Galileo Galilei Institute
Gravitational Scattering, Inspiral, and Radiation

Traditional Post-Newtonian Approach to Compact Binary Inspiral

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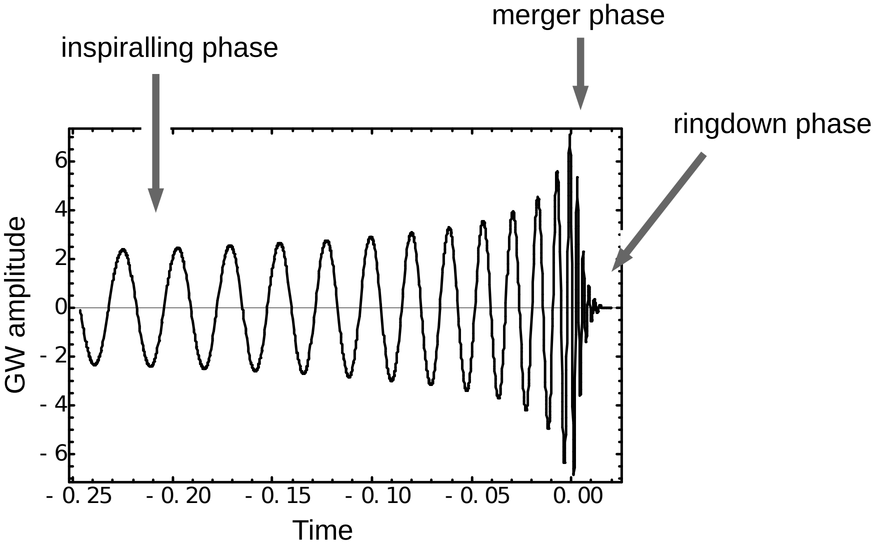
The gravitational chirp of binary black holes



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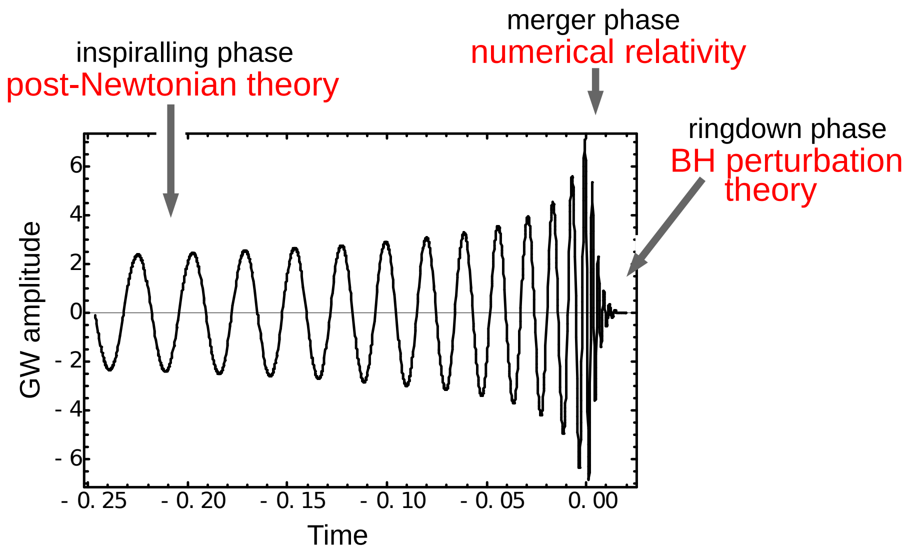
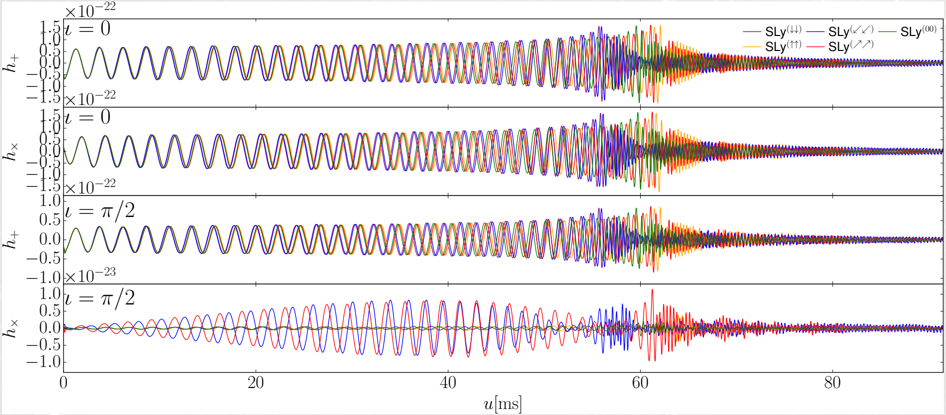
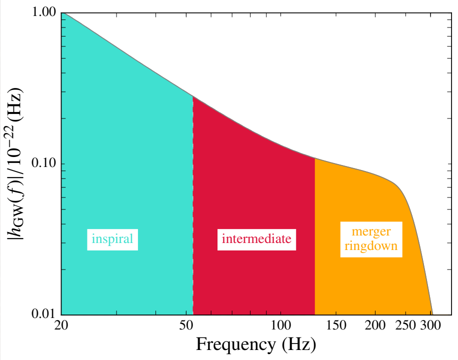


Post-merger waveform of neutron star binaries

[Dietrich, Bernuzzi, Bruegmann, Ujevic & Tichy 2018]



The inspiral-merger-ringdown models

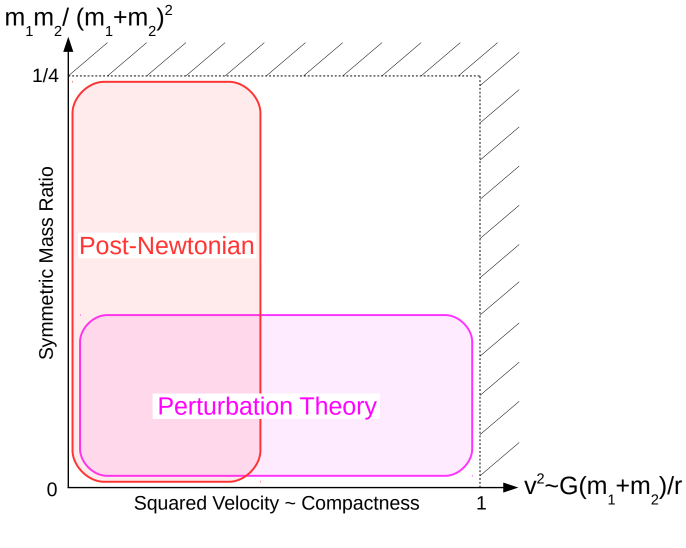


These models interpolate between the different phases play a crucial role

- The effective-one-body (EOB) approach [Buonanno & Damour 1999]
- The inspiral-merger-ringdown (IMR) [Ajith *et al.* 2008]

$$\{\underbrace{\text{PN parameters;}}_{\text{inspiral}} \; \underbrace{\beta_2, \beta_3}_{\text{intermediate}} \; ; \; \underbrace{\alpha_2, \alpha_3, \alpha_4}_{\text{merger-ringdown}} \}$$

Methods to compute GW templates



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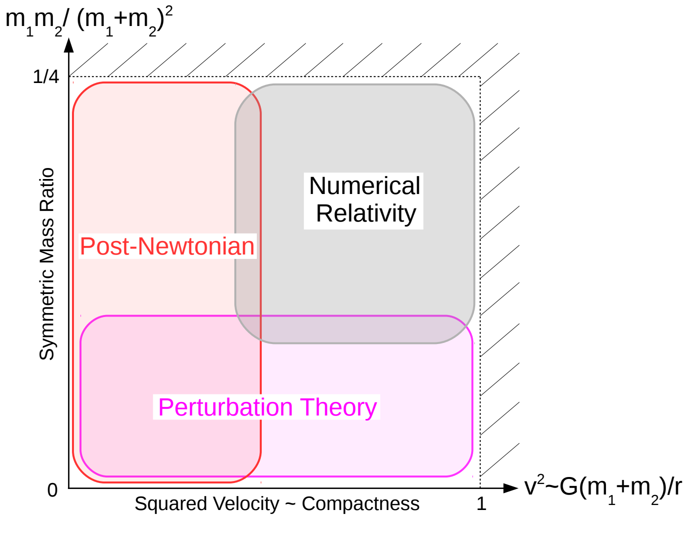
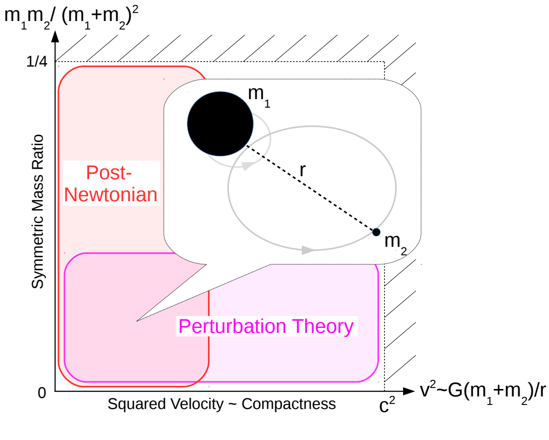
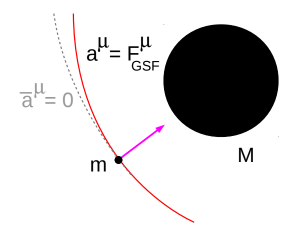


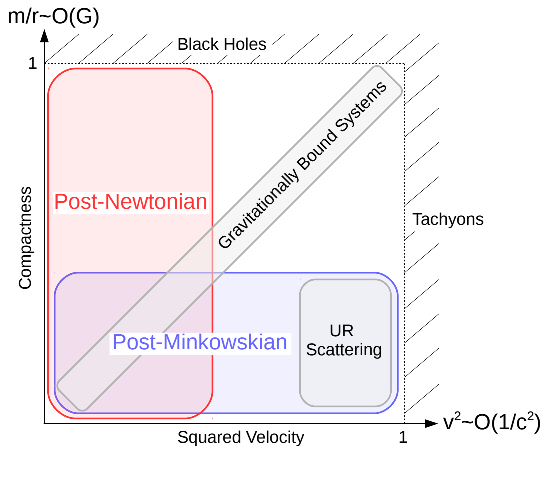




$$a^\mu = F^\mu_{\mathsf{GSF}} = \mathcal{O}\left(\frac{m}{M}\right)$$

PN predictions for the conservative dynamics are consistent with linear GSF calculations up to high order [Detweiler 2008; Blanchet, Detweiler, Le Tiec & Whiting 2010]

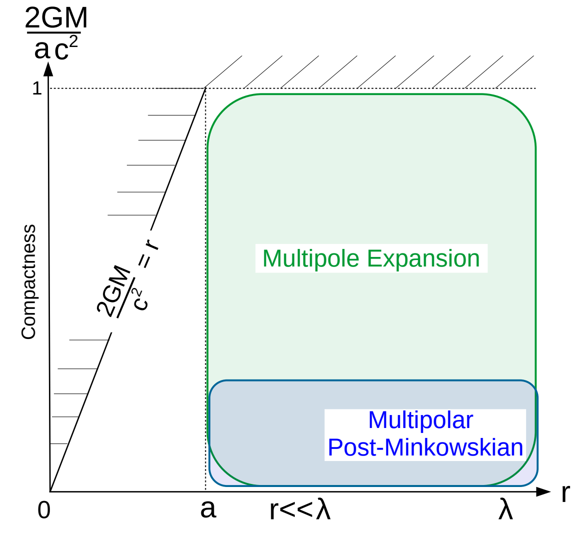
Post-Newtonian versus post-Minkowskian



The post-Minkowskian 3PM two-body Hamiltonian [Bern, Cheung, Solon *et al.* 2019] has been checked with the post-Newtonian 4PN two-body equations of motion

PN-matched Multipolar-Post-Minkowskian

[Blanchet & Damour 1986; Blanchet 1998]



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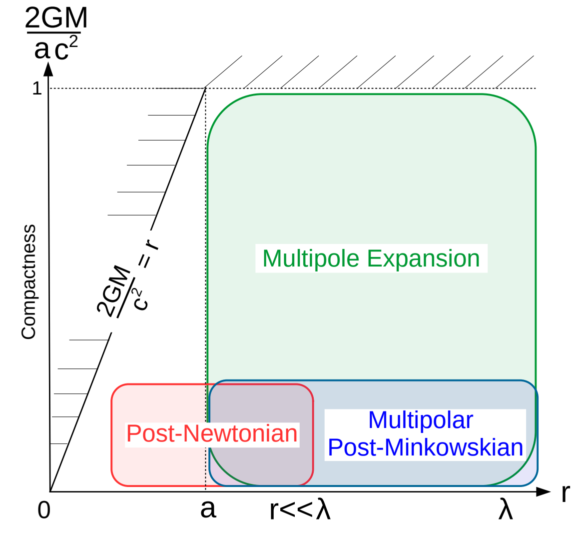
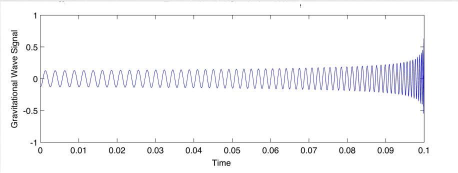


PN parameters in the orbital phase evolution



- The PN parameters come from a mixture of conservative and dissipative effects through the energy balance equation

$$\frac{\mathrm{d}\overbrace{E}^{\text{conservative energy}}}{\mathrm{d}t} = - \underbrace{\mathcal{F}^{\mathrm{GW}}}_{\text{dissipative energy flux}}$$

- The orbital phase $\phi = \int \omega \, \mathrm{d}t$ is obtained as a function of $x = \left(\frac{GM\omega}{c^3}\right)^{2/3}$ and the mass ratio $\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$

$$\phi(x) = \phi_0 - \frac{x^{-5/2}}{32\nu} \sum_p \left(\varphi_{p\mathrm{PN}}(\nu) + \varphi^{(l)}_{p\mathrm{PN}}(\nu) \log x \right) x^p + \mathcal{O}\left[(\log x)^2\right]$$

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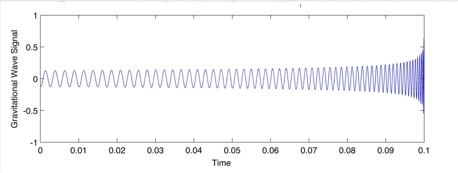

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$$\boxed{\phi(x) = \phi_0 - \frac{x^{-5/2}}{32\nu} \sum_p \Big(\varphi_{p\text{PN}}(\nu) + \varphi^{(l)}_{p\text{PN}}(\nu) \, \log x \Big) \, x^p + \mathcal{O}[(\log x)^2]}$$

The known 3.5PN parameters [Blanchet 2014 for a review]

They were computed using the Multipolar-post-Minkowskian-PN approach

$$\varphi_{0\mathrm{PN}} = 1 \qquad \Longleftarrow \text{ Einstein quadrupole formula}$$

$$\varphi_{1\mathrm{PN}} = \frac{3715}{1008} + \frac{55}{12}\nu$$

$$\varphi_{1.5\mathrm{PN}} = -10\pi$$

$$\varphi_{2\mathrm{PN}} = \frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2$$

$$\varphi^{(l)}_{2.5\mathrm{PN}} = \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi$$

$$\varphi_{3\mathrm{PN}} = \frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}\gamma_\mathrm{E} - \frac{3424}{21}\ln 2 + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3$$

$$\varphi^{(l)}_{3\mathrm{PN}} = -\frac{856}{21}$$

$$\varphi_{3.5\mathrm{PN}} = \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi$$

Fokker action versus effective action

$$S_{\rm g}[\boldsymbol{x},h] = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\Big[\overbrace{R}^{\text{Einstein-Hilbert Lagrangian}} \underbrace{-\frac{1}{2}\Gamma^\mu\Gamma_\mu}_{\text{gauge-fixing term}}\Big] - \sum_a \underbrace{m_a\int \mathrm{d}\tau_a}_{\text{point particles}}$$

- **Traditional PN approach:** compute the Fokker action by inserting an explicit iterated PN solution of the Einstein field equations

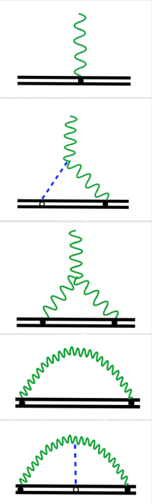
$$h^{\mu\nu}(\mathbf{x},t) \longrightarrow \overline{h}^{\mu\nu}(\mathbf{x};\boldsymbol{x}_a(t),\boldsymbol{v}_a(t),\cdots)$$
$$S_{\text{Fokker}}[\boldsymbol{x}] = S_{\rm g}[\boldsymbol{x},\overline{h}(\boldsymbol{x})]$$

- **Effective field theory:** compute the effective action by integrating over the gravitational degrees of freedom

$$\mathrm{e}^{\mathrm{i}S_{\text{eff}}[\boldsymbol{x}]} = \int \mathcal{D}[h]\,\mathrm{e}^{\mathrm{i}S_{\rm g}[\boldsymbol{x},h]}$$

Diagrammatic expansion in EFT

Effective Field Theory



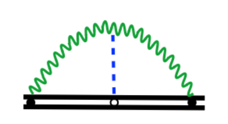
$\Longleftrightarrow$

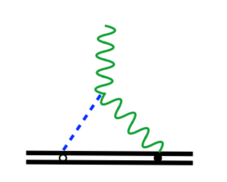
Post-Newtonian

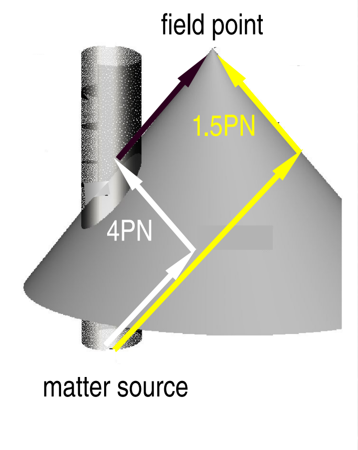
- emission from a quadrupole source
- tail effect in radiation field (1.5PN)
- non-linear memory effect (2.5PN)
- radiation reaction (2.5PN)
- tail in radiation reaction (4PN)

The EFT is equivalent to the traditional PN at the level of tree diagrams

The gravitational wave tail effect [Blanchet & Damour 1988, 1992]







- In the near zone (4PN effect)

$$S^{\text{tail}} = \frac{G^2 M}{5c^8} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} I_{ij}^{(3)}(t)\, I_{ij}^{(3)}(t')$$

- In the far zone (1.5PN effect)

$$h_{ij}^{\text{tail}} = \frac{4G}{c^4 r} \frac{GM}{c^3} \int_{-\infty}^{u} \mathrm{d}u' I_{ij}^{(4)}(u') \ln\left(\frac{u-u'}{P}\right)$$

Tail effects in PN parameters

tail terms

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$$\varphi^{(l)}_{3\text{PN}} = -\frac{856}{21}$$

$$\varphi_{3.5\text{PN}} = \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi$$

Tail effects in PN parameters

tail terms

tail-of-tail terms

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The 4.5PN radiative quadrupole moment

$$U_{ij}(t) = I^{(2)}_{ij}(t) + \underbrace{\frac{GM}{c^3}\int_0^{+\infty} d\tau I^{(4)}_{ij}(t-\tau)\left[2\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{11}{6}\right]}_{\text{1.5PN tail integral}}$$

$$+\frac{G}{c^5}\left\{\underbrace{-\frac{2}{7}\int_0^{+\infty} d\tau I^{(3)}_{a<i}I^{(3)}_{j>a}(t-\tau)}_{\text{2.5PN memory integral}} + \text{instantaneous terms}\right\}$$

$$+\underbrace{\frac{G^2M^2}{c^6}\int_0^{+\infty} d\tau I^{(5)}_{ij}(t-\tau)\left[2\ln^2\left(\frac{\tau}{2\tau_0}\right)+\frac{57}{35}\ln\left(\frac{\tau}{2\tau_0}\right)+\frac{124627}{22050}\right]}_{\text{3PN tail-of-tail integral}}$$

$$+\underbrace{\frac{G^3M^3}{c^9}\int_0^{+\infty} d\tau I^{(6)}_{ij}(t-\tau)\left[\frac{4}{3}\ln^3\left(\frac{\tau}{2\tau_0}\right)+\cdots+\frac{129268}{33075}+\frac{428}{315}\pi^2\right]}_{\text{4.5PN tail-of-tail-of-tail integral}}$$

$$+\mathcal{O}\left(\frac{1}{c^{10}}\right)$$

Toward 4.5PN parameters

- The 4.5PN term is also known and due to the 4.5PN tail-of-tail-of-tail integral for circular orbits [Marchand, Blanchet & Faye 2017; Messina & Nagar 2017]

$$\varphi_{\text{4.5PN}} = \Big(-\frac{93098188434443}{150214901760} + \frac{80}{3}\pi^2 + \frac{1712}{21}\gamma_\text{E} + \frac{3424}{21}\ln 2 + \left[\frac{1492917260735}{1072963584} - \frac{2255}{48}\pi^2\right]\nu - \frac{45293335}{1016064}\nu^2 - \frac{10323755}{1596672}\nu^3 \Big)\pi$$

$$\varphi^{(l)}_{\text{4.5PN}} = \frac{856}{21}\,\pi$$

tail-of-tail-of-tail terms

- However the 4PN term is only known from perturbative BH theory in the test-mass limit $\nu \to 0$ [Tagoshi & Sasaki 1994; Tanaka, Tagoshi & Sasaki 1996]

$$\varphi_{\text{4PN}} = \frac{2550713843998885153}{2214468081745920} - \frac{45245}{756}\pi^2 - \frac{9203}{126}\gamma_\text{E} - \frac{252755}{2646}\ln 2 - \frac{78975}{1568}\ln 3 + \mathcal{O}(\nu)$$

$$\varphi^{(l)}_{\text{4PN}} = -\frac{9203}{252} + \mathcal{O}(\nu)$$

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The source type multipole moments

They are obtained from a matching between the near zone and the exterior zone

$$I_L = \mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \int_{-1}^{1} \mathrm{d}z \biggl\{ \delta_\ell \, \hat{x}_L \, \overline{\Sigma} - \frac{4(2\ell+1)}{c^2(\ell+1)(2\ell+3)} \, \delta_{\ell+1} \, \hat{x}_{iL} \, \overline{\Sigma}_i^{(1)} + \frac{2(2\ell+1)}{c^4(\ell+1)(\ell+2)(2\ell+5)} \, \delta_{\ell+2} \, \hat{x}_{ijL} \, \overline{\Sigma}_{ij}^{(2)} \biggr\} \left(\boldsymbol{x}, t - \frac{rz}{c}\right)$$

$$J_L = \mathop{\mathrm{FP}}_{B=0} \int \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B \varepsilon_{ab\langle i_\ell} \int_{-1}^{1} \mathrm{d}z \biggl\{ \delta_\ell \hat{x}_{L-1\rangle a} \Sigma_b - \frac{2\ell+1}{c^2(\ell+2)(2\ell+3)} \delta_{\ell+1} \hat{x}_{L-1\rangle ac} \overline{\Sigma}_{bc}^{(1)} \biggr\} \left(\boldsymbol{x}, t - \frac{rz}{c}\right)$$

$$\overline{\Sigma} = \frac{\overline{\tau}^{00} + \overline{\tau}^{ii}}{c^2} \qquad \overline{\Sigma}_i = \frac{\overline{\tau}^{0i}}{c} \qquad \overline{\Sigma}_{ij} = \overline{\tau}^{ij}$$

where $\overline{\tau}^{\mu\nu}$ is the PN expansion (*a priori* valid only in the near zone) of the pseudo matter + gravitation stress-energy tensor

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The 4PN mass type quadrupole moment

[Marchand, Henry, Larrouturou, Marsat, Faye & Blanchet, 2020]

- Using dimensional regularisation for UV but Hadamard regularization for IR

$$I_{ij} = \mu\, A\, x_{\langle i}x_{j\rangle} + \cdots + \mathcal{O}\left(\frac{1}{c^9}\right)$$

$$\boxed{\begin{aligned} A &= 1 + \gamma\left(-\frac{1}{42} - \frac{13}{14}\nu\right) + \gamma^2\left(-\frac{461}{1512} - \frac{18395}{1512}\nu - \frac{241}{1512}\nu^2\right) \\ &\quad + \underbrace{\gamma^3\left(\frac{395899}{13200} - \frac{428}{105}\ln\left(\frac{r}{r_0}\right) + \left[\frac{3304319}{166320} - \frac{44}{3}\ln\left(\frac{r}{r'_0}\right)\right]\nu + \cdots\right)}_{\text{3PN terms}} \\ &\quad + \underbrace{\gamma^4\left(-\frac{1023844001989}{12713500800} + \frac{31886}{2205}\ln\left(\frac{r}{r_0}\right) + \cdots\right)}_{\text{4PN terms}} \\ C &= \overbrace{\frac{48}{7} + \gamma\left(-\frac{4096}{315} - \frac{24512}{945}\nu\right)}^{\text{2.5PN and 3.5PN terms}} \end{aligned}}$$

- This result has to be completed by dimensional regularization for the IR

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UV scale (r'_0)

IR scale (r_0)

- This result has to be completed by dimensional regularization for the IR

The 3PN current type quadrupole moment

[Henry, Faye & Blanchet, in preparation]

- In this case we need dimensional regularisation for the UV but (probably) Hadamard regularization is sufficient for the IR
- To apply dimensional regularization we define the decomposition of a tensor into irreducible pieces in d dimensions (where we do not have the usual ε_{ijk} to define the current moment). The mass moment I_L is given by the usual STF moment, but the generalization of the current moment involves two tensors $J_{i|L}$ and $K_{ij|L}$ having the symmetries of mixed Young tableaux

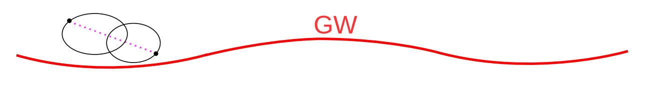
$$I_L = \begin{array}{|c|c|c|}\hline i_\ell & \dots & i_1 \\ \hline \end{array}$$

$$J_{i|L} = \begin{array}{|c|c|c|c|}\hline i_\ell & i_{\ell-1} & \dots & i_1 \\ \hline i & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad K_{ij|L} = \begin{array}{|c|c|c|c|c|}\hline i_\ell & i_{\ell-1} & i_{\ell-2} & \dots & i_1 \\ \hline j & i & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

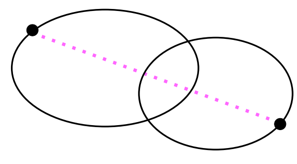
- From J_{ij} one can obtain the radiative moment V_{ij} by adding the tails and tail-of-tails and compute the relevant GW mode h_{21} to 3PN order

Effective action for compact binary systems

- Hierarchy of length scales in a compact binary system



$$\Longleftarrow \boxed{a \ll \lambda_{\mathsf{GW}} \sim \frac{ac}{v}}$$

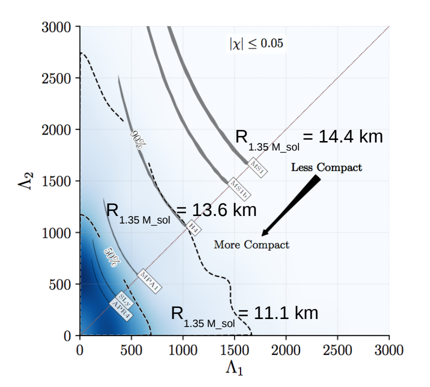


$$\Longleftarrow \boxed{R_1\,,\; R_2 \;\ll\; a}$$

- The Newtonian effective action for compact binaries with quadrupolar tidal effects (neglecting tidal dissipation) is

$$\boxed{S_{\text{eff}} = \sum_a \int \mathrm{d}t \Bigg[\overbrace{\frac{1}{2} m_a \boldsymbol{v}_a^2 + \frac{1}{2}\sum_{b\neq a}\frac{G m_b}{r_{ab}}}^{\text{point-particle action}} + \overbrace{\underbrace{\frac{\mu_a}{4}\mathcal{G}_a^{ij}\,\mathcal{G}_a^{ij}}_{\text{5PN}}}^{\text{internal structure effect}} \Bigg]}$$

Dominant quadrupole tidal effect in BNS



Tidal contribution to the GW chirp

$$x(t) = \frac{1}{4}\theta^{-1/4}\left[1 + \frac{39}{8192}\tilde{\Lambda}\,\theta^{-5/4}\right]$$

$$\phi(t) = \phi_0 - \frac{x^{-5/2}}{32\nu}\left[1 + \underbrace{\frac{39}{8}\tilde{\Lambda}\,x^5}_{\text{5PN effect}}\right]$$

with $x = \left(\frac{Gm\omega}{c^3}\right)^{2/3}$ and $\theta = \frac{\nu c^3}{5Gm}(t_c - t)$
[Flanagan & Hinderer 2008]

Effective field theory for extended compact objects

[Goldberger & Rothstein 2006; Damour & Nagar 2009]

- Matter action with non-minimal world-line couplings

$$S_{\text{eff}} = \sum_a \int \mathrm{d}\tau_a \left\{ -m_a + \sum_{\ell=2}^{+\infty} \frac{1}{2\ell!} \left[\underbrace{\mu_a^{(\ell)}}_{\substack{\text{mass type}\\ \text{polarizability}}} \left(\mathcal{G}^a_{\hat{L}}\right)^2 + \frac{\ell}{\ell+1} \underbrace{\sigma_a^{(\ell)}}_{\substack{\text{current type}\\ \text{polarizability}}} \left(\mathcal{H}^a_{\hat{L}}\right)^2 \right] + \cdots \right\}$$

- Tidal multipole moments [Thorne & Hartle 1985; Zhang 1986]

$$\mathcal{G}^a_{\hat{L}} = -\left[\nabla_{\langle \hat{i}_1} \cdots \nabla_{\hat{i}_{\ell-2}} C_{\hat{i}_{\ell-1}\underline{\hat{0}}\hat{i}_\ell\rangle\hat{0}}\right]_a$$

$$\mathcal{H}^a_{\hat{L}} = 2\left[\nabla_{\langle \hat{i}_1} \cdots \nabla_{\hat{i}_{\ell-2}} C^*_{\hat{i}_{\ell-1}\underline{\hat{0}}\hat{i}_\ell\rangle\hat{0}}\right]_a$$

where $C_{\hat{i}0\hat{j}0}$ are the components of the Weyl tensor $C_{\mu\nu\rho\sigma}$ projected on a local tetrad and evaluated at the location of the particle using a self-field regularization

High-order PN tidal effects in the orbital phasing

[Damour, Nagar, Villain 2012; Vines & Flanagan 2013; Landry 2018; Abdelsalhin *et al.* 2018]

- A recent result [Henry, Faye & Blanchet 2020abc] is the orbital SPA phase complete at the next-to-next-to-leading order for compact binaries on circular orbit

$$\psi_{\text{tidal}} = -\frac{117}{2}v^5\Bigg\{\widetilde{\mu}^{(2)} + \overbrace{\left(\frac{3115}{1248}\widetilde{\mu}^{(2)} + \frac{370}{117}\widetilde{\sigma}^{(2)}\right)v^2}^{\text{NLO}}$$
$$-\pi\widetilde{\mu}^{(2)}v^3 + \underbrace{\left(\frac{379931975}{44579808}\widetilde{\mu}^{(2)} + \frac{935380}{66339}\widetilde{\sigma}^{(2)} + \frac{500}{351}\widetilde{\mu}^{(3)}\right)v^4}_{\text{NNLO}}$$
$$-\pi\left(\frac{2137}{546}\widetilde{\mu}^{(2)} + \frac{592}{117}\widetilde{\sigma}^{(2)}\right)v^5\Bigg\}$$

- Unexplained disagreement on one coefficient (tail term at 7.5PN order) with the previous work [Damour, Nagar, Villain 2012]
- On the other hand the PN conservative dynamics is consistent with the PM tidal Hamiltonian [Cheung & Solon 2020; Kälin, Liu & Porto 2020]

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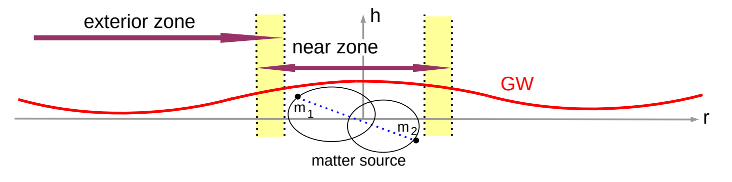
Near zone/exterior zone split in PN expansions



- Multipole expansion in the exterior zone

$$\mathcal{M}(h) = \underset{B=0}{\mathrm{FP}}\, \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \mathcal{M}(\Lambda) \right] + \underbrace{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{F_L(t-r/c)}{r} \right\}}_{\substack{\text{general retarded homogeneous solution} \\ \text{(with no incoming radiation)}}}$$

- Post-Newtonian expansion in the near zone

$$\bar{h} = \underset{B=0}{\mathrm{FP}}\, \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \bar{\tau} \right] + \underbrace{\sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{R_L(t-r/c) - R_L(t+r/c)}{r} \right\}}_{\substack{\text{general homogeneous retarded-advanced solution} \\ \text{(regular when } r \to 0)}}$$

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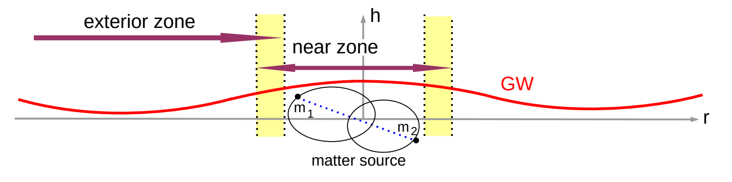

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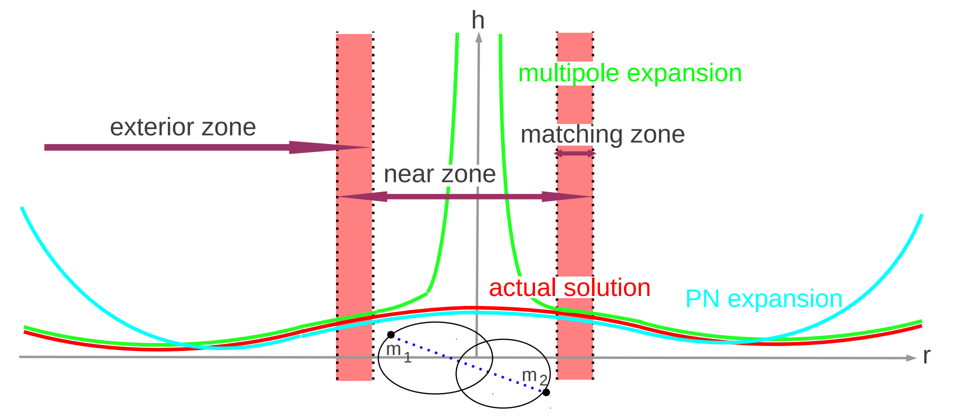
Problem of the matching

[Lagerström *et al.* 1967; Burke & Thorne 1971; Kates 1980; Anderson *et al.* 1982; Blanchet 1998]



matching equation $\Longrightarrow$ $\overline{\mathcal{M}(h)} = \mathcal{M}(\bar{h})$

Near-zone expansion of the multipole expansion

Lemma 1

$$\overline{\underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\text{ret}}\left[\left(\frac{r}{r_0}\right)^B\mathcal{M}(\Lambda)\right]} = \underset{B=0}{\mathrm{FP}}\,\Box^{-1}_{\text{sym}}\left[\left(\frac{r}{r_0}\right)^B\overline{\mathcal{M}(\Lambda)}\right] - \underbrace{\frac{4G}{c^4}\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{\mathcal{R}_L(t-r/c)-\mathcal{R}_L(t+r/c)}{2r}\right\}}_{\text{antisymmetric type homogeneous solution}}$$

where the radiation reaction multipole moments are

$$\mathcal{R}_L(u) = \underset{B=0}{\mathrm{FP}}\int \mathrm{d}^3\mathbf{x}\left(\frac{r}{r_0}\right)^B\hat{x}_L\int_1^{+\infty}\mathrm{d}z\,\gamma_\ell(z)\,\underbrace{\mathcal{M}(\tau)(\mathbf{x},t-zr/c)}_{\text{multipole expansion of the pseudo-tensor}}$$

The finite part at $B=0$ plays the role of an UV regularization $(r\to 0)$

Far-zone expansion of the PN expansion

Lemma 2

$$\mathcal{M}\left(\mathop{\mathrm{FP}}_{B=0}\Box^{-1}_{\text{sym}}\left[\left(\tfrac{r}{r_0}\right)^B\bar{\tau}\right]\right) = \mathop{\mathrm{FP}}_{B=0}\Box^{-1}_{\text{sym}}\left[\left(\tfrac{r}{r_0}\right)^B\mathcal{M}(\bar{\tau})\right]$$
$$\underbrace{- \frac{1}{4\pi}\sum_{\ell=0}^{+\infty}\partial_L\left\{\frac{\mathcal{F}_L(t-r/c)+\mathcal{F}_L(t+r/c)}{2r}\right\}}_{\text{symmetric type homogeneous solution}}$$

$$\boxed{\mathcal{F}_L(u) = \mathop{\mathrm{FP}}_{B=0}\int \mathrm{d}^3\mathbf{x}\left(\tfrac{r}{r_0}\right)^B\hat{x}_L\int_{-1}^{1}\mathrm{d}z\,\delta_\ell(z)\underbrace{\bar{\tau}(\mathbf{x},t-zr/c)}_{\text{PN expansion of the pseudo-tensor}}}$$

The finite part at $B=0$ plays the role of an IR regularization ($r\to+\infty$)

General solution of the matching equation

[Blanchet 1998; Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]

1. In the far zone

$$\mathcal{M}(h) = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \mathcal{M}(\Lambda) \right] - \underbrace{\frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{F}_L(t-r/c)}{r} \right\}}_{\text{source's multipole moments}}$$

2. In the near zone

$$\bar{h} = \mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \bar{\tau} \right] - \underbrace{\overline{\frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{R}_L(t-r/c) - \mathcal{R}_L(t+r/c)}{r} \right\}}}_{\text{non-local tail term (4PN+ order)}}$$

Potential modes versus radiation modes

$$\bar{h} = \underbrace{\mathop{\mathrm{FP}}_{B=0} \Box^{-1}_{\text{ret}} \left[\left(\frac{r}{r_0}\right)^B \bar{\tau} \right]}_{\text{potential modes}} - \underbrace{\frac{4G}{c^4} \sum_{\ell=0}^{+\infty} \partial_L \left\{ \frac{\mathcal{R}_L(t-r/c) - \mathcal{R}_L(t+r/c)}{r} \right\}}_{\text{radiation modes}}$$

- The potential modes are responsible for conservative near zone effects and can be computed with the symmetric propagator (when neglecting radiation reaction effects)
- The radiation modes are conservative effects coming from gravitational waves propagating at infinity and re-expanded in the near zone. The first radiation effect is the non local tail effect at 4PN order

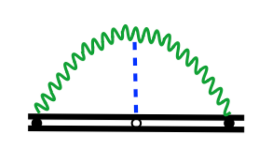
Equivalence between PN and the EFT at 4PN order

1. Computing the $M \times I_{ij}$ multipole interaction with dimensional regularization yields a piece in the 4PN conservative action [Marchand, Bernard, Blanchet & Faye 2018]

$$S^{\text{tail}} = K_d \frac{G^2 M}{c^8} \iint \frac{\mathrm{d}t\,\mathrm{d}t'}{|t-t'|^{2d-5}} I_{ij}^{(3)}(t)\, I_{ij}^{(3)}(t')$$

with $$K_d = \frac{12-12d+5d^2-4d^3+d^4}{8(d-1)^2(d+2)} \left(\frac{2\ell_0^2}{\pi}\right)^{d-3} \frac{\Gamma\left(-\frac{d}{2}\right)}{\Gamma\left(\frac{7}{2}-d\right)\Gamma\left(\frac{5}{2}-\frac{d}{2}\right)}$$

2. This result is identical to the Feynman diagram derivation in Fourier space by the EFT community [Foffa & Sturani 2012; Galley, Leibovich, Porto & Ross 2016]



Equivalence between PN and the EFT at 4PN order

1. The limit $\varepsilon \to 0$ takes the form of Hadamard's "Partie finie" (Pf) integral

$$S^{\text{tail}} = \frac{G^2 M}{5c^8} \underset{\tau_0}{\text{Pf}} \iint \frac{\mathrm{d}t\,\mathrm{d}t'}{|t-t'|} I^{(3)}_{ij}(t)\, I^{(3)}_{ij}(t')$$

$$\text{with} \quad \tau_0 = \frac{\ell_0}{c\sqrt{\pi}} \exp\Big[\underbrace{\frac{1}{2\varepsilon}}_{\text{UV type pole}} - \tfrac{1}{2}\gamma_\text{E} - \tfrac{41}{60} \Big]$$

2. The UV pole is cancelled by the IR pole coming from the potential modes
3. The coefficient $-\frac{41}{60}$ is equivalent to the ambiguity parameter C in the ADM Hamiltonian approach [Damour, Jaranowski & Schäfer 2014, 2016]
4. Two complete/ambiguity-free derivations of the 4PN equations of motion:
   - Traditional PN derivation based on the Fokker Lagrangian [Bernard, Blanchet, Bohé, Faye & Marsat 2016, 2017ab; Marchand *et al.* 2018]
   - Diagrammatic expansion in EFT [Foffa & Sturani 2019; Foffa, Porto, Rothstein & Sturani 2019; Blümlein, Maier, Marquard & Schäfer 2020]

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Cancellation of arbitrary scales between NZ and FZ

- Source quadrupole moment at the 3PN order [Blanchet, Joguet & Iyer 2002]

$$I_{ij}(t) = Q_{ij}(t) + \beta_2 \frac{G^2 M^2}{c^6} Q^{(2)}_{ij} \ln\left(\frac{r_{12}}{2\, r_0}\right) + \cdots$$

where $\beta_2 = -\frac{214}{105}$

- Radiative moment versus source moment including 3PN tail-of-tail [Anderson, Kates, Kegeles & Madonna 1982; Blanchet & Damour 1988; Blanchet 1988]

$$\begin{aligned} U_{ij} &= I^{(2)}_{ij} + \frac{1}{c^3}\,[\text{tail}] + \frac{1}{c^5}\,[\text{memory}] \\ &+ \frac{G^2 M^2}{c^6} \int_0^{+\infty} \mathrm{d}\tau I^{(5)}_{ij}(t-\tau) \left[\beta_2 \ln\left(\frac{c\tau}{2\, r_0}\right) + \cdots\right] \end{aligned}$$

- More generally $\beta_\ell = -2\frac{15\ell^4+30\ell^3+28\ell^2+13\ell+24}{\ell(\ell+1)(2\ell+3)(2\ell+1)(2\ell-1)}$

Renormalization group equations from EFT

- In the EFT the cancellation of r_0 is ruled by the renormalization group equations [Goldberger & Ross 2010, Goldberger, Ross & Rothstein 2014]
- The RG equations for mass and angular momentum are (with $\mu \equiv r_0$ the renormalization scale)

$$\frac{\mathrm{d}\log M(\mu)}{\mathrm{d}\log\mu} = -\frac{2G^2}{5}\left[2I_{ij}^{(1)}I_{ij}^{(5)} - 2I_{ij}^{(2)}I_{ij}^{(4)} + I_{ij}^{(3)}I_{ij}^{(3)}\right]$$

$$\frac{\mathrm{d}J^i(\mu)}{\mathrm{d}\log\mu} = -\frac{8G^2 M}{5}\varepsilon^{ijk}\left[I_{jl}I_{kl}^{(5)} - I_{jl}^{(1)}I_{kl}^{(4)} + I_{jl}^{(2)}I_{kl}^{(3)}\right]$$

- The quadrupole moment itself undergoes a logarithmic renormalization under the RG flow (in the Fourier domain)

$$\boxed{\tilde{I}_{ij}(\omega,\mu) = \bar{\mu}^{\beta_2 (GM\omega)^2}\tilde{I}_{ij}(\omega,\mu_0)}$$

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Leading powers of logarithms in the energy function

[Blanchet, Foffa, Larrouturou & Sturani 2020]

1. We integrate the previous RG equations, average them over one orbital scale, then specialize to quasi-circular orbits

$$E = \frac{1}{2}m\nu\, r^2\omega^2 - \frac{Gm^2\nu}{r} - 8m\nu^2\frac{\gamma^2}{\beta_2}\sum_{n=1}^{+\infty}\frac{1}{n!}\left(8\beta_2\gamma^3\log v\right)^n$$

$$J = m\nu\, r^2\omega - \frac{48}{5}G^2m^3\nu^2\frac{\omega}{\beta_2\gamma}\sum_{n=1}^{+\infty}\frac{1}{n!}\left(8\beta_2\gamma^3\log v\right)^n$$

2. For circular orbits the two invariants $E(\omega)$ and $J(\omega)$ are linked by the "thermodynamic" relation or first law of binary mechanics

$$\frac{\mathrm{d}E}{\mathrm{d}\omega} = \omega\frac{\mathrm{d}J}{\mathrm{d}\omega}$$

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[Blanchet, Foffa, Larrouturou & Sturani 2020]

This gives three relations for the three unknowns $E(\omega)$ and $J(\omega)$ and $r(\omega)$

$$E^{\text{leading }(\log)^n} = -\frac{m\nu\, x}{2}\left[1 + \frac{64\nu}{15}\sum_{n=1}^{+\infty}\frac{6n+1}{n!}(4\beta_2)^{n-1}\, x^{3n+1}(\log x)^n\right]$$

$$J^{\text{leading }(\log)^n} = \frac{m^2\nu}{\sqrt{x}}\left[1 - \frac{64\nu}{15}\sum_{n=1}^{+\infty}\frac{3n+2}{n!}(4\beta_2)^{n-1}\, x^{3n+1}(\log x)^n\right]$$

Leading powers of logarithms in the energy function

[Blanchet, Foffa, Larrouturou & Sturani 2020]

The result agrees with high-order analytical GSF calculations up to 22PN order !

[Kavanagh, Ottewill & Wardell 2015]