Localization of Supersymmetric gauge theories in 3d

Theory Lectures by Young Researchers

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References

* Original paper
  - 0909.4559

* Usefull review for 3d
  - 1608.02958
  - 1104.0783
  - 1608.02960 (F-maximization)

* General review
  - 1608.02952
  - 9608068

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$$I_\lambda = \int_a^b du \; g(u)\, e^{-\lambda f(u)}$$

$$\lambda \sim \frac{1}{\hbar}$$

$\lambda \gg 1$

$$\begin{cases} f'(u^*) = 0 \\ f''(u^*) > 0 \end{cases}$$

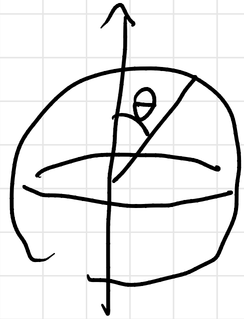
$g = 1 \qquad u^* \in (a, b)$

$$f(u) \cong f(u^*) + (u-u^*)^2 \frac{1}{2} f''(u^*)$$

$$I_\lambda \cong \underset{\substack{\uparrow \\ \text{classical}}}{e^{-\lambda f(u^*)}} \int_a^b du \; e^{-\lambda/2\,(u-u^*)\, f''(u)} =$$

$$= e^{-\lambda f(u^*)} \sqrt{\frac{2\pi}{\lambda f''(u^*)}}$$

$$I_\lambda = \int_{S^2} ds^2 \; e^{i\lambda\cos\theta} =$$



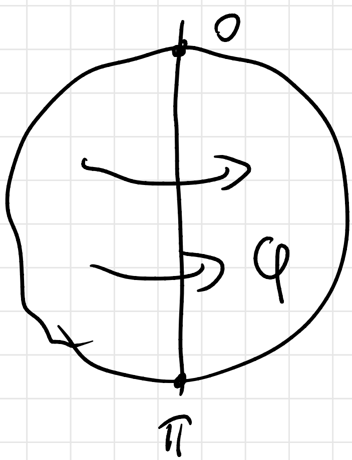
$$= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \; e^{i\lambda\cos\theta} \simeq$$

$$\simeq \frac{2\pi i}{\lambda} \left(-e^{i\lambda} + e^{-i\lambda} \right) = \frac{4\pi}{\lambda} \sin\lambda$$

$\downarrow$

EXACT !

WHY ?



$\varphi \to \varphi + \beta \qquad U(1)$

(M, g) no boundary $\qquad \dim M = 2\ell$

$V = V^\mu \partial_\mu \qquad \nabla_{(\mu} V_{\nu)} = 0$

$U(1)_V \hookrightarrow M$

$$\Lambda^m M = \left\{ \omega = \frac{1}{m!} \omega_{\mu_1 \dots \mu_m} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_m} \right\}$$

$$\Lambda M = \bigoplus_{k=1}^{2\ell} \Lambda^k M$$

$\alpha \in \Lambda M \qquad \alpha = \alpha_{2\ell} + \dots + \alpha_0$

$$\int_M \alpha = \int_M \alpha_{2\ell}$$

- $d : \Lambda^m M \to \Lambda^{m+1} M$

$$d\omega = \frac{1}{m!} \partial_\lambda \omega_{\mu_1 \dots \mu_m} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_m}$$

$$d^2 = 0$$

- $\iota_v : \Lambda^m M \to \Lambda^{m-1} M$

$$\iota_v \omega = \frac{1}{m!} V^\mu \omega_{\mu \nu_1 \dots \nu_{m-1}} dx^{\nu_1} \wedge \dots \wedge dx^{\nu_{m-1}}$$

$$\iota_v^2 = 0$$

$\partial M = \emptyset$

Stokes theorem $\int_M d\omega = \int_{\partial M} \omega = 0$

- α closed if $d\alpha = 0$
- α exact if $\alpha = d\beta$

α exact $\Rightarrow$ α is closed $\quad d\alpha = d^2\beta = 0$

$$H^m(M) = \operatorname{Ker} d|_{\Lambda^m M} / \operatorname{Im} d|_{\Lambda^{m-1} M}$$

$$\int \alpha = \int \alpha + d\beta$$

$U(1) \hookrightarrow M$ $\qquad$ $M/U(1)$ not a manifold

fixed point $p \in M$ $\qquad$ $gp = p$ $\qquad$ $g \neq \mathrm{Id}$

- equivariant differential

$d_v := d - i_v \qquad d_v : \Lambda M \to \Lambda M$

$d_v^2 = d^2 + i_v^2 - \{d, i_v\} = -\mathcal{L}_v$

$\Lambda_v M = \{ \alpha \in \Lambda M \mid \mathcal{L}_v \alpha = 0 \}$

$d_v^2 = 0$ on $\Lambda_v M$

- equiv. closed if $d_v \alpha = 0$
- equiv exact if $\alpha = d_v \beta$

$H_v^\bullet(M) = \operatorname{Ker} d_v|_{\Lambda_v M} / \operatorname{Im} d_v|_{\Lambda_v M}$

$$\int_M d_V \beta = \int (d - i_V)(\beta_{2\ell} + \beta_{2\ell-1} + \cdots + \beta_0) =$$

$$= \int_M d\beta_{2\ell-1} = 0$$

$$\int_M \alpha = I \qquad \alpha \in \Lambda_V M \qquad d_V \alpha = 0$$

$$\alpha_t := \alpha e^{-t d_V \beta} \qquad t \in \mathbb{R}$$

$$\beta \in \Lambda_V M$$

$$I(t) = \int_M \alpha_t \qquad I(0) = I$$

$$d_V \alpha = 0$$

$$\mathcal{L}_V \beta = d_V^2 \beta = 0$$

$$\frac{d}{dt} I(t) = -\int \alpha \, d_V \beta \, e^{-t d_V \beta} =$$

$$= -\int d_V (\cdots) = 0$$

$$I = I(t) = \lim_{t\to\infty} \int \alpha_t = \int \alpha \, e^{-t(d\beta - |V|^2)}$$

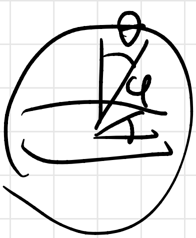
$$\beta = V_\mu \, dx^\mu$$

$$d_V \beta = d\beta - V^\mu V_\mu = d\beta - |V|^2$$

Integral localizes $|V|^2 = 0$

$$M_V = \{ x \in M \mid V(x) = 0 \}$$ localization locus

$$I = \int_{S^2} dS^2 \, e^{i\lambda\cos\theta} \qquad V = \partial_\varphi$$



$$\alpha = d\varphi \wedge d\cos\theta \, e^{i\lambda\cos\theta}$$

$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$$

$$\beta = g_{\mu\nu} V^\nu dx^\mu = \sin^2\theta \, d\varphi$$

$$\beta_1 \qquad M_v \qquad I = \int \dots$$

$$\qquad\qquad\qquad\qquad \parallel$$

$$\beta_2 \qquad M_v \qquad I = \int \dots$$

$$dx^\mu \wedge dx^\nu = - dx^\nu \wedge dx^\mu$$

$$dx^\mu \to \psi^\mu \qquad \mu = 1 \dots 2\ell$$

$$\omega(x,\psi) = \frac{1}{m!} \omega_{\mu_1 \dots \mu_m} \psi^{\mu_1} \dots \psi^{\mu_m}$$

$$x^\mu \qquad \delta \qquad \begin{cases} \delta x^\mu = \psi^\mu \\ \delta \psi^\mu = V^\mu(x) \end{cases}$$

$$\psi^\mu$$

$$\delta^2 \sim \mathcal{L}_V \qquad d_V \longleftrightarrow \delta$$

$$\int_M \alpha \sim \int d^{2\ell} x \, \alpha(x) \to \int d^{2\ell} x \, d^{2\ell} \psi \, \alpha(x, \psi)$$

$$I = \int \mathcal{L} = \int d^{2\ell}x\, d^{2\ell}\psi\, \mathcal{L}(x,\psi)$$

$$\delta \mathcal{L} = 0 \qquad \psi^\mu \leftrightarrow dx^\mu$$

$$I(t) = \int d^{2\ell}x\, d^{2\ell}\psi\, \mathcal{L}(x,\psi)\, e^{-t\delta W}$$

$$\frac{dI}{dt} = 0 \Longleftrightarrow \delta^2 W = 0$$

$$W = V^\mu g_{\mu\nu} \psi^\nu = \delta\psi^\mu g_{\mu\nu} \psi^\nu$$

$$\delta W = \partial_\lambda (V_\nu)\, \psi^\lambda \psi^\nu + |V|^2$$

$$\delta^2 W = \psi^\rho V^\mu (\mathcal{L}_V g)_{\rho\mu} = 0$$

$$M_V = \{ x_k \}$$

$$\delta W_{(2)} \cong \frac{1}{2} \left(H_{\mu\nu} x^\mu x^\nu + S_{\mu\nu} \psi^\mu \psi^\nu \right) +$$

$$\begin{cases} \psi^\mu = 1/\sqrt{t}\, \psi'^\mu \\ x^\mu = 1/\sqrt{t}\, x'^\mu \end{cases}$$

$x^* \in M_V$

$x^* = 0$

$$I = \int d^{2\ell} x' \, d^{2\ell} \psi' \, \alpha\left(x/\sqrt{t}, \psi/\sqrt{t} \right) \times$$

$$\times e^{-\frac{1}{2} H_{\mu\nu} x'^\mu x'^\nu - \frac{1}{2} \psi'^\mu S_{\mu\nu} \psi'^\nu} =$$

$$= \alpha_0(0) \frac{\mathrm{Pf}(S)}{\sqrt{\det H}} \leftarrow$$

δ_L linearized

$$\delta_L x^\mu = \psi^\mu$$

$$\delta_L \psi^\mu = x^\nu \partial_\nu V^\mu(0)$$

$$\delta_L \, \delta W_{(2)} = 0$$

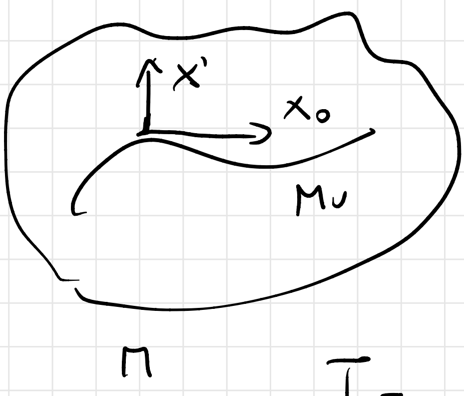
$$\downarrow$$

$$H_{\mu\nu} = S_{\mu\lambda} \partial_\nu V^\lambda(0)$$

$$\frac{\mathrm{Pf}(S)}{\sqrt{\det H}} = \left[\det \partial_\nu V^\lambda(0)\right]^{-1/2} \sqrt{\frac{\det H}{\det H}}$$

$$\int_M \alpha = (-2\pi)^e \sum_{p \in M_V} \frac{\alpha_0(p)}{\sqrt{\det \partial_\mu V^\nu(p)}}$$

M_V is a submanifold



$$x = x_0 + x'/\sqrt{t}$$

$$\psi = \psi_0 + \psi'/\sqrt{t}$$

$$I = \int_{M_V} dx_0 \, d\psi_0 \; \alpha\big|_{M_V} \, (\text{1-loop det})$$

$$Z = \int_M d^n x \, d^n \varphi \; e^{-S(x,\varphi)}$$

$$V \sim \frac{\partial}{\partial x_1} \qquad VS = 0 = \frac{\partial}{\partial x_1} S(x,\varphi)$$

$$Z = \mathrm{Vol}\,G \int d^{n-1} x \, d^n \varphi \; e^{-S(x,\varphi)}$$

$$V \sim \frac{\partial}{\partial \varphi_1} \qquad \mathrm{Vol}\,G = \int d\varphi_1 = 0$$

$$M \setminus \{M_V\}$$

$M \to F = \{$ space fields

$dv \to \delta \quad , \quad \delta^2 = U(1)_B$

dv closed $\to$ BPS observables $\delta \mathcal{O} = 0$

$$\langle \mathcal{O} \rangle = \int D[\Phi]\, \mathcal{O}\, e^{-S[\Phi]}$$

$$\langle \delta \mathcal{O} \rangle = 0$$

$$\langle \mathcal{O}[\Phi + \delta\Phi] - \mathcal{O}[\Phi] \rangle \qquad \frac{\partial}{\partial \Phi}\mathcal{O}$$

① $D[\Phi]$ not anomalous

② there's no contribution from boundary space of fields

$$\langle \mathcal{O} \rangle_t = \int D\Phi \, e^{-S[\Phi] - t\delta W[\Phi]}$$

$$\langle \mathcal{O} \rangle_{t=0} = \langle \mathcal{O} \rangle$$

$$\frac{d}{dt} \langle \mathcal{O} \rangle_t = \int D[\Phi] \, \delta(\ldots)$$

$$\text{if} \quad \delta^2 W = B W = 0$$

$$W_{an}[\Phi] = \int d^d x \sum_A \psi_A (\delta \psi_A)^\dagger$$

$$\delta W_{an}[\Phi] = \int d^d x \sum_A |\delta \psi_A|^2 + \psi_A \delta^2 \psi_A$$

bosonic

$\rightarrow$ localize $\boxed{\delta \psi_A = 0}$

$\rightarrow \psi_A = 0$

$$\delta^2 = \mathcal{B} + \text{e.o.m.}\,(\psi_A)$$

$$\psi_A \quad \text{e.o.m.} \quad \psi_A \qquad\qquad \Phi_0 = \text{const}$$

$$\Phi_0 \qquad\qquad \Phi = \Phi_0 + \frac{\delta\Phi}{\sqrt{t}}$$

$$\langle \mathcal{O} \rangle = \int D\Phi_0 \, e^{-S[\Phi_0]} \, \mathcal{O}(\Phi_0) \times$$

$$\times \, \mathrm{Sdet}\left(\frac{\delta^2 S_{eoc}[\Phi_0]}{\delta\Phi^2} \right)$$

$\epsilon = \epsilon(x)$

$$\delta S = \int d^dx \, J^\mu_\alpha \, \partial_\mu \epsilon^\alpha \Rightarrow \partial_\mu \epsilon = 0$$

$$\downarrow$$

$$\int d^dx \, J^\mu_\alpha \, \nabla_\mu \epsilon^\alpha \Rightarrow \nabla_\mu \epsilon \overset{?}{=} 0$$

$$M \rightarrow \ell$$

$$\begin{cases} S_{\text{feat}}[\varphi, \psi] \\ \delta_\epsilon \varphi, \delta_\epsilon \psi \end{cases} \longrightarrow \begin{cases} S_{\text{feat}}\big|_{\substack{\eta \to g \\ \partial \to \nabla}} + \sum_n a_n \ell^{-n} \\ \delta_\epsilon \varphi \big|_{\text{m.c.}} + \sum_n b_n \ell^{-n} \end{cases}$$

$$[\ell^{-1}] = 1$$

- $\{\delta_{\epsilon_1}, \delta_{\epsilon_2}\}\varphi$ SUSY algebra
- $\delta_\epsilon S = 0$

"Sistematic method"

SUSY QFT on M is a rigid limit of a SUGRA

$G_N \to 0$ $\qquad$ $g_{\mu\nu} \to (g_{\mu\nu})_{\text{fixed}}$

- non SUSY QFT

- flat space theory
- couple to gravity $\to$ differ invariant
- fix the metric $g_{\mu\nu}$ decouple fluct ($G_N \to 0$)

$$\delta g_{\mu\nu} = 0 \qquad \nabla_{(\mu} V_{\nu)} = 0$$

- add SUSY $U(1)$ current

$\partial_\mu J^\mu = 0$

seagulls

$$\mathcal{L}_0 \to \mathcal{L}_0 + J_\mu a^\mu + \mathcal{O}(a^2)$$

J_μ sits in a e.g. 4d $\mathcal{N}=1$

$$\mathcal{J} = (K, J_\alpha, \bar{J}_{\dot\alpha}, J_\mu)$$

$$V = (a_\mu, D, \lambda_\alpha, \bar{\lambda}_{\dot\alpha})$$

$$\mathcal{L}_0 \to \mathcal{L}_0 + J_\mu a^\mu + \lambda^\alpha J_\alpha + \bar{\lambda}^{\dot\alpha} \bar{J}_{\dot\alpha} + DK$$
$$+ \ldots$$

V non dyn. preserving som SUSY

$$\delta \mathcal{L}_{\text{coupe}} = 0 \quad \longleftrightarrow \quad \delta V_{\text{bock}} = 0$$

$- \quad \lambda = \bar{\lambda} = 0$

$$\delta \lambda = \underset{\uparrow \atop q}{\epsilon} (\underbrace{F_{\mu\nu} \gamma^{\mu\nu} + D}_{\text{susy backgr.}}) = 0$$

$$\delta \text{ bos} = \text{ferm} = 0$$

- SUGRA

$$T = (T_{\mu\nu} , \overset{\text{current}}{J_i} , f_i)$$

$$\mathcal{H} = (g_{\mu\nu} , B_i , \psi_{\mu\alpha} , \cdots)$$

backgrounds

$$\mathcal{L}_0 \to \mathcal{L}_0 + \tfrac{1}{2} h_{\mu\nu} T^{\mu\nu} + B^i J_i + \cdots$$

$g_{\mu\nu} , G_N \to 0$ breaks all local susy

$$\psi_{\mu\alpha} = 0$$

$$\delta \psi_{\mu\alpha} = 0 \qquad \text{off shell closed}$$

$$\nabla_\mu \epsilon_\alpha = M_\mu (g, B^i)_\alpha{}^\beta \epsilon_\beta \qquad \text{GKS}$$

$$g,\ B^i,\ \epsilon_\alpha$$

3d $\mathcal{N}=2$ SUSY theories

λ_α Dirac spinor $\alpha = 1,2$ $\mathrm{spin}(3) \cong SU(2)$

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -\epsilon_{\alpha\beta}$$

$\gamma_{\mu\,\alpha}{}^{\beta}$ = Pauli matrices

$$\gamma^\mu \gamma^\nu = \delta^{\mu\nu} + i\,\epsilon^{\mu\nu\lambda}\gamma_\lambda$$

$$\lambda\chi = \lambda^\alpha \chi_\alpha \quad , \quad \lambda\gamma_\mu\chi = \lambda^\alpha (\gamma_\mu)_\alpha{}^\beta \chi_\beta$$

$$\left.\begin{aligned} \{Q_\alpha, \tilde{Q}_\beta\} &= 2\gamma^\mu_{\alpha\beta} P_\mu + 2i\,\epsilon_{\alpha\beta} Z \\ \{Q_\alpha, Q_\beta\} &= \{\tilde{Q}_\alpha, \tilde{Q}_\beta\} = 0 \\ [Q_\alpha, P_\mu] &= [\tilde{Q}_\alpha, P_\mu] = 0 \end{aligned}\right\} \mathcal{N}=2 \text{ algebra}$$

- chiral : $\Phi = (\phi, \psi_\alpha, F)$
- vector : $V = (A_\mu, \sigma, \lambda_\alpha, \tilde{\lambda}_\alpha, D)$

$$\mathcal{L}_{sym} = \frac{1}{g^2} \mathrm{Tr} \Big(\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} + \tfrac{1}{2} (D_\mu \sigma)^2$$
$$- i \tilde{\lambda} \not{D} \lambda + i \tilde{\lambda} [\sigma, \lambda] - \tfrac{1}{2} D^2 \Big)$$

$$\mathcal{L}_{chir} = D_\mu \tilde{\phi} D^\mu \phi + \tilde{\phi} (\sigma^2 + D) \phi +$$
$$- i \tilde{\psi} \not{D} \psi - i \tilde{\psi} \sigma \psi + \sqrt{2} i (\tilde{\phi} \lambda \psi + \tilde{\psi} \tilde{\lambda} \phi)$$
$$- \tilde{F} F$$

$$\delta = \epsilon Q + \tilde{\epsilon} Q$$

$$\delta A_\mu = -i (\epsilon \gamma_\mu \lambda + \tilde{\epsilon} \gamma_\mu \lambda)$$
$$\delta \sigma = -\epsilon \tilde{\lambda} + \tilde{\epsilon} \lambda$$
$$\delta \lambda = (i D - \tfrac{1}{2} \gamma^{\mu\nu} F_{\mu\nu} - i \gamma^\mu D_\mu \sigma) \epsilon$$
$$\delta \tilde{\lambda} = (-i D - \tfrac{1}{2} \gamma^{\mu\nu} F_{\mu\nu} + i \gamma^\mu D_\mu \sigma) \tilde{\epsilon}$$

$$\delta D = \epsilon\gamma^\mu D_\mu \bar\lambda - \tilde\epsilon\gamma^\mu D_\mu \lambda - [\epsilon\bar\lambda, \sigma] + $$
$$- [\tilde\epsilon\lambda, \sigma]$$

$\mathcal{L}_v$

$$\delta^2 \sigma \sim i\tilde\epsilon\gamma^\mu\epsilon D_\mu\sigma \sim i v^\mu \partial_\mu \sigma + \sigma\wedge\sigma$$

$$\Lambda \sim i_v A + \epsilon\tilde\epsilon\sigma$$

$$\mathcal{L}_{cs} = \frac{ik}{4\pi} \mathrm{Tr}\left(\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho\right)$$

- topological $\quad \mathcal{L} \sim A\wedge dA + \frac{2i}{3}A^3$

- not gauge inv.

$$e^{iS_{cs}} \rightarrow e^{ik\mathbb{Z}} \qquad k\in\mathbb{Z}$$

$\delta\mathcal{L}_{cs}$

$$\mathcal{L}_{scs} = \frac{ik}{4\pi}\mathrm{Tr}\left(\epsilon^{\mu\nu\rho}(A_\mu\partial_\nu A_\rho + \frac{2i}{3}A_\mu A_\nu A_\rho) + 2iD\sigma + 2\bar\lambda\lambda\right)$$

S^3

$$e^\mu_a e^\nu_b g_{\mu\nu} = \eta_{ab}$$

$$e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}$$

$$\nabla_\mu e_j = e_i (\omega_\mu)^i_j$$

$$\underline{de^a + \omega^a_b \wedge e^b = 0}$$

$$\nabla_\mu \psi = \left(\partial_\mu + \tfrac{i}{8} \omega_{\mu i j} [\gamma^i, \gamma^j] \right) \psi$$

$$\gamma_\mu = e^a_\mu \gamma_a \qquad \nabla_\mu \gamma_\nu = 0$$

$$S^3 \rightarrow \theta, \varphi, \tau \qquad \theta \in [0,\pi] \qquad \varphi, \tau \in [0, 2\pi)$$

toroidal coordinates

$$ds^2 = d\theta^2 + \sin^2\theta \, d\varphi^2 + \cos^2\theta \, d\tau^2$$

left-invariant frame e^a_μ

$$\omega_{ab,\mu} = -\tfrac{1}{r} \epsilon_{abc} e^c_\mu \Rightarrow \nabla_\mu = \partial_\mu + \tfrac{i}{2r} \gamma_\mu$$

$[A] = [\sigma] = 1$ $\quad [Q] = 1/2 \quad [\delta] = 0$

$[\lambda] = [\tilde{\lambda}] = 3/2$ $\quad [\epsilon] = [\tilde{\epsilon}] = -1/2$

$[D] = 2$ $\quad [\nabla_\mu \epsilon] = 1/2$

$$\delta A_\mu = -i(\epsilon \gamma_\mu \tilde{\lambda} + \tilde{\epsilon} \gamma_\mu \lambda)$$

$$\delta \sigma = \delta \sigma$$

$$\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$$

$$\delta \lambda = (iD - 1/2\, \gamma^{\mu\nu} F_{\mu\nu} - i \cancel{D} \sigma) \epsilon +$$

$$+ a_1 \sigma \gamma^\mu \nabla_\mu \epsilon$$

$$\delta \tilde{\lambda} = \delta \tilde{\lambda}_{flat} + a_2 \sigma \gamma^\mu \nabla_\mu \tilde{\epsilon}$$

$$\delta D = \delta D_{flat} + a_3 (\nabla_\mu \epsilon \gamma^\mu \lambda - \nabla_\mu \tilde{\epsilon} \gamma^\mu \lambda)$$

$$\{\delta_\epsilon, \delta_{\tilde{\epsilon}}\} \Phi = \text{reasonable}$$

$$\{\delta_\epsilon, \delta_{\tilde\epsilon}\} A_\mu = i \mathcal{L}_v A_\mu + D_\mu(\Lambda)$$

$$a_1 = -a_2 = -2i/3$$

$$\{\delta_\epsilon, \delta_{\tilde\epsilon}\} \lambda, \tilde\lambda \rightarrow a_3 = 1/3$$

$$\{\delta_\epsilon, \delta_{\tilde\epsilon}\} D = i \mathcal{L}_v D - i[\Lambda, D]$$
$$-2\rho D + \tfrac{1}{3}\sigma\big(\epsilon \gamma^\rho \gamma^\mu D_\rho D_\mu \tilde\epsilon +$$
$$- \tilde\epsilon \gamma^\rho \gamma^\mu D_\rho D_\mu \epsilon\big)$$

$$\begin{cases} \nabla_\mu \epsilon = \dfrac{i}{2l} \gamma_\mu \epsilon \\ \nabla_\mu \tilde\epsilon = \dfrac{i}{2l} \gamma_\mu \tilde\epsilon \end{cases}$$

Killing spinor eq.

$$\rho = \frac{i}{3}\left(\nabla_\mu \tilde\epsilon \gamma^\mu \epsilon + \tilde\epsilon \gamma^\mu \nabla_\mu \epsilon\right) = 0$$

$$S^2 = U(1)$$

$$v^\mu = \tilde\epsilon \gamma^\mu \epsilon \qquad \nabla_{(\mu} v_{\nu)} = 0$$

$$\delta \mathcal{L} = \bar{\lambda}\lambda/\ell \qquad \delta \mathcal{L} = \sigma^2/\ell^2 \qquad \delta \mathcal{L} = \frac{D\sigma}{\ell}$$

$$\mathcal{L}_{SYM} = \frac{1}{g^2} \mathrm{Tr}\Big(\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2} D_\mu \sigma^2$$
$$- \tfrac{1}{2}\left(D + \tfrac{i}{\ell}\sigma\right)^2 - i\bar{\lambda} \not{D} \lambda +$$
$$- \tfrac{1}{2\ell}\bar{\lambda}\lambda + i\tilde{\lambda}[\sigma, \lambda]\Big)$$

$$\mathcal{L}_{SCS} = \frac{ik}{4\pi} \mathrm{Tr}\Big(\epsilon^{\mu\nu\rho}\left(A_\mu \partial_\nu A_\rho + \tfrac{2i}{3} A_\mu A_\nu A_\rho\right)$$
$$+ 2D\sigma + 2\lambda\bar{\lambda} \Big)$$

Φ has R-charge r

$$\mathcal{L}_{MAT} = D_\mu \bar{\Phi} D^\mu \phi - i\bar{\psi}\not{D}\psi - \bar{F}F$$
$$+ \bar{\Phi}\left(\sigma^2 + i(2r-1)/\ell\, \sigma + D - \frac{r(2-r)}{\ell^2}\right)\phi$$
$$- i\tilde{\psi}\left(\sigma + \tfrac{i}{\ell}\left(r - \tfrac{1}{2}\right)\right)\psi + i\sqrt{2}\left(\bar{\Phi}\lambda\psi + \tilde{\psi}\tilde{\lambda}\phi\right)$$

$$\delta\phi = \sqrt{2}\,\epsilon\psi$$

$$\delta\psi = \sqrt{2}\,\epsilon F - \sqrt{2}\,i\,\gamma^\mu\tilde{\epsilon}\,D_\mu\phi +$$
$$+ \sqrt{2}\,i\,\epsilon\,\sigma\,\tilde{\epsilon} - \sqrt{2}\,r/\ell\,\tilde{\epsilon}\,\phi$$

$$\delta F = -\sqrt{2}\,i\,\tilde{\epsilon}\,\gamma^\mu D_\mu\psi - i\sqrt{2}\,\tilde{\epsilon}\,\sigma\psi + 2i\,\tilde{\epsilon}\,\tilde{\lambda}\phi$$
$$+ \sqrt{2}/\ell\,(r - 1/2)\,\tilde{\epsilon}\,\psi$$

$$[\delta_\epsilon, \delta_{\tilde{\epsilon}}]\,\phi = -2i\left(\mathcal{L}_v + \tilde{\epsilon}\epsilon\,\frac{ir}{\ell}\right)\phi$$

R-multiplet $\sim$ $\mathcal{T} = (T_{\mu\nu}, J_\mu^{(R)}, \ldots)$

$$\mathcal{H} = (g_{\mu\nu}, A_\mu^{(R)}, \psi_{\mu\alpha}, \ldots)$$

$$(\nabla_\mu - i A_\mu^{(R)})\,\epsilon = -\tfrac{1}{2} H\gamma_\mu\epsilon + \cdots$$

$$(\nabla_\mu + i A_\mu^{(R)})\,\tilde{\epsilon} = -\tfrac{1}{2} H\gamma_\mu\tilde{\epsilon} + \cdots$$

$$A_\mu^{(R)} = 0 \qquad H = \frac{-i}{\ell} \longrightarrow \text{KS eq}$$

$$\{\delta_\epsilon, \delta_{\tilde\epsilon}\} \sim \mathcal{L}_v + \epsilon\tilde\epsilon\, rH$$

Localization

$$\nabla_\mu \epsilon = \frac{i}{2\ell}\gamma_\mu \epsilon = \left(\partial_\mu + \frac{i}{2\ell}\gamma_\mu\right)\epsilon$$

$$\partial_\mu \epsilon = 0$$

$$\epsilon = \begin{pmatrix}1\\0\end{pmatrix} \qquad \tilde\epsilon = \begin{pmatrix}0\\1\end{pmatrix}$$

$$v = \tilde\epsilon\gamma^\mu\epsilon\,\partial_\mu \propto \partial_\tau + \partial_\varphi$$

$$\delta^2 \sim \mathcal{L}_v + ir + G_\Lambda \qquad \Lambda = i_v A + \epsilon\tilde\epsilon\sigma$$

$$S_{SYM} = \frac{1}{2}\delta^2 \int d^3x \sqrt{g}\, \mathrm{Tr}\left(\frac{1}{2}\bar\lambda\lambda + i\sigma D\right)$$

$$S_{MAT} = 1/2\, \delta^2 \int d^3x \sqrt{g}\, \left(\tilde\psi\psi - 2i\tilde\phi\sigma\phi + \frac{2(r-1)}{r}\tilde\phi\phi\right)$$

Localizing terms S_{SYM}, S_{MAT}

- gauge cond. $\nabla_\mu A^\mu = 0$

$$S_{g} = \int d^3x \sqrt{g}\ \mathrm{Tr}\left(\bar{c}\,\nabla_\mu D^\mu c + b\,\nabla^\mu A_\mu\right)$$

$$\delta' = \delta + \delta_{BRST}$$

$$W' = W_{SYM} + \bar{c}\,\nabla_\mu A^\mu$$

$$\delta' W' = S_{SYM} + S_{gh}$$

$$\bar{c}\,\nabla_\mu \delta A_\mu \sim \bar{c}\,\lambda + \bar{c}\,\tilde{\lambda}$$

$$S_{SYM} \geqslant 0 \quad \Longleftarrow \quad \text{“}A_\mu, \sigma \in \mathbb{R}\text{”} \quad \tilde{\lambda} = \lambda^\dagger$$

$$S_{SYM} \sim F_{\mu\nu}^2 + D_\mu \sigma^2 + (D + i\sigma/\ell)^2$$

Locus
$$\begin{cases} F_{\mu\nu}=0 & A_\mu = 0 \\ D_\mu G = 0 \to \partial_\mu G = 0 & G = \text{Const} = \frac{G_0}{e} \\ D = -iG/e & D = -i\frac{G_0}{e^2} \end{cases}$$

$$i\tilde\lambda \not{D}\lambda + \tilde\lambda\lambda \to \begin{cases} \tilde\lambda = 0 \\ \lambda = 0 \end{cases}$$

$$\Phi = \Phi_0 + 1/\sqrt{t}\, \Phi'$$

$$tS'_{SYM} = \int d^3x \sqrt{g}\, \Big(F'^{\mu\nu}F'_{\mu\nu} + \tfrac{1}{2}(\partial_\mu G')^2$$
$$-\tfrac{1}{2e}[A'_\mu, G_0]^2 - \tfrac{1}{2}(D' + i/e\, G')^2 +$$
$$-i\tilde\lambda' \not{\partial}\lambda - \tfrac{1}{2e}\tilde\lambda'\lambda + i/e\, \tilde\lambda'[G_0, \lambda']\Big)$$
$$+ O(1/t) \qquad F'^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

(1) $A'^\mu = B_\mu + \partial_\mu \varphi \qquad \nabla_\mu B^\mu = 0$

② $G_0 \in \mathrm{Lie}(G) \longrightarrow G_0 \in$ Cartan

g.r. $\downarrow$

determinant

③ $F = F_0 + \sum_{\alpha \in \text{roots}} F^\alpha X_\alpha$

$\uparrow$

Π

$\mathrm{Tr}(X_\alpha X_\beta) = \delta_{\alpha+\beta,0}$

$[G_0, F_0] = 0 \qquad [X_\alpha, G_0] = \alpha(G_0) X_\alpha$

$\int D[A'_\mu] \to \int D B_\mu D\varphi$

$$\int \cancel{D\varphi}\, \cancel{Db}\, \cancel{Dc}\, \cancel{D\bar{c}}\, \cancel{DG'}\, \cancel{DD}\; e^{-S_{gh} + S_{G;D,\varphi}} =$$

$$\delta(\nabla_\mu A'^\mu) = \delta(\nabla^2 \varphi) = \frac{\delta(\varphi)}{\sqrt{\det \nabla^2}}$$

$$= \frac{\det \nabla^2}{\sqrt{\det \nabla^2}\sqrt{\det \nabla^2}}$$

$$\alpha_0 \equiv \alpha(G_0)$$

$$S = \int d^3x \sqrt{g} \sum_{\alpha \in \text{roots}} \Big\{ \tfrac{1}{2} B^m_{-\alpha} \big(-\nabla^2 + \tfrac{1}{\ell^2}\alpha_0^2\big) B^m_\alpha$$

$$+ \tilde{\lambda}'_{-\alpha} \big(-i \not\nabla + i/\ell \, \alpha_0 - 1/2\ell\big) \lambda'_\alpha \Big\}$$

$$Z^{\text{vec}}_{1\text{-loop}} = \prod_{\alpha \in \text{roots}} \frac{\det\big(-i\not\nabla + i/\ell \, \alpha_0 - 1/2\ell\big)}{\det\big(-\nabla^2 + 1/\ell^2 \, \alpha_0^2\big)}$$

$\hookrightarrow$ div. vect on S^3

$J = 0, 1, \ldots$

MODES	eigenV.	deg.
scalars	$1/\ell^2 \, J(J+2)$	$(J+1)^2$
spinors	$\pm 1/\ell \, (J + 3/2)$	$(J+1)(J+2)$
div. vect.	$1/\ell^2 (J+2)^2$	$2(J+1)(J+3)$

$$Z^{vec}_{1\text{-loop}}(G_0) = \prod_\alpha \prod_{j=0}^{+\infty} \frac{\left((-j-2+i\alpha_0)(j+1+i\alpha_0)\right)^{(j+1)(j+2)}}{\left(\alpha_0^2 + (j+2)^2\right)^{(j+1)(j+3)}}$$

$$= \dots = \prod_\alpha \prod_j \frac{(j+i\alpha_0)^{j+1}}{(j-i\alpha_0)^{j-1}} =$$

$$= \prod_{\alpha>0} \prod_j \frac{\left[(j+i\alpha_0)(j-i\alpha_0)\right]^{j+1}}{\left[(j-i\alpha_0)(j+i\alpha_0)\right]^{j-1}} =$$

$$= \prod_{\alpha>0} \prod_j (j^2+\alpha_0^2)^2 =$$

$$\frac{\sinh \pi\alpha_0}{\pi\alpha_0}$$

$$\downarrow$$

$$= \prod_j j^4 \prod_{\alpha>0} \prod_j \left(1+\alpha_0^2/j^2\right)^2 =$$

$$Z^{vec}_{1\text{-loop}} = \prod_{\alpha>0} \left(\frac{2\sinh \pi\alpha_0}{\pi\alpha_0}\right)^2$$

$$S_{CS} = \frac{ik}{4\pi} \int d^3x \sqrt{g}\, \mathrm{Tr}\left(2i \left(\sigma_0/\ell \right) \left(\frac{\sigma_0}{i\ell^2} \right) \right) =$$

$$= \pi i k \,\mathrm{Tr}\, \sigma_0^2$$

$$\sigma_0 \in \mathfrak{t} \longrightarrow \det{}_V = \prod_{\alpha>0} (\pi \alpha_0)^2$$

$$Z_{vec} = \det{}_{Ad}\, 2\sinh \pi\sigma$$

$$\det{}_R f(\sigma_0) = \prod_{\rho \in R} f(\rho(\sigma_0))$$

$$Z_{CS}^{S^3} = \int d\sigma_0 \; e^{-ik\pi \mathrm{Tr}\sigma_0^2} \det{}_{Ad} \sinh \pi\sigma$$

$$\mathcal{L}_{chir} = \partial_\mu \tilde{\phi} \partial^\mu \phi + \frac{1}{\ell^2} \tilde{\phi} \left(\hat{\tilde{\sigma}}_0^2 + 2i(r-1) + r(2-r) \right) \phi - i \tilde{\psi} \not{\nabla} \psi$$

$$- \frac{i}{\ell} \tilde{\psi} \left(\sigma_0 + i(r - 1/2) \right) \psi - \tilde{F} F$$

$$\phi, \psi = 0 \qquad\qquad \rho_0 \equiv \rho(\sigma_0)$$

$$Z^{chiral}_{1\text{-loop}}(\sigma_0) = \prod_{\rho \in R} \frac{\det\left(-i \not{\nabla} - \frac{i}{\ell}\left(\rho_0 + i(r - 1/2)\right)\right)}{\det\left(-\nabla^2 + \frac{1}{\ell^2}\left(\rho_0^2 + 2i(r-1)\rho_0 + r(2-r)\right)\right)}$$

$$= \dots = \prod_\rho \prod_{j=1}^{+\infty} \left(\frac{j + i\rho_0 + 1 - r}{j - i\rho_0 + r - 1} \right)^j \longleftarrow \begin{matrix} r = 1 \\ \rho_0 = \alpha_0 \end{matrix}$$

$$S_b(x) = \prod_{m,n \geq 0} \frac{mb + nb^{-1} + Q/2 - ix}{mb + nb^{-1} + Q/2 + ix}$$

$$Q = b + 1/b \qquad\qquad m,n \begin{matrix} \nearrow m+n = j \\ \searrow m-n \end{matrix}$$

$$S_{b=1} = \prod_{mn} \frac{m+n+1-ix}{m+n+1+ix}$$

$$Z^{chir}_{1-loop}(\hat{\sigma}_0) = \det_R S_{b=1}(i(1-r)-\sigma) =$$

$$= \prod_{\rho \in R} S_{b=1}(i(1-r)-\rho(\sigma_0))$$

$$Z_{SCS} = \int d\sigma_0 \, e^{-ik\pi \mathrm{Tr}\sigma_0^2} \prod_{\alpha>0} (2\sinh\pi\alpha_0)^2$$

$$\prod_{(i)} Z^{chir\,(i)}(\sigma_0)$$

- $N \geq 3$

hypr : two chirals of $\Delta = 1/2$

$$Z^{1-loop}_{hyper} = S_{b=1}(u/2+\rho_0)\, S_{b=1}\left(\frac{u}{2}-\rho_0\right) =$$

$$= \frac{1}{2\cosh\pi\rho_0}$$

- $r=1 \quad R = \mathrm{Adj}$

$$Z_{chir}^{r=1,\,\mathrm{Adj}} = 1$$

$\mathcal{N}=4$ vector $= \mathcal{N}=2$ vect $\oplus$ chir $r=1$

$$Z_{vec}^{\mathcal{N}=4} = Z_{vec}^{\mathcal{N}=2}$$

$$Z = e^{-F}$$

ABJM matrix model $\qquad AdS_4/CFT_3$

ABJ(M) $\Longleftrightarrow$ M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$

$\mathcal{N}=6$ SCS

$U(N)_k \times U(N)_{-k}$

matter in bifund

superpotential

- 2 vector fields
- 2 bifund chirals
- 2 a bifund chirals
- superp. $\mathcal{Q}$-exact

$\sigma \in U(N)_k$

$\hat{\sigma} \in U(N)_{-k}$

$\sigma = \mathrm{diag}(\sigma_1 \dots \sigma_N)$

$\hat{\sigma} = \mathrm{diag}(\hat{\sigma}_1 \dots \hat{\sigma}_N)$

$$\alpha_{ij}(\sigma) = \sigma_i - \sigma_j \qquad \alpha_{ij}(\hat{\sigma}) = \hat{\sigma}_i - \hat{\sigma}_j$$

$$\rho^{N,\bar{N}}_{ij} = \sigma_i - \hat{\sigma}_j \qquad \rho^{\bar{N},N} = \hat{\sigma}_i - \sigma_j$$

$$Z = \int d^N\sigma \, d^N\hat{\sigma} \; e^{ik\pi \sum_i (\sigma_i^2 - \hat{\sigma}_i^2)} \times$$

$$\times \frac{\prod_{i<j} (2\sinh \pi \sigma_i - \sigma_j)^2 (2\sinh \pi \hat{\sigma}_i - \hat{\sigma}_j)}{\prod_{i,j} (2\cosh \pi(\sigma_i - \hat{\sigma}_j))^2}$$

$$Z = e^{-F} \qquad F \propto N^{3/2}$$

$$R \to R + F$$