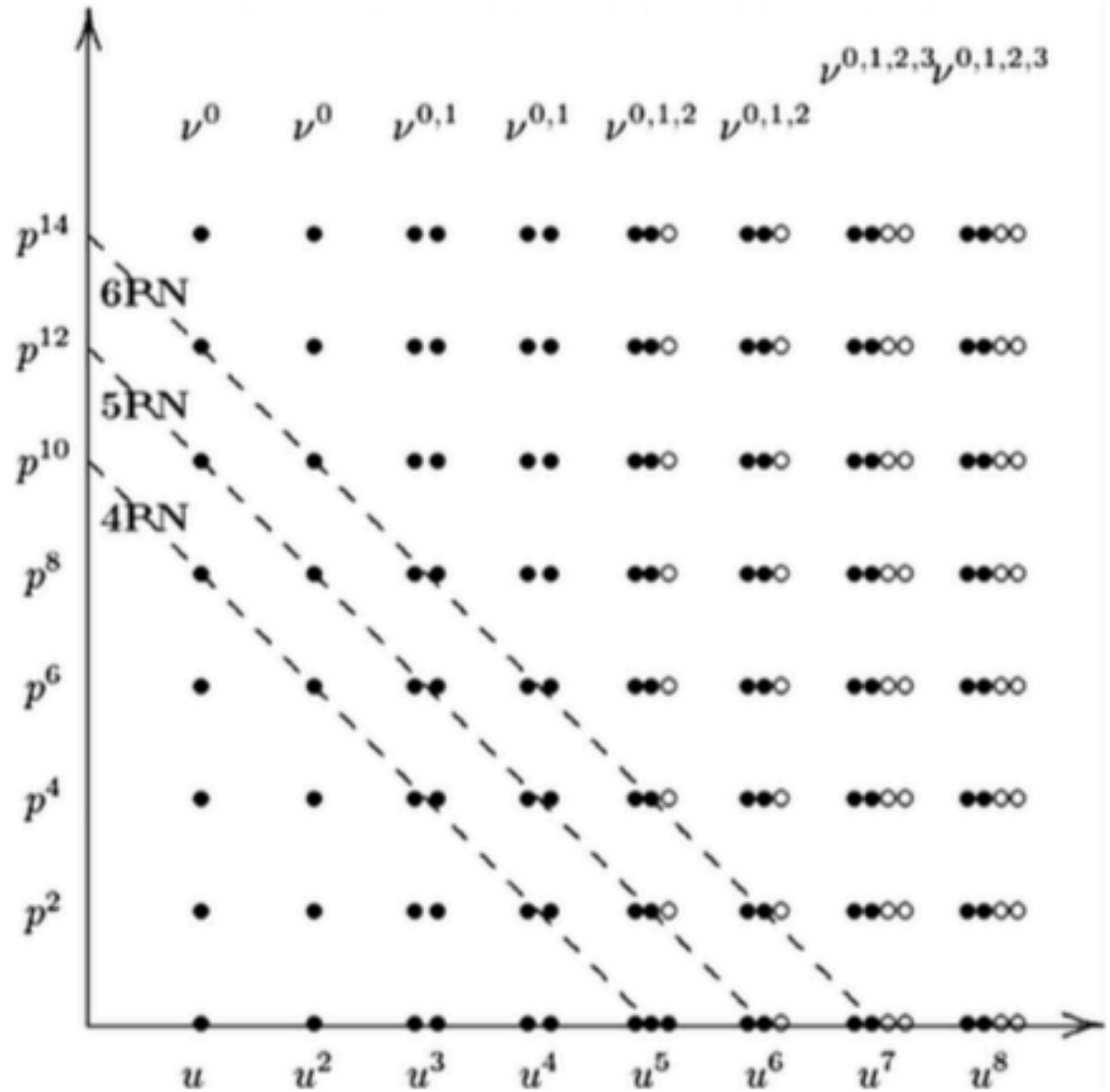


FIG. 1. Schematic representation of the irreducible information contained, at each post-Minkowskian level (keyed by a power of $u = GM/r$), in the local dynamics. Each vertical column of dots describes the post-Newtonian expansion (keyed by powers of p^2) of an energy-dependent function parametrizing the scattering angle. The various columns at a given post-Minkowskian level correspond to increasing powers of the symmetric mass-ratio ν. See text for details.

6PN dynamics complete at 3PM and 4PM

Inclusion of conservative nonlocal effects in EOB

Done using Delaunay-averaging and expansions in e or p_r

(TD-Jaranowski-Schaefer'15,Bini-TD-Geralico…)

Starting at 4PN, dynamics contains a nonlocal action

$$S^{4+5\mathrm{PN}}_{\mathrm{nonloc}}[x_1(s_1),x_2(s_2)] = \frac{G^2\mathcal{M}}{c^3}\int dt\,\mathrm{PF}_{2r^h_{12}(t)/c} \times \int \frac{dt'}{|t-t'|}\mathcal{F}^{\mathrm{split}}_{1\mathrm{PN}}(t,t').$$

$$\mathcal{F}^{\mathrm{split}}_{1\mathrm{PN}}(t,t') = \frac{G}{c^5}\left(\frac{1}{5}I^{(3)}_{ab}(t)I^{(3)}_{ab}(t') + \frac{1}{189c^2}I^{(4)}_{abc}(t)I^{(4)}_{abc}(t') + \frac{16}{45c^2}J^{(3)}_{ab}(t)J^{(3)}_{ab}(t')\right). \quad (($$

For elliptic motions, the 4PN nonlocal EOB Hamiltonian reads

$$A(u) = 1-2u+2\nu u^3+\left(\frac{94}{3}-\frac{41\pi^2}{32}\right)\nu u^4+\left(\left(\frac{2275\pi^2}{512}-\frac{4237}{60}+\frac{128}{5}\gamma_E+\frac{256}{5}\ln 2\right)\nu+\left(\frac{41\pi^2}{32}-\frac{221}{6}\right)\nu^2+\frac{64}{5}\nu\ln u\right)u^5, \quad (8.1a)$$

$$\bar{D}(u) = 1+6\nu u^2+(52\nu-6\nu^2)u^3+\left(\left(-\frac{533}{45}-\frac{23761\pi^2}{1536}+\frac{1184}{15}\gamma_E-\frac{6496}{15}\ln 2+\frac{2916}{5}\ln 3\right)\nu + \left(\frac{123\pi^2}{16}-260\right)\nu^2+\frac{592}{15}\nu\ln u\right)u^4, \quad (8.1b)$$

$$\hat{Q}(\mathbf{r}',\mathbf{p}') = \left(2(4-3\nu)\nu u^2+\left(\left(-\frac{5308}{15}+\frac{496256}{45}\ln 2-\frac{33048}{5}\ln 3\right)\nu-83\nu^2+10\nu^3\right)u^3\right)(\mathbf{n}'\cdot\mathbf{p}')^4 + \left(\left(-\frac{827}{3}-\frac{2358912}{25}\ln 2+\frac{1399437}{50}\ln 3+\frac{390625}{18}\ln 5\right)\nu-\frac{27}{5}\nu^2+6\nu^3\right)u^2(\mathbf{n}'\cdot\mathbf{p}')^6+\mathcal{O}[\nu u(\mathbf{n}'\cdot\mathbf{p}')^8].$$

Using classical and/or quantum gravitational scattering

Extracting PN-expanded dynamics from quantum scattering amplitudes:
Corinaldesi '56 '71, Barker-Gupta-Haracz 66, Barker-O'Connell 70, Iwasaki 71, Hiida-Okamura72,Okamura-Ohta-Kimura-Hiida 73,…,Bjerrum-Bohr-Donoghue-Vanhove 2014,….

Comparing EOB dynamics to NR simulations of classical scattering:
TD-Guercilena-Hopper etal '14; CheungRothsteinSolon'18; BCRSSZ'19;….

Extracting dynamical information from SF computations (TD'09, Barack et al'19)

Extracting PM-expanded dynamics from classical and/or quantum scattering:
TD'16,18,19; CheungRothsteinSolon'18; BCRSSZ'19;….

Several aspects:

dictionary classical scattering <—> Hamiltonian
dictionary quantum scattering <—> Hamiltonian

using either PN-expansion or PM-expansion

EOB, NR and (radiation-reacted)* scattering

*Bini-TD'12

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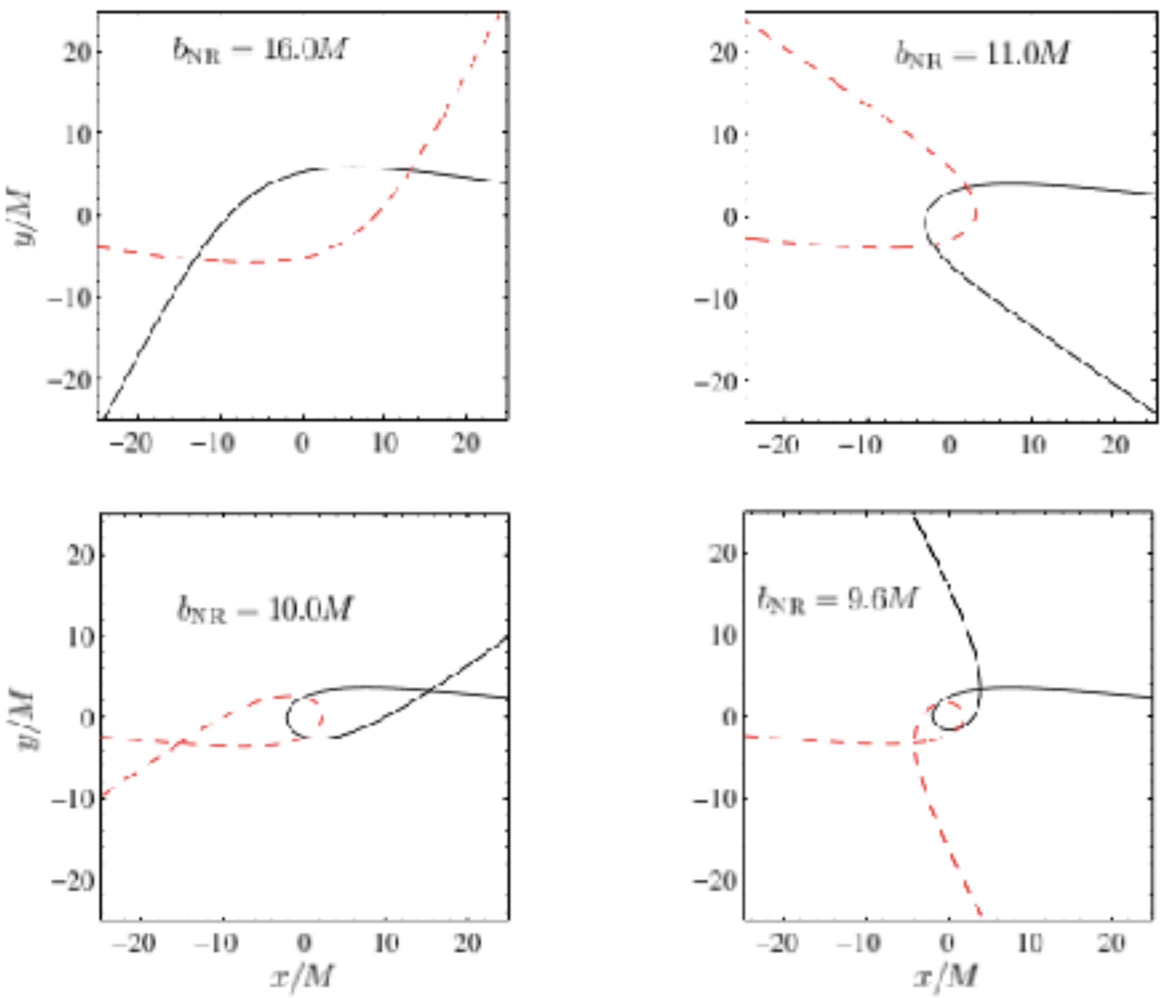
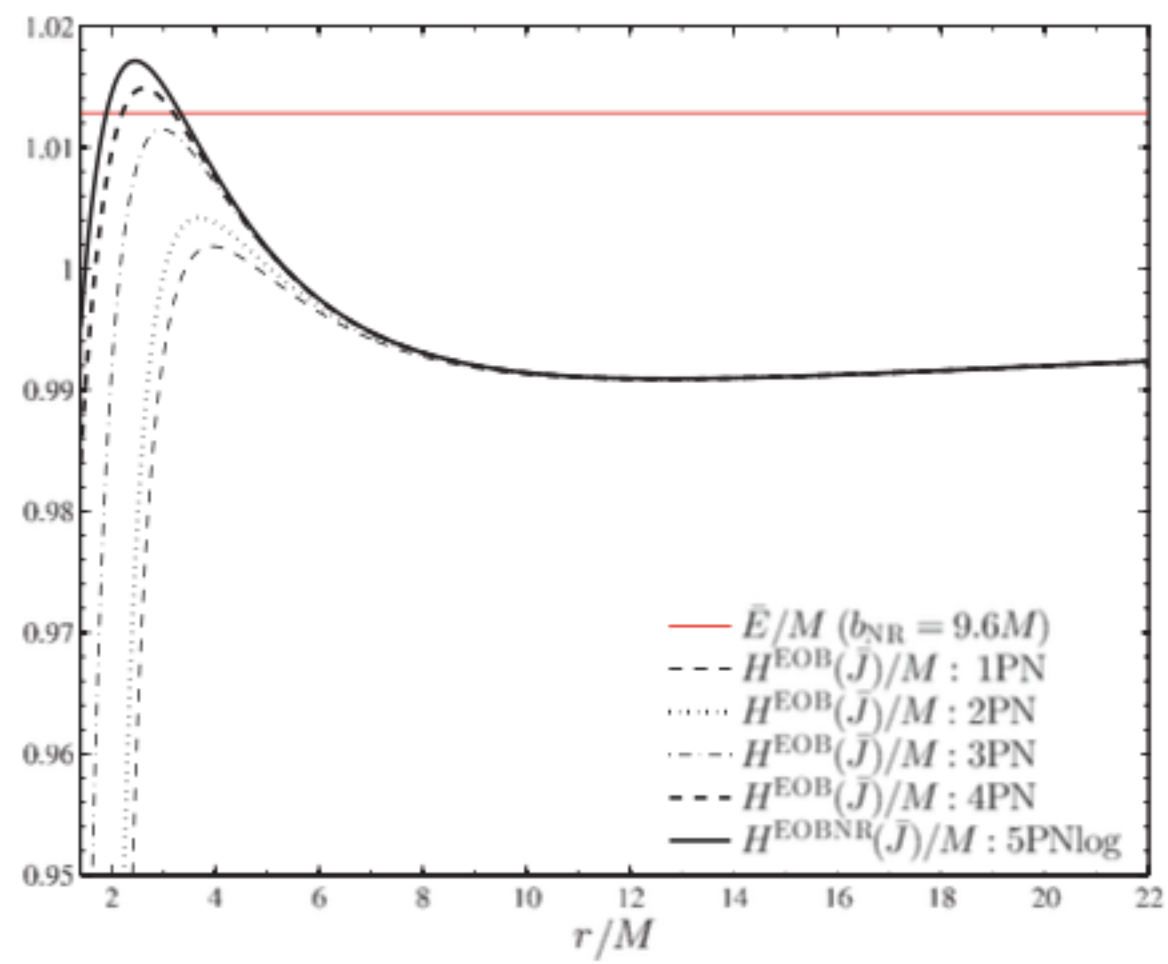
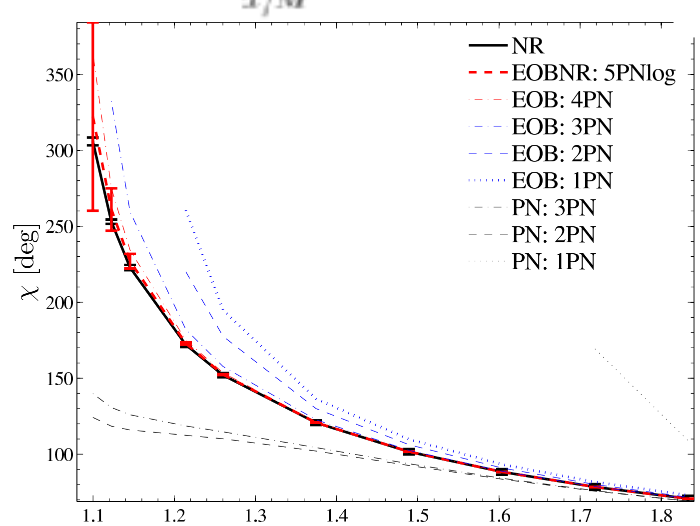
Strong-field scattering of two black holes: Numerics versus analytics

Thibault Damour,[1] Federico Guercilena,[2,3] Ian Hinder,[2] Seth Hopper,[2] Alessandro Nagar,[1] and Luciano Rezzolla[3,2]



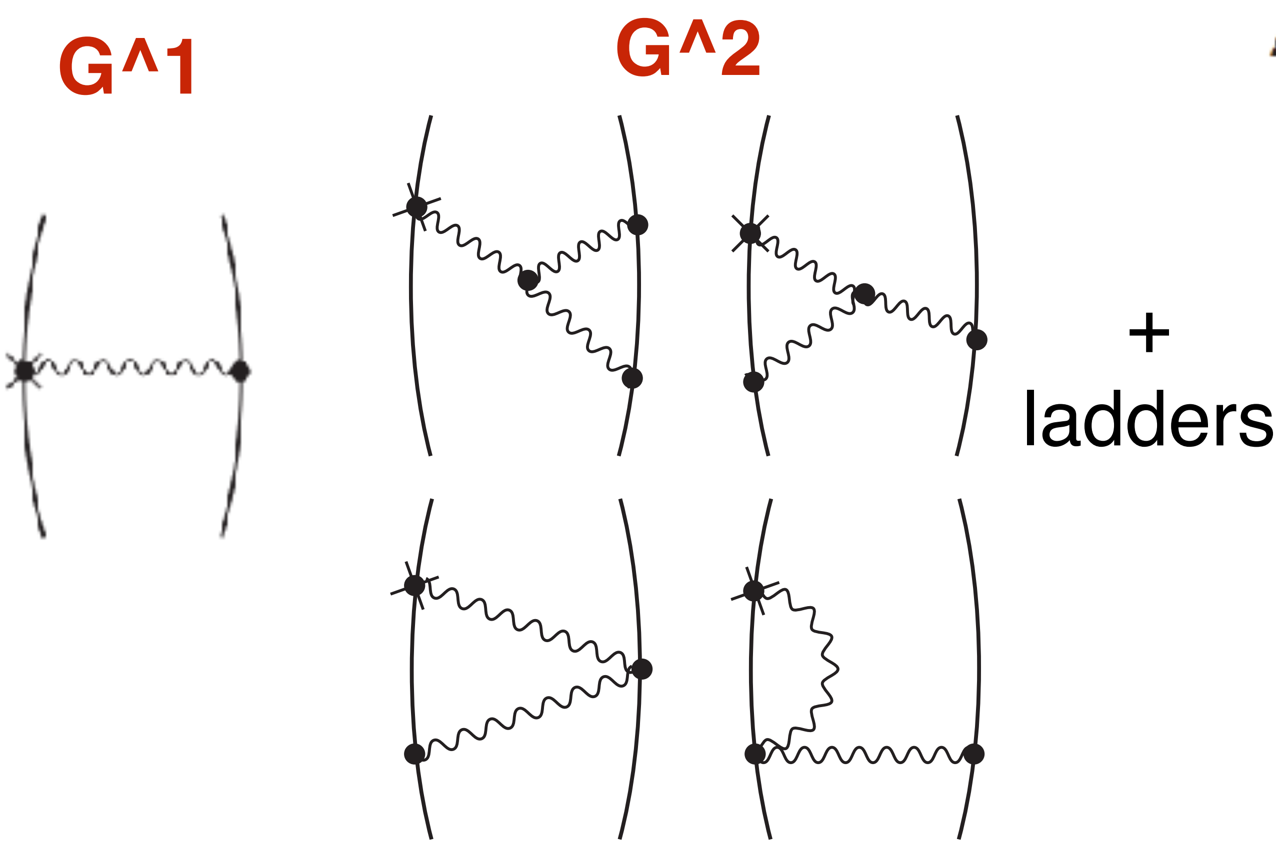




NR challenge:
Ultra-Relativistic scattering would be particularly interesting

PM Perturbation Theory for Classical Gravitational Scattering

Bel-Martin ’75-’81, Portilla ’79, Westpfahl ’85, Damour’16’18,…,Kälin-Porto’20



$$\Delta p_{1\mu} = 2G \int d\sigma_1 d\sigma_2 p_{1\alpha} p_{1\beta} \times \partial_\mu \mathcal{P}^{\alpha\beta;\alpha'\beta'}(x_1(\sigma_1) - x_2(\sigma_2)) p_{2\alpha'} p_{2\beta'} + 2G \int d\sigma_1 d\sigma'_1 p_{1\alpha} p_{1\beta} \partial_\mu \mathcal{P}^{\alpha\beta;\alpha'\beta'}(x_1(\sigma_1) - x_1(\sigma'_1)) p_{1\alpha'} p_{1\beta'} + O(G^2)$$

$$\Delta p_{1\mu} = 8\pi G \int \frac{d^4k}{(2\pi)^4} i k_\mu p_1^\alpha p_1^\beta \frac{P_{\alpha\beta;\alpha'\beta'}}{k^2} p_2^{\alpha'} p_2^{\beta'} \times \int d\sigma_1 \int d\sigma_2 e^{ik.(x_1(\sigma_1) - x_2(\sigma_2))}.$$

$$\frac{1}{2}\chi_{\rm class}(E,J) = \frac{1}{j}\chi_1(\hat{E}_{\rm eff},\nu) + \frac{1}{j^2}\chi_2(\hat{E}_{\rm eff},\nu) + O(G^3)$$

$$j \equiv \frac{J}{Gm_1m_2}$$

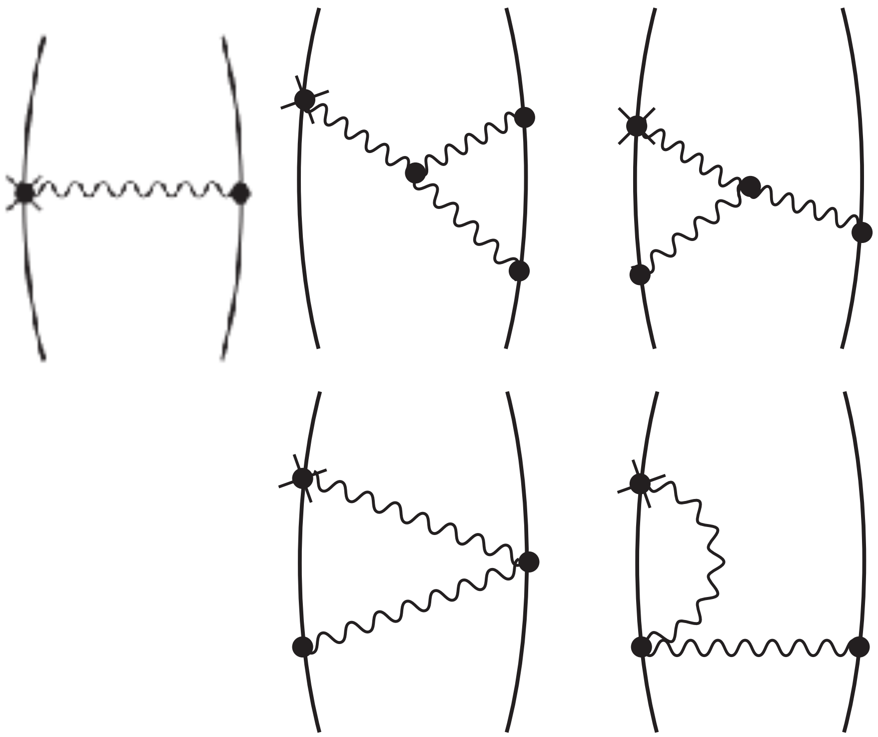
$$\frac{1}{2}\chi^{\rm real}_{1PM} = \frac{G}{J}\frac{2(p_1.p_2)^2 - p_1^2 p_2^2}{\sqrt{(p_1.p_2)^2 - p_1^2 p_2^2}}.$$

$$\hat{\mathcal{E}}_{\rm eff} \equiv \frac{\mathcal{E}_{\rm eff}}{\mu} \equiv \frac{(E_{\rm real})^2 - m_1^2 - m_2^2}{2m_1m_2} = \frac{s - m_1^2 - m_2^2}{2m_1m_2}.$$

$$\chi_1(\hat{\mathcal{E}}_{\rm eff},\nu) = \frac{2\hat{\mathcal{E}}_{\rm eff}^2 - 1}{\sqrt{\hat{\mathcal{E}}_{\rm eff}^2 - 1}}, \qquad \chi_2(\hat{\mathcal{E}}_{\rm eff},\nu) = \frac{3\pi}{8}\frac{5\hat{\mathcal{E}}_{\rm eff}^2 - 1}{\sqrt{1 + 2\nu(\hat{\mathcal{E}}_{\rm eff} - 1)}}.$$

Simple Map: Scattering angle <-> EOB dynamics

TD'16-18
Bini-TD-Geralico'20



$$\frac{1}{2}\chi = \Phi(E_{\rm real}, J; m_1, m_2, G)$$

$$0 = g_{\rm eff}^{\mu\nu} P_\mu P_\nu + \mu^2 + Q$$

$g_{\rm eff}^{\mu\nu}$
Schwarzschild metric M=m1+m2

$$Q = \left(\frac{GM}{R}\right)^2 q_2(E) + \left(\frac{GM}{R}\right)^3 q_3(E) + O(G^4)$$

$$q_2(\hat{\mathcal{E}}_{\rm eff}, \nu) = -\frac{4}{\pi}[\chi_2(\hat{\mathcal{E}}_{\rm eff}, \nu) - \chi_2^{\rm Schw}(\hat{\mathcal{E}}_{\rm eff})].$$

$$q_3(\hat{\mathcal{E}}_{\rm eff}, \nu) = \frac{4}{\pi}\frac{2\hat{\mathcal{E}}_{\rm eff}^2 - 1}{\hat{\mathcal{E}}_{\rm eff}^2 - 1}(\chi_2(\hat{\mathcal{E}}_{\rm eff}, \nu) - \chi_2^{\rm Schw}(\hat{\mathcal{E}}_{\rm eff})) - \frac{\chi_3(\hat{\mathcal{E}}_{\rm eff}, \nu) - \chi_3^{\rm Schw}(\hat{\mathcal{E}}_{\rm eff})}{\sqrt{\hat{\mathcal{E}}_{\rm eff}^2 - 1}}.$$

$$\mathcal{E}_{\rm eff} = \frac{(\mathcal{E}_{\rm real})^2 - m_1^2 - m_2^2}{2(m_1 + m_2)}.$$

$$\frac{\mathcal{E}_{\rm eff}}{\mu} = \gamma = -\frac{p_1 \cdot p_2}{m_1 m_2}$$

$$q_2(\gamma, \nu) = \frac{3}{2}(5\gamma^2 - 1)\left(1 - \frac{1}{h(\gamma, \nu)}\right)$$

$$h(\gamma, \nu) = \sqrt{1 + 2\nu(\gamma - 1)}$$

Linear combinations of the scattering coefficients!

Application to the ACV eikonal scattering phase (massless or ultra-relativistic scattering)

Amati-Ciafaloni-Veneziano'90+ Ciafaloni-Colferai'14+ Bern et al'20+ DiVecchia et al'20

$$\delta^{\rm eikonal} = \frac{1}{\hbar}(\delta^{\rm R} + i\delta^{\rm I}) + \text{quantum corr.}$$

$$\frac{1}{2}\chi^{\rm eikonal} = 2\frac{\gamma}{j} + \frac{16}{3}\frac{\gamma^3}{j^3} + \cdots$$

valid in the HE limit gamma-> infty

Using the chi—> Q dictionary this corresponds to the HE limits:

$$q_2^{\rm HE} = \frac{15}{2}\gamma^2$$
$$q_3^{\rm HE} = \gamma^2$$

i.e. an HE limit for the EOB mass-shell condition (TD'18)

$$0 = g_{\rm eff}^{\mu\nu}(X)P_\mu P_\nu + \mu^2 + Q(X,P)$$

$$0 = g^{\mu\nu}_{\rm Schw}P_\mu P_\nu + \left(\frac{15}{2}\left(\frac{GM}{R}\right)^2 + \left(\frac{GM}{R}\right)^3\right)P_0^2$$

High-energy gravitational scattering and the general relativistic two-body problem

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A technique for translating the classical scattering function of two gravitationally interacting bodies into a corresponding (effective one-body) Hamiltonian description has been recently introduced [Phys. Rev. D **94**, 104015 (2016)]. Using this technique, we derive, for the first time, to second-order in Newton's constant (i.e. one classical loop) the Hamiltonian of two point masses having an arbitrary (possibly relativistic) relative velocity. The resulting (second post-Minkowskian) Hamiltonian is found to have a tame high-energy structure which we relate both to gravitational self-force studies of large mass-ratio binary systems, and to the ultra high-energy quantum scattering results of Amati, Ciafaloni and Veneziano. We derive several consequences of our second post-Minkowskian Hamiltonian: (i) the need to use special phase-space gauges to get a tame high-energy limit; and (ii) predictions about a (rest-mass independent) linear Regge trajectory behavior of high-angular-momenta, high-energy circular orbits. Ways of testing these predictions by dedicated numerical simulations are indicated. We finally indicate a way to connect our classical results to the quantum gravitational scattering amplitude of two particles, and we urge amplitude experts to use their novel techniques to compute the two-loop scattering amplitude of scalar masses, from which one could deduce the third post-Minkowskian effective one-body Hamiltonian.

3PM computation (Bern-Cheung-Roiban-Shen-Solon-Zeng'19)

using a combination of techniques: generalized unitarity; BCJ double-copy; 2-loop amplitude of quasi-classical diagrams; **EFT transcription** (Cheung-Rothstein-Solon'18); **resummation of PN-expanded integrals for potential-gravitons**

$$\chi_3^{\rm cons} = \chi_3^{\rm Schw} - \frac{2\nu\sqrt{\gamma^2-1}}{h^2(\gamma,\nu)}\,\bar{C}^{\rm cons}(\gamma)$$

$$q_3^{\rm cons} = \frac{3}{2}\frac{(2\gamma^2-1)(5\gamma^2-1)}{\gamma^2-1}\left(\frac{1}{h(\gamma,\nu)}-1\right) + \frac{2\nu}{h^2(\gamma,\nu)}\bar{C}^{\rm cons}(\gamma)$$

$$\overline{C}^{\rm cons}(\gamma) = \frac{2}{3}\gamma(14\gamma^2+25) + 2(4\gamma^4-12\gamma^2-3)\frac{\mathcal{A}(v)}{\sqrt{\gamma^2-1}}$$

$$h(\gamma,\nu) \equiv \frac{\sqrt{s}}{M} = \sqrt{1+2\nu(\gamma-1)}$$

$$\mathcal{A}(v) \equiv \text{arctanh}(v) = \frac{1}{2}\ln\frac{1+v}{1-v} = 2\,\text{arcsinh}\sqrt{\frac{\gamma-1}{2}}$$

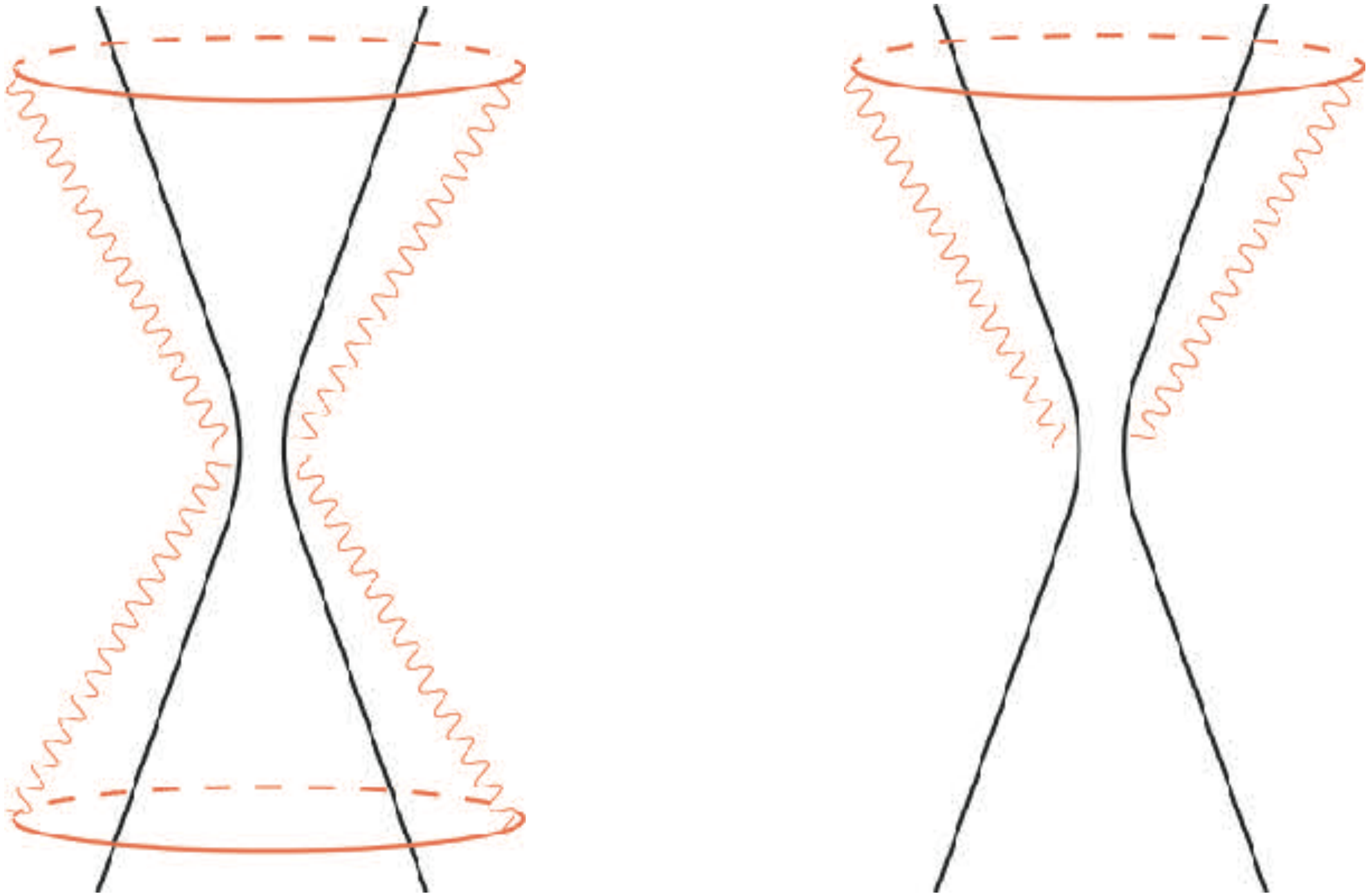
puzzling HE limits when compared to ACV and Akcay et al'12

$$\frac{1}{2}\chi^{\rm cons} = 2\frac{\gamma}{j} + (12 - 8\ln(2\gamma))\frac{\gamma^3}{j^3} + O(G^4)$$

$q_3^{\rm cons} \approx +8\ln(2\gamma)\gamma^2$ instead of $q_3^{\rm ACV} \approx +1\gamma^2$

confirmations: 5PN (Bini-TD-Geralico'19); 6PN (Blümlein-Maier-Marquard-Schäfer'20, Bini-TD-Geralico'20); 3PM (Cheung-Solon'20, Kälin-Porto'20)

Conservative vs Radiation-reacted
Classical Gravitational Scattering



Radiation-reaction effects enter scattering at G^3/c^5 (Bini-TD'12)

$$\frac{1}{2}\chi^{\rm rad} = +\frac{8G^3}{5c^5}\frac{m_1^3 m_2^3}{J^3}\nu v^2 + \cdots$$

Radiation-reaction effects in scattering play a crucial role at **high-energy**

(DiVecchia-Heissenberg-Russo-Veneziano'21, TD'21, Hermann-Parra-Martinez-Ruf-Zeng'21,….)

Radiation-Reaction Contribution to the Classical Scattering Angle at G^3 (TD 2010.01641)

$$\chi^{\rm tot} = \chi^{\rm cons} + \chi^{\rm rad}$$

where, to first order in Rad-Reac, one has (Bini-TD'12)

$$\chi^{\rm rad}(E,J) = -\frac{1}{2}\frac{\partial \chi^{\rm cons}}{\partial E} E^{\rm rad} - \frac{1}{2}\frac{\partial \chi^{\rm cons}}{\partial J} J^{\rm rad}$$

O(G^2) [TD-Deruelle'81]

chi^cons=O(G^1)

O(G^3)

O(G^4)

O(G^3)

$$h_{ij}^{\rm TT} = \frac{f_{ij}(t-r,\theta,\phi)}{r} + O\left(\frac{1}{r^2}\right)$$

$$J_k^{\rm rad} = \frac{\epsilon_{kij}}{16\pi G}\int du\, d\Omega \left[f_{ia}\partial_u f_{ja} - \frac{1}{2} x^i \partial_j f_{ab} \partial_u f_{ab} \right]$$

DeWitt'71, Thorne'80
Kovacs-Thorne'77, Bel et al'81,
Westpfahl'85

$$\mathcal{I}(v) = -\frac{16}{3} + \frac{2}{v^2} + \frac{2(3v^2-1)}{v^3}\mathcal{A}(v) \qquad \mathcal{A}(v) \equiv {\rm arctanh}(v) = \frac{1}{2}\ln\frac{1+v}{1-v}$$

$$\frac{1}{2}\chi^{\rm rad}(\gamma,j,\nu) = +\frac{\nu}{h^2(\gamma,\nu)j^3}(2\gamma^2-1)^2\,\mathcal{I}(v) + O(G^4)$$

$$\frac{1}{2}(\chi^{\rm cons} + \chi^{\rm rad}) = 2\frac{\gamma}{j} + \frac{16}{3}\frac{\gamma^3}{j^3} = \chi^{\rm ACV}$$

Challenge: Radiative Contributions to the Classical Scattering at G^4

Recent amplitude computation of **potential-graviton** contribution to **conservative** 4PM (G^4) dynamics (Bern et al ’21)

Need to add several types of radiation-related contributions:

radiation-graviton conservative nonlocal contribution
radiation-reaction contributions

relevant works: Foffa-Sturani’19, Bluemlein et al ’21, Hermann-Parra-Martinez-Ruf-Zeng’21,Bini-TD-Geralico’21

only missing 5PN parameters

$$a_6^{\nu^2} = r_{a_6} + \frac{25911}{256}\pi^2 ,$$
$$\bar{d}_5^{\nu^2} = r_{\bar{d}_5} + \frac{306545}{512}\pi^2$$

preliminary results for r6,r5:

$$r_{a_6} = -\frac{12969847}{12075} - \frac{256}{27}C_1 - \frac{21760}{621}C_2$$
$$r_{\bar{d}_5} = -\frac{6520279}{1035} - \frac{1216}{27}C_1 - \frac{91456}{621}C_2 .$$

Challenges in translating quantum scattering amplitudes into classical dynamical information

$$\mathcal{M}(s,t) = \mathcal{M}^{(\frac{G}{\hbar})}(s,t) + \mathcal{M}^{(\frac{G^2}{\hbar^2})}(s,t) + \cdots,$$

$$\mathcal{M}^{(\frac{G}{\hbar})}(s,t) = 16\pi \frac{G}{\hbar} \frac{2(p_1 \cdot p_2)^2 - p_1^2 p_2^2}{-t}.$$

$$\alpha_g \equiv \frac{G E_1 E_2}{\hbar}$$

Problem: The domain of validity of the Born-Feynman expansion is GE_1 E_2/(hbar v) << 1, while the domain of validity of the classical scattering is GE_1 E_2/(hbar v) >> 1! (Bohr 1948)

$$\mathcal{M} \sim \frac{Gs}{\hbar} + \left(\frac{Gs}{\hbar}\right)^2 + \left(\frac{Gs}{\hbar}\right)^3 + \ldots$$
$$\sim \alpha_g + \alpha_g^2 + \alpha_g^3 + \ldots$$

$$\hbar \to \infty \text{ vs } \hbar \to 0$$

Ways of recovering the classical information from M(s,t) ?

Focus on non-analytic terms in q =sqrt(-t) in the q-> limit ? (Donoghue'94,….,Neill-Rothstein'13,…,Cachazo-Guevara'17,Damour'17, Cheung et al'18,Bern et al.'19,…)

Control and resum the exponentiated terms in an eikonal-like approximation? ('tHooft'86, ACV'86-90,…,Akhoury et al'13,Bjerrum-Bohr et al'18,Koemans-Collado et al'19

Compute the quasi-classical impulse Delta p from amplitude ? (Kosower-Maybee-O'Connell'19)

Not very efficient way of including radiation-reaction effects?

Henri Poincaré

**«Il n'y a pas de problèmes résolus,
il y a seulement des problèmes
plus ou moins résolus »**

**«There are no solved problems,
there are only
more or less solved problems »**