Post-Newtonian, Near and Far zone & All That

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S. Foffa, RS arXiv:2103.03190, submitted to PRD
S. Foffa, RS, arXiv:1907.02869 PRD (2020)

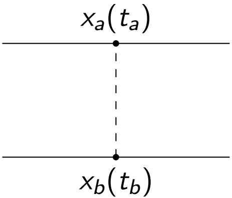
Apr 30th, 2021 Gravitational scattering, inspiral, and radiation – GGI

1. Usefulness of PN approximation (EFT point of view)
2. Near-Far Zone Interplay
3. Green's function boundary conditions
4. Up-to-date results in NRGR
5. Conclusions & prospects

Outline

1PM potential

Out of different ways of computing 2-body Post-Minkowskian expansion e.g. 1PM $O(G_N^1)$ potential gravity coupled to particle world-lines:



$$V_{PM}^{(1)}(x_a^\mu - x_b^\mu) = G_N T^a_{\mu\nu} T^b_{\rho\sigma} \Delta^{\mu\nu,\rho\sigma} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik^\mu(x_{a\mu} - x_{b\mu})}}{|\mathbf{k}|^2 - k_0^2 + \epsilon \,\text{terms}}$$

PM at higher orders

It is a formidable task to go to higher order: complete result so far at 3PM $O(G_N^3)$ (2PM beyond Newtonian interaction) with "particle physics" approach by

1. *Scattering amplitude* method by Bern, Cheung, Roiban, Shen, Solon PRL '19 and partial result at 4PM by

   Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng 2101.07254
2. *EFT+Boundary2Bound* by Kälin, Porto PRL '20

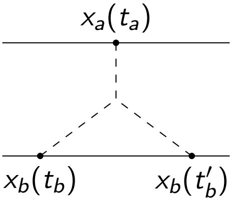
and up to 4PM $O(G_N^4)$ via the

3. "syncretic" *TuttiFrutti* method initiated by Bini, Damour, Geralico, PRL '20

This talk is about PN approximation to 2-body motion in GR in EFT flavour, aka Non-Relativistic General Relativity initiated by Goldberger& Rothstein PRD '05 (name inspired by NRQCD)

PM gets complicated

At higher order things rapidly complicate



$$
\begin{aligned}
V^{(2PM)} &\supset G_N^2 m_1 m_2^2 \int d^4 p e^{ip_\mu(x_a^\mu(t_a)-x_b^\mu(t_b))}\frac{p^\alpha p^\beta}{p^2}\int d^4 k \frac{e^{ik^\mu(x_b^\mu(t_b)-x_b^\mu(t'_b))}}{(p-k)^2 k^2} \\
&= G_N^2 m_1 m_2^2 \int d^4 p e^{ip_\mu(x_a^\mu(t_a)-x_b^\mu(t_b))}\frac{p^\alpha p^\beta}{p^2}\Delta(p^\mu(x_{2\mu}(t_b)-x_{2\mu}(t'_b)))
\end{aligned}
$$

These kinds of conservative diagrams computed up to 4PM order

PN simplifies: Near

- Near Zone: consider $|\mathbf{k}| \gg k_0$, with $\frac{k_0^2}{|\mathbf{k}|^2} \sim v^2$

$$V_{PN-Near} = \int \frac{dk_0}{2\pi}\frac{d^3k}{(2\pi)^3} e^{ik_0(t_a-t_b)} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}_a-\mathbf{x}_b)}}{|\mathbf{k}|^2}\left(1+\frac{k_0^2}{|\mathbf{k}|^2}+\cdots \not\epsilon\right)$$

k_0 dependence factorizes $\to \int dk_0 k_0^{2n} e^{ik_0 t_{ab}} \sim \frac{d^n}{dt^n}\delta^{(2n)}(t_{ab})$
Near-Zone approximation $V_{PM} - V_{PN-Near}$ under control for $k_0^2 < |\mathbf{k}|^2$
Effects for $k_0 \simeq |\mathbf{k}|$ are missed: internal gravitons going on-shell

PN Simplifies: Far

- Far Zone, expand the numerator:

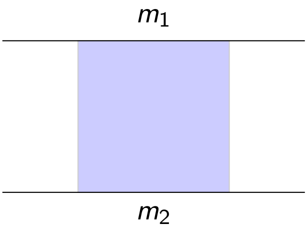
$$V_{PN-Far} \propto \int \frac{d^4k}{(2\pi)^4} e^{ik_0 t_{12}} \sum_n \frac{(i\mathbf{k}\cdot\mathbf{x}_{ab})^n}{n} \frac{1}{|\mathbf{k}|^2 - k_0^2 \pm iak_0 (A,R)}$$

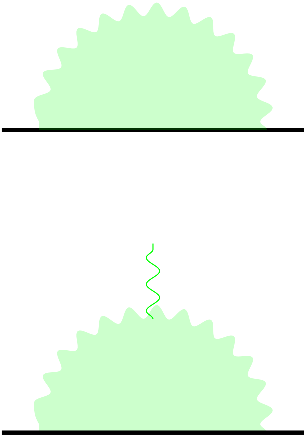
$$G_{A,R} = -\frac{1}{4\pi}\frac{\delta(t\pm r)}{r} \qquad \tilde{G}^*_{A,R}(k_0) = \tilde{G}_{A,R}(-k_0)$$

1. From individual world-lines to multipole expansions $Q_{i_1\ldots i_n}$, $L_{i_1\ldots i_n}$ with small parameter approximation $\mathbf{k}\cdot\mathbf{x}_{ab} \sim \frac{v}{r} \times r = v$
2. Time-symmetric process determined by $G_R + G_A$ (see later in this talk)
3. Longitudinal modes are present in Far Zone too, sourced by M, P_i, L_i

- Old friend of particle physicists: method of regions

Near vs. Far zone graphs (topology)





And 1 pt diagrams → radiation
In this talk only conservative sector
No radiation to ∞

Controlling the approximation

$$
\begin{aligned}
I &\equiv \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2-k_0^2+\epsilon} \\
N &\equiv \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\mathbf{k}^2}\sum_{n\geq 0}\left(\frac{k_0^2}{\mathbf{k}^2}\right)^n \\
F &\equiv \frac{1}{\mathbf{k}^2-k_0^2+\epsilon}\sum_{r\geq 0}\frac{(i\mathbf{k}\cdot\mathbf{x})^r}{r!} \\
D &\equiv \frac{1}{\mathbf{k}^2}\sum_{n,r\geq 0}\left(\frac{k_0^2}{\mathbf{k}^2}\right)^n\frac{(i\mathbf{k}\cdot\mathbf{x})^r}{r!}
\end{aligned}
$$

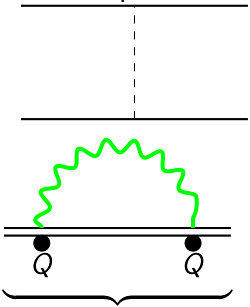
Using scale separation $\frac{v}{r}<\kappa<\frac{1}{r}$ and dim. reg.: $\int_{\mathbf{k}}|\mathbf{k}|^\alpha=0$, $H:\kappa<k$, $S:k<\kappa$

(Manohar+ PRD '07, Jantzen JHEP '12)

$$
\begin{aligned}
\int_{\mathbf{k}} I-(N+F) &= \int_H (PM-N-F)+\int_S (PN-N-F)+\underbrace{\int_k D}_{=0} \\
&= \int_H\Big[\underbrace{I-N}_{=0}-\underbrace{(F-D)}_{=0}\Big]+\int_S\Big[\underbrace{I-F}_{=0}-\underbrace{(N-D)}_{=0}\Big]
\end{aligned}
$$

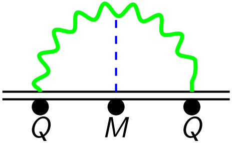
Diagram topology

- $V_{PM} = V_{PN-Near} + V_{PN-Far}$
- Parameter of expansion in NZ and FZ approximation are related



$\sim \ddot{Q}_{ij}\,\dddot{Q}_{ij} = 0(A,R)\,,\ Im \neq 0\,(Feynman)$

$$V_{PN-N} \sim \frac{G_N m_1 m_2}{r}\left[1 + v^2 + v^4 \dots\right]$$



Lowest-order non 0 FZ diag:

$G_N M \times \dddot{Q}_{ij}^2$

NZ diag expansion parameter $G_N M k_0 \sim v^3$ (Kepler's law)
Multipole couplings: Mh_{00}, $Lh_{0i,j} \sim Mv^2 h$, $\ddot{Q} h_{ij} \sim Mv^2 h_{ij}$

Near + Far with no mixing

3 Green's functions remainder

$$\int_{\mathbf{kqp}} \delta(k+q+p)\left(I^3-N^3-F^3\right) = \left[\int_{HHH}+\int_{SSS}+\int_{HHS}+\int_{HSH}+\int_{SHH}+\cancel{\int_{HSS}}\right]\delta(k+q+p)(I^3-N^3-F^3+\underbrace{D^3}_{\int_{\mathbf{k}} D=0})$$

$$\int_{HHH}\delta\left(I^3-N^3-F^3+D^3\right) = \int_{HHH}\delta[\underbrace{I^3-N^3}_{=0}-\underbrace{(F^3-D^3)}_{=0}]$$

$$\int_{SSS}\delta\left(I^3-N^3-F^3+D^3\right) = \int_{SSS}\delta[\underbrace{I^3-F^3}_{=0}-\underbrace{(N^3-D^3)}_{=0}]$$

$$\int_{HHS}\delta\left(I^3-N^3-F^3+D^3\right) = \int_{HHS}\delta[(\underbrace{I^2-N^2}_{=0}+N^2)(\underbrace{I-F}_{=0}+F) -NN(\underbrace{N-D}_{=0}+D)-(\underbrace{FF-DD}_{=0}+DD)F+D^3]$$

$$= \int_{HHS}\delta\,(NN-DD)\,(F-D)$$

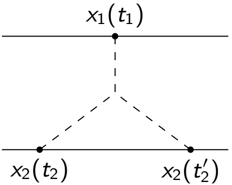
$$\int_{\mathbf{kqp}}\delta\,(NN-DD)\,(F-D) = \int_{\mathbf{kqp}} NNF$$

p-dependence has been factorized apart from $F(p)$, imaginary integral $\Longrightarrow$ remainder is $O(G_N^2)$ correction to flux, not to conservative potential QED

Outline

NZ calls for FZ processes

NZ produces spurious IR divergences



In the full theory:

$$\begin{aligned} V &\supset G_N^2 m_1 m_2^2 \int dt_{1,2,2'} d^4p\, e^{ip_\mu(x_1^\mu(t_1)-x_2^\mu(t_2))}\frac{p^\alpha p^\beta}{p^2}\int d^4k \frac{e^{ik^\mu(x_2(t_2)-x_2(t_2'))}}{(p-k)^2k^2} \\ &= G_N^2 m_1 m_2^2 \int dt_{1,2,2'} d^4p e^{ip_\mu(x_1^\mu(t_1)-x_2^\mu(t_2))}\frac{p^\alpha p^\beta}{p^2}\Delta(p^\mu(x_2(t_2)-x_2(t_2'))) \end{aligned}$$

after near/far breaking:

$$\int dt\, d^3p\, e^{i\vec{p}(\vec{x}_1-\vec{x}_2)}\frac{p^i p^j}{|\mathbf{p}^2|}\int d^3k \frac{1}{|\mathbf{k}|^2|\mathbf{p}-\mathbf{k}|^2}\left(1+\ldots+\frac{\omega^4}{|\mathbf{k}|^4}+\ldots\right)$$

$$=\int dt\, d^3p e^{i\vec{p}\cdot\vec{x}_{12}}\frac{p^i p^j}{|\mathbf{p}|^3}\left\{1+\ldots+\frac{1}{|\mathbf{p}|^4}\left[(\vec{p}\cdot\vec{v}_1)^3(\vec{p}\cdot\vec{v}_2)^3+\ldots+\underbrace{\vec{p}\cdot\dot{\vec{a}}_1\vec{p}\cdot\dot{\vec{a}}_2}_{IR\ \ divergence}\right]\right\}$$

Divergence and finite terms

- Near IR/Far UV

$$V \supset G^2 M \dddot{Q}_{ij}^2 \left(\frac{1}{\epsilon_{UV}} + 2\log\left(\frac{k_0}{\mu}\right) - \frac{41}{30} + \cancel{i\pi sgn(k_0)} \right)$$
$$+ m_1 m_2^2 r^2 \dot{a}_1^i \dot{a}_{2i} \left(-\frac{1}{\epsilon_{IR}} + \log(\mu r) + \ldots \right) + 1 \leftrightarrow 2$$

Theory at short and large distances have compensating spurious divergences, finite terms derived straightforwardly (S. Foffa, RS PRD '13)
Effect $G_N^2 M^3 v^6 \to G_N^4 M^5 v^2$ using e.o.m. (4PN)

- Near zone UV divergences canceled by local counterterms:

$$G^2 m_a^3 \int d\tau \left(a^\mu \dot{v}_\mu + R_{\mu\nu} v^\mu v^\nu \right)$$

S. Foffa, R. Porto, I. Rothstein, RS PRD '19

$a^\mu = 0 = R_{\mu\nu}$ on the equations of motion

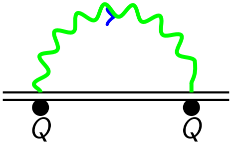
- No far zone IR divergences
- FZ alone → leading UV logs in the Energy at all orders via Ren. Group flow

W. Goldberger, A. Ross PRD '10; W. Goldberger, A. Ross, I. Rothstein PRD '14; L. Blanchet, S. Foffa, F. Larrouturou, RS PRD '20

Outline

First emission, then absorption

- Diagram with 1 full propagator

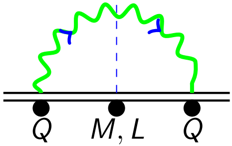


Averaging over $1 \to 2 + 2 \to 1 \implies 1/2(G_R + G_A)$ *Green's pag. 29 of these slides*

$$
\begin{aligned}
G_F &= \frac{1}{2}(G_A + G_R) - \frac{i}{2}(\Delta_+ + \Delta_-) \\
\Delta_\pm(t, \mathbf{x}) &= \int_k \frac{dk_0}{2\pi} \theta(\pm k_0) \delta(k_0^2 - \mathbf{k}^2) e^{-ik_0 t + i\mathbf{k}\cdot\mathbf{x}}
\end{aligned}
$$

G_F gives correct conservative result + bonus: "probability loss" (optical theorem)

- Diagram with 2 full propagators

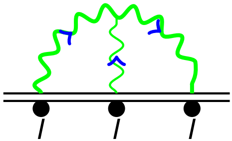


$Q(k_0)\frac{1}{2}\left(\tilde{G}_R(k_0)\tilde{G}_R(k_0) + \tilde{G}_A(k_0)\tilde{G}_A(k_0)\right) Q(-k_0)$, using $\tilde{G}_F$ same result because

$\tilde{G}_F^2 - \frac{1}{2}\left(\tilde{G}_R^2 + \tilde{G}_A^2\right) = \frac{i}{2}\left(\tilde{G}_A + \tilde{G}_R\right)\left(\tilde{\Delta}_+ + \tilde{\Delta}_-\right)$ (dissipative bonus)

3 Radiative Green's functions

- With 3 full propagators

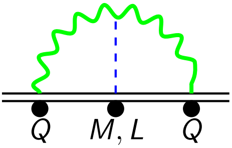


$Q(k_{01})Q(k_{02})Q(-k_{01}-k_{02})\tilde{G}_R(k_{01})\tilde{G}_R(k_{02})\tilde{G}_A(-k_{01}-k_{02})$
$+$ symmetrization not expressible in terms of product of $\tilde{G}_F$
Diagram involves UV divergent 2-loop master integral which however drops out when adding all polarizations: it is finite

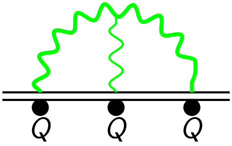
Outline

5PN ✓

- 5PN Near Zone recently computed by
  J. Blümlein, A. Maier, P. Marquard, G. Schäfer PLB '21
  Computed 188533 diagrams with same master integrals as 4PN
  All UV divergences vanish on e.o.m. $\rightarrow$
  BHs still looking for Love $\sim \dot{R}^2_{\mu\nu\rho\sigma}$
  Also results at NZ 6PN up to G_N^4 Blümlein+ PLB '21
- 5PN Far Zone done in S. Foffa, RS PRD '20 (+ Erratum '21)



Tail



Memory

Poles at G_N^2 from Far Zone known for all multipoles

(S. Foffa,RS arXiv:2103.03190)

Far zone Self Energy results at 5PN and beyond

Real part $\to$ conservative dynamics (to be added to near zone results, starting 4PN order)

Imaginary part matches into flux formula $F \propto \dddot{Q}_{ij}^2 + \dots$

Divergent graphs regularized in dim. reg.:

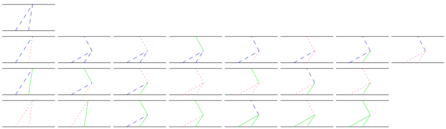
divergence (and coeff. of logarithmic term) linked to imaginary part

$$
\begin{aligned}
S_{5PN\,tail} &= G_N^2 M \int \frac{dk_0}{2\pi} k_0^6 \left[-\frac{1}{5}\left(\frac{1}{\epsilon} + \log\left(k_0^2/\bar{\mu}^2\right) - i\pi + \frac{41}{30}\right) k_0^2 |Q_{ij} \right. \\
&\quad -\frac{1}{189}\left(\frac{1}{\epsilon} + \log\left(k_0^2/\bar{\mu}^2\right) - i\pi + \frac{163}{35}\right) |O_{ijk}|^2 \\
&\quad \left. -\frac{16}{45}\left(\frac{1}{\epsilon} + \log\left(k_0^2/\bar{\mu}^2\right) - i\pi - \frac{127}{60}\right) |J_{ij}|^2 \right] \\
S_{5PN\,Ltail} &= \frac{8}{15} G_N^2 \int dt\, \ddddot{Q}_{il} \dddot{Q}_{jl} \epsilon_{ijk} L_k \\
S_{5PN\,memory} &= -\frac{G_N^2}{15} \int dt \left[\ddddot{Q}_{il} \ddddot{Q}_{jl} Q_{ij} + \frac{4}{7} \dddot{Q}_{il} \dddot{Q}_{jl} \ddot{Q}_{ij} \right]
\end{aligned}
$$

Diagram proliferation at higher orders

For PN expansion it is useful to distinguish gravity polarizations, e.g. G_N^2

2 topologies
(Post-Minkowskian, G_N-expansion)



23 Feynman diagrams at 4PN
$(G^2 v^3)$

$\sim h_{00} \to v^0$; $\sim h_{0i} \to v^i$; $\sim h_{ij} \to v^i v^j$;

However only 1 of the two topologies is intrinsically G_N^2

Factorizable vs. prime diagrams

Number grows exponentially

3PN	prime top	prime dgrs	fac	4PN	prime dgrs	fac
G_N	1	3	0		3	0
G_N^2	1	16	3		18	23
G_N^3	5	31	22		158	54
G_N^4	12	0	8		171	146
G_N^5				25	25	25

S. Foffa, RS PRD'19

At 5PN the $G_N^5 v^2$ sector has 40 prime topologies ($\sim$ 700 prime diagrams) and 1232 fac. diags

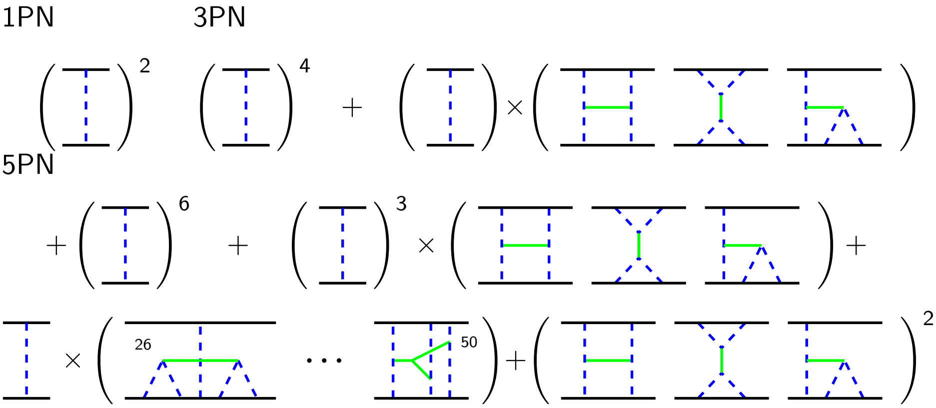
Effective action does not efficiently store perturbative G_N information

NRGR can help tackling the high n $G_N^n v^{0,2}$ side (orthogonal to PM)

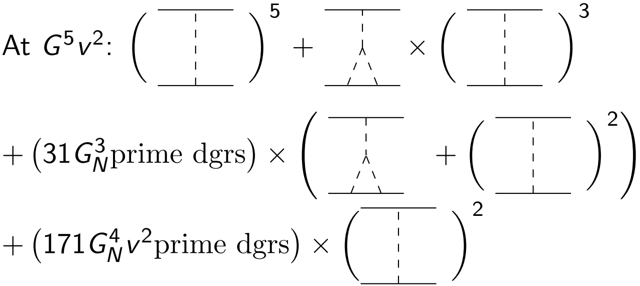
Factorizable diagrams at G^6 5PN

5PN G^6 static sector has 0 prime toplgs and 154 fac dgrs
Static sectors at $2n+1$-PN order have no prime sector:
impossible to build prime digrs with $2n+1$ m(ass) insertions and m-ϕ and $\phi^2\sigma$ vertices

Foffa, Mastrolia, RS, Sturm, W. Torres Bobadilla, PRL '19



Factorizable diagrams at G^5 5PN



They amount to 1232 out of 1907 $G^5 v^2$ diagrams

Foffa, RS. W. J. Torres Bobadilla, JHEP '20 in agreement with subset of full 5PN by BMMS PLB '21

Outline

Take home message

- PN approximation breaks difficult integrals into simpler ones
- This introduces some spurious divergences UV/IR divergences, which allow non-trivial sanity checks of the calculation, now available at all PN order at $O(G_N^2)$ (but be careful with G_N off-shell $\rightarrow$ on-shell power counting)
- Near zone UV divergences are short-distance singularities absorbable by counterterms vanishing on the e.o.m. up to at least 5PN included
- Going to higher order one has to face two kinds of problem:

  1. diagram proliferation 
  2. solving new master integrals possibly at $G^6 v^2$, very likely at G_N^7 (6PN)

Summary of 2 body dynamics expansions (spin-less)

Post-Minkowskian expansion parameter is $G_N M/r$, vs PN expansion

$$\mathcal{L} = -Mc^2 + \frac{\mu v^2}{2} + \frac{GM\mu}{r} + \frac{1}{c^2}[\ldots] + \frac{1}{c^4}[\ldots]$$

Terms known so far

		N	1PN	2PN	3PN	4PN	5PN	6PN	...
0PM	1	v^2	v^4	v^6	v^8	v^{10}	v^{12}	v^{14}	...
1PM		$1/r$	v^2/r	v^4/r	v^6/r	v^8/r	v^{10}/r	v^{12}/r	...
2PM			$1/r^2$	v^2/r^2	v^4/r^2	v^6/r^2	v^8/r^2	v^{10}/r^3	...
3PM				$1/r^3$	v^2/r^3	v^4/r^3	v^6/r^3	v^8/r^4	...
4PM					$1/r^4$	v^2/r^4	v^4/r^4	v^6/r^5	...
5PM						$1/r^5$	v^2/r^5	v^4/r^5	...
6PM							$1/r^6$	v^2/r^6	...
7PM								$1/r^7$	...
...									...

Spare slides

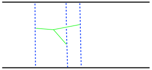
Green's function

$$
\begin{aligned}
G_F &= -i\left[\theta(t)\Delta_+ + \Theta(-t)\Delta_-\right] \\
G_R &= -i\theta(t)\left[\Delta_+ - \Delta_-\right] \\
G_A &= i\theta(-t)\left[\Delta_+ - \Delta_-\right] \\
G_H &= \frac{1}{2}\left[\Delta_+ - \Delta_-\right] \\
G_R - G_A &= -i\left[\Delta_+ - \Delta_-\right]
\end{aligned}
$$

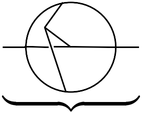
EFT and amplitude: tale of a happy marriage

The other obstruction to scalability of the NRGR PN calculation program is the computation of master integrals

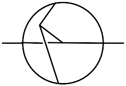
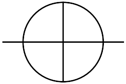
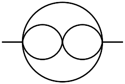
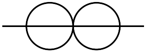
E.g. in the static 4PN sector (i.e. G_N^5) one meets

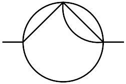
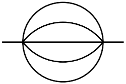


$$= -\mathrm{i}\,(8\pi G_N)^5 \left(\frac{(d-2)}{(d-1)}\; m_1 m_2\right)^3$$



$$\int_{k_{1,2,3,4}} \frac{N_{50}}{k_1^2\;\; k_2^2\;\; k_3^2\;\; k_4^2\;\; k_{12}^2\;\; k_{34}^2\;\; \hat{k}_{24}^2\;\; p_{13}^2\;\; \hat{p}_{14}^2}$$

 $= c_1$  $+c_2$  $+c_3$ 

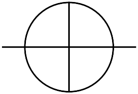
$+c_4$  $+c_5$ 

in terms of 4-loop self-energy diagrams in gauge theory

Reduction in terms of master integrals

No new master integrals at 5PN, 4PN ones did it all

Foffa, Mastrolia, RS, Sturm '17



$$= \frac{e^{2\varepsilon\gamma_E}}{s^{2-2\varepsilon}(4\pi)^{4+2\varepsilon}}\left\{\frac{1}{2\varepsilon^2}-\frac{1}{2\varepsilon}-4+\frac{\pi^2}{24}\right.$$

$$\left.-\varepsilon\left[9-\pi^2\left(\frac{13}{8}-\log 2\right)-\frac{77}{6}\zeta_3\right]+\mathcal{O}\left(\varepsilon^2\right)\right\}$$

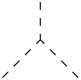
Numerical result obtained via Summertime by Lee& Mingulov
analytic result via PSLQ algorithm, fitting trascendentals to numerical result

Confirmed up to $O(\varepsilon^0)$ by Damour, Jaranowski '18

Double copy for EFT

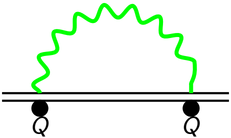
Master integrals have to do with denominators, however numerators can be simplified too by writing $A_{GR} = A_{YM}^2$

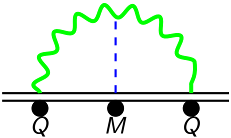
Bern, Carrasco, Johansson PRD '08

On-shell three vertices can be mapped: 

$$
\begin{array}{ccc}
\text{gauge theory} & \to & \text{gravity theory} \\
gf^{abc}\left(\eta_{\mu\nu}\left(k_1-k_2\right)_\rho+\text{cyclic}\right) & \to & \sqrt{G_N}\left(\eta_{\mu\nu}\left(k_1-k_2\right)_\rho+\text{cyclic}\right) \\
 & & \times\left(\eta_{\mu\nu}\left(k_1-k_2\right)_\rho+\text{cyclic}\right) \\
\text{color factors} & \to & \text{kinematic numerator}
\end{array}
$$

Double Copy of Far-Zone amplitudes





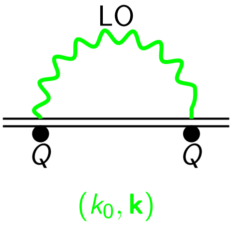
also doubling of world-line vertex  $d_i F_{0i} \to I_{ij} E_{ij}$, $\mu_i F_{ij} \to J_{ij} B_{ij}$

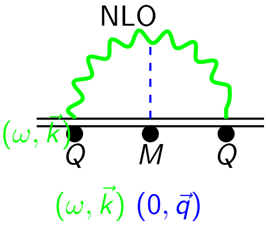
Verified for both electric and magnetic multipoles at NLO in G_N

Goldberger, Ridgway '18, Shen '18, Almeida, RS, Foffa JHEP '20

Yet to be verified derivation how YM→ GR mapping propagates from "microscopic physics" to multipoles

Double Copy Far-Zone continued





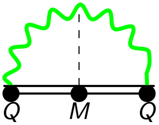
- Electric Self-Energy
  1. LO: $I^{iji_1\dots i_r}(\omega) Q^{iji_1\dots i_r}(-\omega) \frac{k_{i_1}\dots k_{i_r} k_{k_1}\dots k_{k_r}}{k^2-\omega^2} \left(\omega^2\delta_{ik} - k_i k_k\right) \left(\omega^2 \delta jl - k_j k_l\right)$
  2. NLO: $I^{iji_1\dots i_r}(\omega) I^{iji_1\dots i_r}(-\omega) \frac{k_{i_1}\dots k_{i_r} k_{k_1}\dots k_{k_r}}{(k^2-\omega^2)((k+q)^2-\omega^2)q^2} k_0^2$
  $\times \left(\omega^2\delta_{ik} - (k+q)_i k_k + q_i q_k\right) \left(\omega^2\delta_{jl} - (k+q)_j q_l + q_j q_l\right)$
- analogously for the magnetic self-Energy at LO and NLO

Gravity+dilaton+anti-symmetric tensor amplitude matches gauge^2

All leading logs in $E_{circ}(x)$

$$E_{circ} = -\frac{M\nu}{2} x \left(1 + \frac{16\nu x^2}{15\beta_I}\left[\left(1 + 24\beta_I x^3 \log x\right) x^{4\beta_I x^3} - 1\right]\right) \qquad \beta_I = -\frac{214}{105}$$

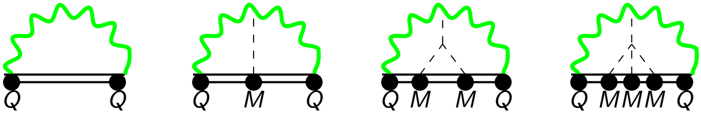
- In PN approximation Log terms arise from tail processes at 4PN order, non-local (but causal) effective term in conservative dynamics ($x \sim v^2 \sim Gm/r \equiv \gamma$):



$$\mathcal{L} = \frac{M\nu}{2} v^2 + \ldots + \frac{2G^2M}{5} \dddot{Q}_{ij}(t) \int d\tau \log(\tau) \ddddot{Q}_{ij}(t-\tau) \ldots$$
$$\to E_{circ} = -\frac{M\nu x}{2}\left(1 + \ldots + \frac{448}{15}\nu x^5 \log x + \ldots\right)$$

which turns local on circular orbits

- Expansion in $GM\omega = \frac{GM}{r} \times r\omega \sim v^3$:



LO tail: Blanchet, Damour PRD ('88)

Leading Logs at all orders

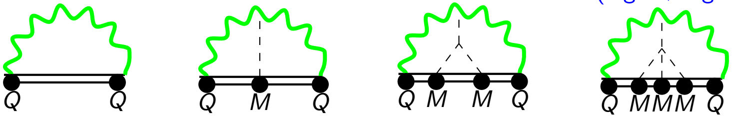
Real part E:

Self-E$\sim$ $(\ddot{Q})(\dddot{Q}) \sim 0$

tail^1 $\sim$ $GM(\dddot{Q})^2 \log t$ $x^4 \log x$

tail^2 $\sim$ $(GM)^2(\dddot{Q})(\ddddot{Q}) \log$ $\sim (GM)^2(\dddot{Q})(\ddddot{Q})$ $x^{11/2}$

tail^3 $\sim$ $(GM)^3 (\log + \log^2) \times$ $(\ddddot{Q})(\ddddot{Q})$ $x^7(\log x + \log^2 x)$



Imaginary $\frac{dE}{dt}$:

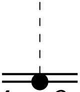
LO flux

tail^1 $\sim \pi x^{3/2}$

tail^2 $\sim x^3 log(x)$

tail^3 $\sim ? \times x^{9/2}$

Other insertions possible:



$M \to Q, L$

but do not contribute to leading E-logs: $\nu^2 x^{3n+1} \log^n x$ from tail^{2n-1}

Renormalization group enables to compute all leading logs: E-logs formula extends logs in $E(x)$ at $O(\nu)$ from self-force expanded up to 22 PN in Kavanagh-Ottewill-Wardell PRD 92 (2015)

Far UV divergences

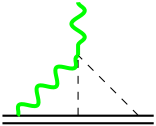
Suppose one had the Far zone theory only: the UV divergence is not compensated by the NZ but it can be renormalized:

- drop the divergence (absorb it with a local counterterm)
- impose $\mu-$independence Goldberger, Ross, Rothstein PRD '14

$$\frac{d\mathcal{L}_{tail}}{d\log\mu}=0 \implies \frac{dM}{d\log\mu}=-\frac{2G^2M}{5}\left(2Q^{(1)}_{ij}Q^{(5)}-2Q^{(2)}_{ij}Q^{(4)}_{ij}+\left(Q^{(3)}_{ij}\right)^2\right)$$

which can be solved by short-circuiting with analog equation for

$$\frac{dQ_{ij}}{d\log\mu}=\frac{214}{105}(GM)^2\,\ddot{Q}_{ij}(t,\mu)$$



Goldberger, Ross PRD '09
(see also Anderson+ '82!)
Adding analogous formula for J (Bernard, Blanchet, Faye, Marchand, Phys. Rev. D97 (2018)) and taking orbital average:

$$M(\mu)=M(\mu_0)-MG^2\sum_{n\geq 1}\frac{(2\log(\mu/\mu_0))^n}{n!}\left(\beta_I G^2M^2\right)^{n-1}\langle Q^{(n+2)}_{ij}Q^{(n+2)}_{ij}\rangle$$

$$L(\mu)=L(\mu_0)-\frac{12MG^2}{5}\sum_{n\geq 1}\frac{(2\log(\mu/\mu_0))^n}{n!}\left(\beta_I G^2M^2\right)^{n-1}\langle Q^{(n+1)}_{ij}Q^{(n+2)}_{ij}\rangle$$

Not quite there for E_{circ}: need for $dE = \omega dL$

Using $Q_{ij}(M,\mu)$ one has (leading log part of) $M(M_0,v,\gamma)$, $L(L_0,v,\gamma)$, adding $dE=\omega dL$ one can compute $r(v)$ on circular orbits: $Energy(r,v) \to E_{circ}(x)$ $(x \equiv (GM\omega)^{2/3})$

$$\gamma \equiv \frac{GM}{r} = x\left[1 + \frac{32\nu}{15}\sum_{n\geq 1}\frac{3n-7}{n!}(4\beta_I)^{n-1}x^{3n+1}(\log x)^n\right]$$

$$\begin{aligned} E &= -\frac{m\nu x}{2}\left[1+\frac{64\nu}{15}\sum_{n\geq 1}\frac{6n+1}{n!}(4\beta_I)^{n-1}x^{3n+1}(\log x)^n\right] \\ J &= \frac{m^2\nu}{\sqrt{x}}\left[1-\frac{64\nu}{15}\sum_{n\geq 1}\frac{3n+2}{n!}(4\beta_I)^{n-1}x^{3n+1}(\log x)^n\right] \end{aligned}$$

Remarkably $E(x)$ agrees 22PN order $x^{3n+1}(\log x)^n$ (up to $n=7$), expanded self-force result by Kavanagh, Ottewill, Wardell, PRD (2015)