The Eikonal Approach to Gravitational Scattering and Radiation

Rodolfo Russo

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Queen Mary University of London

The aim

Use a particle-physicist approach to derive classical observables relevant to gravitational binaries

see Parra-Martinez's lectures and Bern's and Kosower's talk

- Model the celestial bodies as "elementary" particles with known couplings to gravity (massless fields in general)
- Use quantum perturbative amplitudes to describe the large-distance scattering and take the classical PM limit
- Analytically continue the results from open to closed orbits

see Porto's talk

In the eikonal approach it is possible to implement this programme by focusing on gauge invariant quantities

Classical physics is obtained by resumming an infinite set of contributions which leads to exponentiation

It is a general approach applicable to all perturbative gravitational theories (GR, supergravity, string theory; shockwaves, spin ...)

Based on:

2104.03256, 2101.05772, 2008.12743:
$\mathcal{N} = 8$, $m_{1,2} \neq 0$, 3PM, also results in GR

1911.11716, 1908.05603:
$\mathcal{N} = 8$, $m_{1,2} = 0$, 3PM

1904.02667:
GR, $m_{1,2} \neq 0$, general d, 2PM

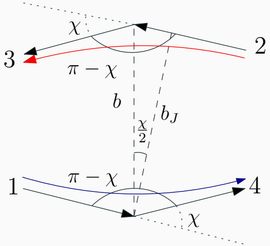
1807.04588:
$\mathcal{N} = 8$, $m_1 \to \infty$, $m_2 = 0$, 2PM

in (various) collaboration with: P. Di Vecchia, C. Heissenberg, A. Koemans Collado, A. Luna, S. G. Naculich, S. Thomas, G. Veneziano, C. D. White

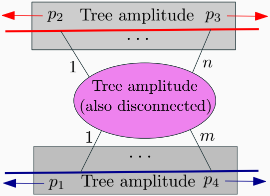
The elastic scattering

The setup

Consider the $2 \to 2$ scattering with $p_1^2 = p_4^2 = -m_1^2$, $p_2^2 = p_3^2 = -m_2^2$



A spacetime picture of the scattering



Diagrammatic picture

Key classical quantities:

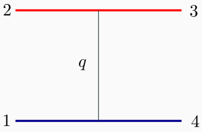
The centre-of-mass energy E, $E^2 = s = -(p_1 + p_2)^2 = (m_1^2 + m_2^2 + 2m_1 m_2 \sigma)$

The angular momentum $J = p\, b_J$, $p = |\vec{p}_i|$, $Ep = m_1 m_2 \sqrt{\sigma^2 - 1}$

The momentum transferred $Q = p_1 + p_4$, $|Q| = 2p \sin\left(\frac{\chi}{2}\right)$

One particle exchange

Let us start from the 1-particle exchange



$$\beta^{\mathcal{N}=8} = 4m_1^2 m_2^2 \sigma^2$$

$$\mathcal{A}_0 = \frac{8\pi G\beta}{q^2} + \dots$$

$$\beta^{GR} = 4m_1^2 m_2^2 \left(\sigma^2 - \frac{1}{D-2}\right)$$

q is quantum and the dots contain analytic terms as $q \to 0$

In terms of classical quantity $b \sim \hbar/q$

$$\widetilde{\mathcal{A}}(s,b) = \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \frac{\mathcal{A}(s,q^2)}{4pE} e^{-ib\cdot q} \,.$$

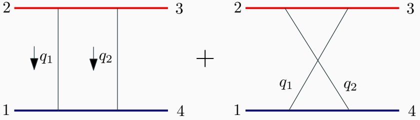
In $D = 4 - 2\epsilon \to 4$ we have

$$i\widetilde{\mathcal{A}}_0^{\mathcal{N}=8} = \frac{2im_1 m_2 G(\pi b^2)^{\epsilon}\sigma^2\Gamma(-\epsilon)}{\sqrt{\sigma^2-1}} \to -i\frac{Gm_1 m_2}{\hbar}\log(b)\frac{4\sigma^2}{\sqrt{\sigma^2-1}}$$

No well defined classical limit?!

Two particle exchange

Consider the two particle exchange. The non-analytic contributions are



This captures the whole $\mathcal{N}=8$ result

$$\mathcal{A}_1(s,q^2) = \frac{a_1^{(1)}(s)}{(q^2)^{1+\epsilon}} + \frac{a_1^{(2)}(s)}{(q^2)^{\frac{1}{2}+\epsilon}} + \frac{a_1^{(3)}(s)}{(q^2)^{\epsilon}} + \dots$$

with $q_1 + q_2 = q$

(I) From $a_1^{(1)}$ we have $\mathcal{O}\left(\frac{1}{\hbar^2}\right)$ term: $i\widetilde{\mathcal{A}}_1^{(1)}(s,b) = \frac{1}{2}(i\widetilde{\mathcal{A}}_0)^2$. Then resumming the leading contributions (as $\hbar \to 0$) we expect $1 + i\widetilde{\mathcal{A}}_0 + i\widetilde{\mathcal{A}}_1^{(1)} + \dots = e^{i\widetilde{\mathcal{A}}_0}$ (eikonal exponentiation)

(II) $a_1^{(2)}$ yield a new contribution $\mathcal{O}\left(\frac{1}{\hbar}\right)$ (which is $\mathcal{O}(\epsilon)$ in $\mathcal{N}=8$)

(III) $a_1^{(\mathrm{n}\geq 3)}$ yields a long-range, but quantum terms $\mathcal{O}(\hbar^{n-3})$

Terms with negative powers of $\hbar$ exponentiate

for $\widetilde{\mathcal{A}}_0$: Kabat, Ortiz; Akhoury, Saotome, Sterman; Parra-Martinez's lecture

The eikonal

The semi-classical limit requires that long range part of $\widetilde{\mathcal{A}}$ takes the form

$$\boxed{1 + i\widetilde{\mathcal{A}}(s,b) = \Big(1 + 2i\Delta(s,b)\Big)\, e^{\frac{i}{\hbar}2\delta(s,b)}}$$

where δ is $\mathcal{O}(\hbar^0)$ and Δ encodes the quantum terms $\mathcal{O}(\hbar^m)$ with $m \geq 0$

δ_k and Δ_k, $k \geq 0$, encode contributions of order G^{k+1} (PM expansion)

$\mathcal{N} = 8$ in $D = 4$: we have $2\delta_0 = -\log(b)\frac{4Gm_1m_2\sigma^2}{\sqrt{\sigma^2-1}}$, $\delta_1 = 0$ Caron-Huot, Zahraee

GR in $D = 4$: we have $2\delta_0 = -\log(b)\frac{2Gm_1m_2(2\sigma^2-1)}{\sqrt{\sigma^2-1}}$ and

$$2\delta_1 = \frac{3\pi G^2 m_1 m_2 (m_1+m_2)}{4b}\frac{5\sigma^2-1}{\sqrt{\sigma^2-1}}$$

Ignoring the quantum terms the inverse FT reads

$$i\,\frac{\mathcal{A}(s,Q^2)}{4pE} = \int d^{D-2}b\left(e^{\frac{i}{\hbar}2\delta(s,b)} - 1\right)\, e^{\frac{i}{\hbar}b\cdot Q}$$

and a stationary phase approximation yields $Q^\mu = -\frac{\partial\,\mathrm{Re}\,2\delta(s,b)}{\partial b^\mu}$ and so χ

Connection to bound orbits

The derivatives of the eikonal give standard observables

$$\text{Time delay} \quad \Delta T = \frac{\partial \operatorname{Re} 2\delta}{\partial E} \,, \qquad \text{Scatt. angle} \quad \chi = \frac{\partial \operatorname{Re} 2\delta}{\partial J}$$

An analytic continuation to $\sigma < 1$ describes bound states ($E < m1 + m2$).
This implies $\sqrt{\sigma^2 - 1} \to i\sqrt{1 - \sigma^2}$, $b \to \pm ib$ so as to have $J \to \pm J$

Kälin, Porto

By using the eikonal $\tilde{\delta}$ after analytic continuation, we can introduce the periastron advance K and the period P

$$P = \left[\frac{\partial \operatorname{Re} 2\tilde{\delta}}{\partial E} - (J \to -J)\right] \,, \quad K - 1 = \frac{1}{2\pi}\left[\frac{\partial \operatorname{Re} 2\tilde{\delta}}{\partial J} + (J \to -J)\right]$$

From $\tilde{\delta}_{0,1}$ we can derive Eqs. (347) for K and $n = \frac{2\pi}{P}$ of Blanchet's review at all orders in ϵ and first subleading order in $j_B = \frac{J^2}{G^2}\frac{\epsilon}{(m_1 m_2)^2}$

The 3-PM eikonal in $\mathcal{N} = 8$

The 2-loop integrand in $\mathcal{N} = 8$ is known in terms of scalar integrals

Extract the first non-analytic terms in the small q expansion

$$\mathcal{A}_2(s, q^2) = \frac{a_2^{\rm sscl}(s)}{(q^2)^{1+2\epsilon}} + \frac{a_2^{\rm scl}(s)}{(q^2)^{\frac{1}{2}+2\epsilon}} + \frac{a_2^{\rm cl}(s)}{(q^2)^{2\epsilon}} + \dots$$

Go to b-space and solve for δ_2. By using also $\delta_{0,1}$ and Δ_1, we get DHRV

radiation reaction

Parra-Martinez, Ruf, Zeng: 2005.04236

$$\begin{aligned}(2\delta_2) = {} & \frac{16 m_1^2 m_2^2 G^3 \sigma^6}{b^2(\sigma^2-1)^2} - \frac{16 m_1^2 m_2^2 \sigma^4 G^3}{b^2(\sigma^2-1)} \cosh^{-1}(\sigma)\left[1 - \frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{3}{2}}}\right] \\ & - i\frac{16 m_1^2 m_2^2 G^3}{\pi b^2}\frac{\sigma^4}{(\sigma^2-1)^2}\left\{\frac{1}{\epsilon}\left(\sigma^2 + \frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{1}{2}}}\cosh^{-1}(\sigma)\right)\right. \\ & - \left(\log(4(\sigma^2-1)) - 3\log(\pi b^2 e^{\gamma_E})\right)\left[\sigma^2 + \frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{1}{2}}}\cosh^{-1}(\sigma)\right] \\ & \left. + (\sigma^2-1)\left[1 + \frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{3}{2}}}\right](\cosh^{-1}(\sigma))^2 + \frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{1}{2}}}\mathrm{Li}_2(1-z^2) + 2\sigma^2\right\}\end{aligned}$$

$z = \sigma - \sqrt{\sigma^2 - 1}$

A consequence of analyticity and crossing – DHRV: 2104.03256

Confirmed independently by Bjerrum-Bohr, Damgaard, Planté, Vanhove

PN limit $v \to 0$, $(b\,v)$ fixed
$\sigma^2 - 1 = v^2(1-v^2)^{-1} \sim v^2$,
$\cosh^{-1}(\sigma) \sim v$

In the UR limit ($\sigma \gg 1$), $\mathrm{Re}(2\delta_2) \to \frac{16 G^3 (m_1 m_2 \sigma)^2}{b^2}$ which is universal!

Amati, Ciafaloni, Veneziano; Ademollo, Bellini, Ciafaloni; DNRVW 1911.11716; Bern, Ita, Parra-Martinez, Ruf; DHRV: 2008.12743

A comment on the integrals

The integrals depend on one scale and can be translated into a set of differential equations for a basis of soft master integrals

…, Parra-Martinez, Ruf, Zeng. Tools: `LiteRed`, `Fire6`, `epsilon`

The full soft contribution $q \to 0$ has been included, i.e. no separation in potential gravitons (near region) and radiation gravitons (far region)

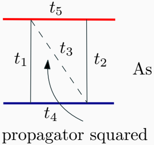
see also Herrmann, Parra-Martinez, Ruf, Zeng

The key step is the evaluation of the $\sigma \to 1$ boundary conditions

One element of the double-box basis of integrals is

$$f_{\mathrm{III},3} = \epsilon^3 q^2 \tau$$

$\tau = \sqrt{\sigma^2 - 1} + \mathcal{O}\left(\frac{q^2}{m^2}\right)$



As $\tau \to 0$ we have two regions

ordinary region: $t_{1,2,3} \sim \mathcal{O}(\tau^0)$

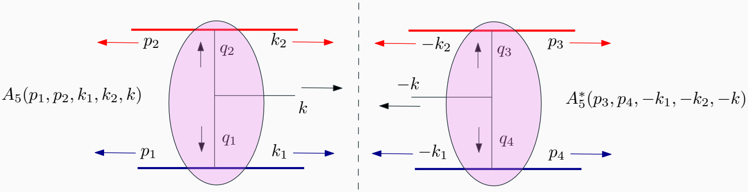
singular region: $t_{1,2} \sim \mathcal{O}(\tau^0)$, $t_3 \sim \mathcal{O}(\tau^{-2})$

The "singular static" contributions is crucial to restore crossing symmetry. It will play a useful role also in the cut version of these integrals

Radiative effects

The 3-particle cut

Why do we have $\mathrm{Im}(2\delta_2) \neq 0$? It is related to the 3-particle cut



Unitarity implies

$$[\mathrm{Im}\, 2A_2]_{3pc} = \int \frac{d^D k}{(2\pi)^D} \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} (2\pi)^D \delta(p_1 + p_2 + k_1 + k_2 + k)$$
$$2\pi\theta(k^0)\,\delta(k^2)\; 2\pi\theta(k_1^0)\,\delta(k_1^2 + m_1^2)\; 2\pi\theta(k_2^0)\,\delta(k_2^2 + m_2^2)\; |A_5|^2$$

In b-space we have $\left[\mathrm{Im}\, \widetilde{A}_2\right]_{3pc} = \mathrm{Im}(2\delta_2)$: so this is a shortcut to the derivation of the imaginary part!

Amati, Ciafaloni, Veneziano

Inelastic amplitudes (3-particle cut)

A unified GR and $\mathcal{N}=8$ expression for the $2\to 3$ classical amplitude

Goldberger, Ridgway; Luna, Nicholson, O'Connell, White; Mogull, Plefka, Steinhoff

$$A_5^{MN} = (8\pi G)^{\frac{3}{2}}\Bigg\{\frac{8\big(P_1kP_2^M - P_2kP_1^M\big)\big(P_1kP_2^N - P_2kP_1^N\big)}{q_1^2q_2^2}$$
$$+8P_1P_2\left[\frac{P_1^MP_1^N\frac{kP_2}{kP_1} - P_1^{(M}P_2^{N)}}{q_2^2} + \frac{P_2^MP_2^N\frac{kP_1}{kP_2} - P_1^{(M}P_2^{N)}}{q_1^2} - 2\frac{P_1kP_2^{(M}q_1^{N)} - P_2kP_1^{(M}q_1^{N)}}{q_1^2q_2^2}\right]$$
$$+\beta\left[-\frac{P_1^MP_1^N(kq_1)}{(P_1k)^2q_2^2} - \frac{P_2^MP_2^N(kq_2)}{(P_2k)^2q_1^2} + 2\left(\frac{P_1^{(M}q_1^{N)}}{(P_1k)q_2^2} - \frac{P_2^{(M}q_1^{N)}}{(P_2k)q_1^2} + \frac{q_1^Mq_1^N}{q_1^2q_2^2}\right)\right]\Bigg\}$$

Leading term in the Weinberg limit $k \ll q_i$

$\mathcal{N}=8$ setup: P_i, K_i are 10D momenta, q and k 4D

$$P_1 = (p_1;0,0,0,0,0,m_1)\,,\quad P_1^2=0$$
$$P_2 = (p_2;0,0,0,0,m_2,0)\,,\quad P_2^2=0$$
$$\cdots$$

GR setup: all vectors are 4D

$$P_1 = (p_1;0,0,0,0,0,0)\,,\quad P_1^2 = p_1^2 = -m_1^2$$
$$P_2 = (p_2;0,0,0,0,0,0)\,,\quad P_2^2 = p_2^2 = -m_2^2$$
$$\cdots$$

This provides explicit integrands for the discontinuity

$$|A_5^{\mathcal{N}=8}|^2 \to A_5^{MN}(P_1,P_2,K_1,K_2,k)\;\eta_{MR}\eta_{NS}\;A_5^{RS}(P_4,P_3,-K_1,-K_2,-k)$$
$$|A_5^{gr}|^2 \to A_5^{\mu\nu}(p_1,p_2,k_1,k_2,k)\left[\eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}\right]A_5^{\rho\sigma}(p_3,p_4,k_4,k_3,-k)$$

The phase-space integrals can be performed by recycling the loop integrals

Anastasiou, Melnikov; Herrmann, Parra-Martinez, Ruf, Zeng

The 3-PM eikonal in GR

The unitarity approach leads to $\mathrm{Im}(2\delta_2)$ in GR (and reproduces the $\mathcal{N}=8$ result). By combining it with previous results we have

radiation reaction
(agrees with Damour's result)

Bern, Cheung, Roiban,
Shen, Solon, Zeng

probe limit

$$
\begin{aligned}
2\delta_2^{(gr)} = \frac{4G^3 m_1^2 m_2^2}{b^2} \Bigg\{ & \frac{(2\sigma^2-1)^2(8-5\sigma^2)}{6(\sigma^2-1)^2} - \frac{\sigma(14\sigma^2+25)}{3\sqrt{\sigma^2-1}} \\
& + \frac{s(12\sigma^4-10\sigma^2+1)}{2m_1 m_2(\sigma^2-1)^{\frac{3}{2}}} + \cosh^{-1}\sigma \left[\frac{\sigma(2\sigma^2-1)^2(2\sigma^2-3)}{2(\sigma^2-1)^{\frac{5}{2}}} + \frac{-4\sigma^4+12\sigma^2+3}{\sigma^2-1} \right] \Bigg\} \\
& + i\, \frac{2m_1^2 m_2^2 G^3}{\pi b^2} \frac{(2\sigma^2-1)^2}{(\sigma^2-1)^2} \Bigg\{ -\frac{1}{\epsilon} \left[\frac{8-5\sigma^2}{3} - \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}} \cosh^{-1}(\sigma) \right] \\
& + \left(\log(4(\sigma^2-1)) - 3\log(\pi b^2 \mathrm{e}^{\gamma_E})\right) \left[\frac{8-5\sigma^2}{3} - \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}} \cosh^{-1}(\sigma) \right] \\
& + (\cosh^{-1}(\sigma))^2 \left[\frac{\sigma\left(3-2\sigma^2\right)}{(\sigma^2-1)^{\frac{1}{2}}} - 2\, \frac{4\sigma^6-16\sigma^4+9\sigma^2+3}{(2\sigma^2-1)^2} \right] \\
& + \cosh^{-1}(\sigma) \left[\frac{\sigma\left(88\sigma^6-240\sigma^4+240\sigma^2-97\right)}{3(2\sigma^2-1)^2(\sigma^2-1)^{\frac{1}{2}}} \right] \\
& + \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}} \mathrm{Li}_2(1-z^2) + \frac{-140\sigma^6+220\sigma^4-127\sigma^2+56}{9(2\sigma^2-1)^2} \Bigg\}
\end{aligned}
$$

A consequence of analyticity
and crossing

the universal UR terms

$$\sigma^2\left(-\tfrac{10}{3} - \tfrac{14}{3} + 12\right) = 4$$

$$\Downarrow$$

$$\mathrm{Re}(2\delta_2) \to \frac{16G^3(m_1 m_2 \sigma)^2}{b^2}$$

Radiation Reaction from Soft Theorems

The radiation-reaction contribution is fully determined by Weinberg's leading term A_5^W in the soft expansion DHRV: 2101.05772

In b-space, the soft limit takes the following form

$$\widetilde{A}_5^W = -i\frac{\kappa^3\beta}{8\pi m_1 m_2}\frac{(\pi b^2)^\epsilon}{b^2\sqrt{\sigma^2-1}}\left[(kb)\left(\frac{\bar{p}_1^\mu\bar{p}_1^\nu}{(\bar{p}_1k)^2}-\frac{\bar{p}_2^\mu\bar{p}_2^\nu}{(\bar{p}_2k)^2}\right)-\frac{\bar{p}_1^\mu b^\nu+\bar{p}_1^\nu b^\mu}{(\bar{p}_1k)}+\frac{\bar{p}_2^\mu b^\nu+\bar{p}_2^\nu b^\mu}{(\bar{p}_2k)}\right]$$

$$p_1^\mu=-\bar{p}_1^\mu+q^\mu/2\,,\qquad p_2^\mu=-\bar{p}_2^\mu-q^\mu/2$$

Use this to calculate $\left[\mathrm{Im}\,\widetilde{A}_2\right]_{3pc} = \mathrm{Im}(2\delta_2)$

- The integral over $\omega=|\vec{k}|$ yields the $1/\epsilon$ divergence
- The integral over the angles is identical to the one appearing in Damour's linear-response approach

Exploit soft-theorems more systematically: do subleading terms play an interesting role? Apply them at 4PM and beyond? BMS group?

Radiated energy

One can use A_5 to derive other quantities, just by inserting the appropriate generator in the integrand. For the emitted energy

$$|A_5^{(\dots)}|^2 \to k^\mu |A_5^{(\dots)}|^2$$

The phase-space integrals can again be performed by using the same technology discussed previously

However, we have access to differential quantities as well. A simple example is the Zero-Frequency-Limit of the energy spectrum

$$\frac{dE^{rad}}{d\omega}(\omega \to 0) = \lim_{\epsilon \to 0} \left[-4\hbar\epsilon(\mathrm{Im}2\delta_2) \right] \xrightarrow[\sigma \to 1]{} \frac{32 G^3 m_1^2 m_2^2}{5\pi b^2}$$

Smarr

Also the angular distribution can be derived with the same approach

Waveforms

One can also consider directly the waveforms (in the CoM frame)

Instead of focusing on $|A_5|^2$, perform the FT to b-space of A_5

$$\widetilde{A}_{5,i}(b,\vec{k}) = \int \frac{d^{D-2}\boldsymbol{\Delta}}{(2\pi)^{D-2}} \frac{e^{-ib\boldsymbol{\Delta}}}{4Ep} A_{5,i}(p_1,p_2,k_1,k_2,k)$$

- $\boldsymbol{\Delta} \equiv \frac{1}{2}(\boldsymbol{q}_1 - \boldsymbol{q}_2)$ is a $(D-2)$-vector; the other momenta (except $\vec{k}$) are fixed by the onshell/conservation conditions
- $A_{5,i} = A_5^{MN}\epsilon^{(i)}_{MN}$ for the physical polarizations $i = +, \times$ (in the $\mathcal{N} = 8$ case one has also other massless particles: dilaton, ...)

One extra FT yields the result in terms of the retarded time $u = t - r$ rather than the frequency ω

$$\widetilde{A}_{5,i}(u,b,\hat{k}) = \int \frac{d\omega}{2\pi} e^{-i\omega u} \widetilde{A}_{5,i}(b,\vec{k})$$

A unified approach for $\mathcal{N} = 8$ and pure GR

Waveforms: an example

In DHRV 2104.03256 we give the $\mathcal{N}=8$ dilaton waveform as an example

The extension to GR is tedious but straightforward

Our result for the $\times$ polarisation in GR is

Alert: perliminary

$$\tilde{A}_{5,\times} = i\frac{\kappa^3(\hat{b}e_\phi)}{4pE(2\pi)}\Bigg\{-\beta\left[\frac{m_2\sin\theta}{\sqrt{s}}e^{ibk/2}K_1(bc_2|\mathbf{k}|)+\frac{m_1\sin\theta}{\sqrt{s}}e^{-ibk/2}K_1(bc_1|\mathbf{k}|)\right]$$
$$+\beta\int_0^1 dx\, e^{i\frac{kb}{2}(1-2x)}\left(i(\hat{b}e_\theta)(b\sqrt{f}|\mathbf{k}|)K_1(b|\mathbf{k}|\sqrt{f})-(x-\tfrac{1}{2})(\mathbf{k}e_\theta)bK_0(b|\mathbf{k}|\sqrt{f})\right)$$
$$+\left[4(p_1p_2)pE\omega\sin\theta-\beta\,\frac{\omega}{s}\sin\theta\left(\frac{(m_1+m_2\sigma)(m_2+m_1\sigma)}{\sqrt{\sigma^2-1}}+\frac{m_1^2-m_2^2}{2}\cos\theta\right)\right]\int_0^1 dx\, e^{i\frac{kb}{2}(1-2x)}bK_0(b|\mathbf{k}|\sqrt{f})\Bigg\}$$

Bold vectors are and b (D-2)-dimensional

$$|\mathbf{k}|\sqrt{f}=\sqrt{\mathbf{k}^2\Big(x(1-x)+c_1^2x+c_2^2(1-x)\Big)}\,,\qquad |\mathbf{k}|c_1=\frac{(p_2\,k)}{m_2\sqrt{\sigma^2-1}}\,,\quad |\mathbf{k}|c_2=\frac{(p_1\,k)}{m_1\sqrt{\sigma^2-1}}\,,\quad |\mathbf{k}|=|\omega\sin\theta|$$

$$e^\mu_\phi=(0,-\sin\phi,\cos\phi,0)\,,\qquad e^\mu_\theta=(0,\cos\theta\cos\phi,\cos\theta\sin\phi,-\sin\theta)\,,\qquad k^\mu=\omega(1,\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$$

This result is written in the center of mass frame

The FT to *u*-space is straightforward and then the *x*-integral is simple

We are in the process of comparing with Jakobsen, Mogull, Plefka, Steinhoff; Mougiakakos, Riva, Vernizzi

Conclusions

The eikonal provides a conceptually simple approach to gravitational binaries: it leads directly to classical, observable quantities starting from a full quantum framework

The state of the art is the full 3PM analysis

It can be extended to different situations of experimental (spin) or theoretical (susy, string theory) interest

What next?

- Derive the 4PM eikonal ($\mathcal{N} = 8$ is again the perfect laboratory)
- Clarify the exponentiation of the radiative part: promote δ to a Hermitean operator, extend the analysis of the waveforms ...
- Consider more complicated objects: here string theory can (again) be a useful guide to analyse spin, tidal effects, ...