A systematic approach to understand all order soft theorems and tail memories

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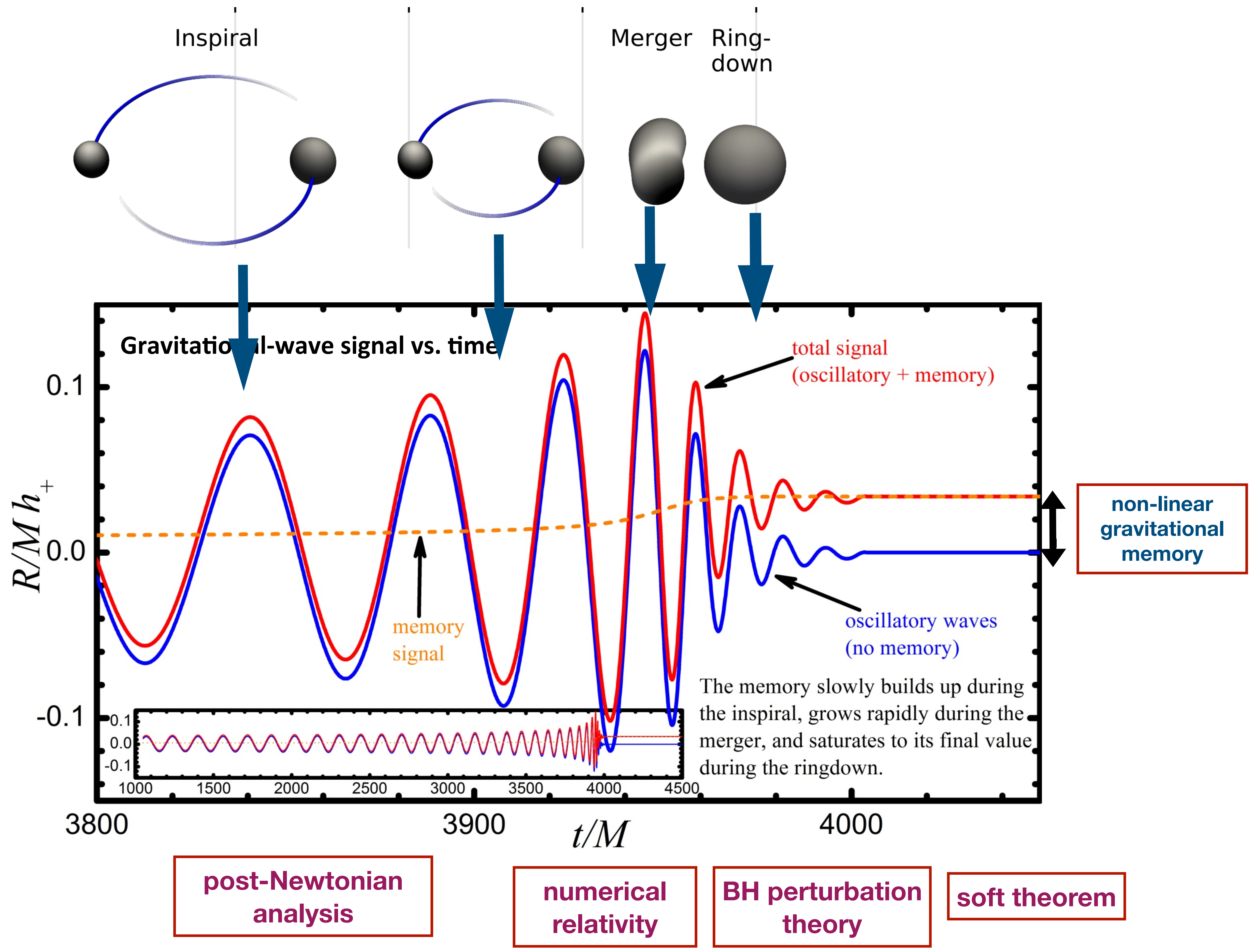
Outline

- **Soft Theorem in D>4**
- **Soft Theorem in D=4**
- **Classical limit**
- **Classical Soft Theorem**
- **Gravitational tail memory**

Theme of the workshop



All the results have analogous photon/electromagnetic contributions, which will not be discussed in this talk.

Based on the works

- arXiv: 1808.03288 with Ashoke Sen
- arXiv: 1912.06413 with Arnab Priya Saha and Ashoke Sen
- arXiv: 2008.04376
- “Spin dependent tail memory” will appear with Debodirna Ghosh.

Soft Graviton Theorem in D>4

What is Soft Graviton Theorem in terms of S-matrices?

An amplitude with arbitrary number of finite energy particles (hard particles) with arbitrary mass and spin and arbitrary number of small energy gravitons (soft gravitons) is related to the amplitude without the soft gravitons, via expansion in powers of soft momenta.

For one soft graviton:

Weinberg ; Cachazo, Strominger ; Sen ; …

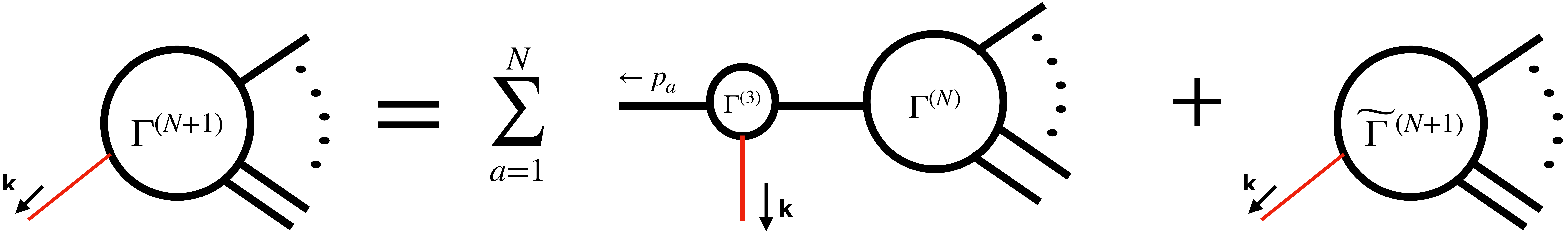
$$\Gamma^{(N+1)}(\varepsilon, k; \{p_a, \Sigma_a\}) = \Big[\ \underbrace{\mathbf{S}^{(0)}}_{\mathcal{O}\left(\frac{1}{\omega}\right)} + \underbrace{\mathbf{S}^{(1)}}_{\mathcal{O}(\omega^0)} + \underbrace{\mathbf{S}^{(2)}}_{\mathcal{O}(\omega)} + \cdots \Big]\ \Gamma^{(N)}(\{p_a, \Sigma_a\})$$

— in D=4 tree level ,
— in D>4 for loop amplitude.

- **Here (ε, k) are the polarisation and momentum of outgoing graviton.**
- **This expansion is valid only when the graviton energy $\omega = |\vec{k}|$ is small compare to the other finite energy particles' momenta $\{p_a\}$.**

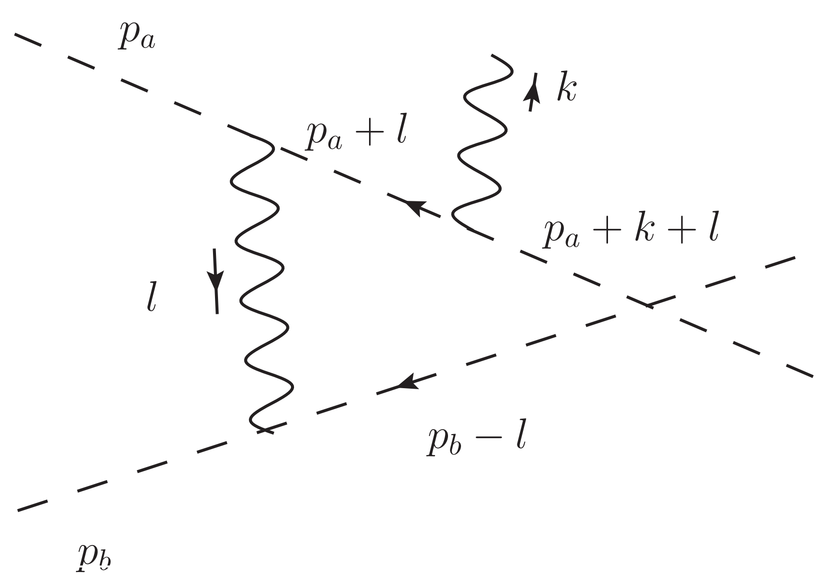
Suppose the theory is described by a general coordinate invariant one particle irreducible (1PI) effective action.

—— tree level amplitude computed from this give the full quantum result.



Above $\Gamma^{(3)}$ and $\widetilde{\Gamma}^{(N+1)}$ can be evaluated by covariantizing the 1PI action in the soft graviton background if we assume $\Gamma^{(3)}$ and $\widetilde{\Gamma}^{(N+1)}$ don't contribute soft momenta in the denominator.

Example:



—— Breaks down in D=4 when massless particle runs in loop.

$$\sim \int_{|l| \approx |k|} \frac{d^4 l}{(2\pi)^4} \frac{1}{p_a \cdot l} \frac{1}{l^2} \frac{1}{p_b \cdot l} \frac{1}{p_a \cdot (k+l)} \sim \frac{1}{|k|}$$

Single soft graviton theorem:

Weinberg ; Cachazo, Strominger ; Sen; Laddha, Sen ; …

$$\Gamma^{(N+1)}(\varepsilon, k;\ \{\epsilon_i, p_i\})$$

$$= \sum_{i=1}^{N} \epsilon_i^T \left[\underbrace{\frac{\varepsilon_{\mu\nu} p_i^\mu p_i^\nu}{p_i \cdot k}}_{S^{(0)}} + \underbrace{\frac{\varepsilon_{b\mu} p_i^\mu\, k_a}{p_i \cdot k} \left\{ p_i^b \frac{\partial}{\partial p_{ia}} - p_i^a \frac{\partial}{\partial p_{ib}} + (\Sigma_i^{ab})^T \right\}}_{S^{(1)}} \right] \Gamma^{(i)}(p_i)$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \frac{\varepsilon_{ac} k_b k_d}{p_i \cdot k}\, \epsilon_i^T \left[p_i^b \frac{\partial}{\partial p_{ia}} - p_i^a \frac{\partial}{\partial p_{ib}} + (\Sigma_i^{ab})^T \right] \left[p_i^d \frac{\partial}{\partial p_{ic}} - p_i^c \frac{\partial}{\partial p_{id}} + (\Sigma_i^{cd})^T \right] \Gamma^{(i)}(p_i)$$

$$+ \frac{1}{2} \sum_{i=1}^{N} \frac{1}{p_i \cdot k} R_{\mu\rho\nu\sigma}(k)\, \epsilon_i^T \mathcal{N}_{(i)}^{\mu\rho\nu\sigma}(-p_i)\, \Gamma^{(i)}(p_i) \qquad + \mathcal{O}(\omega^2)$$

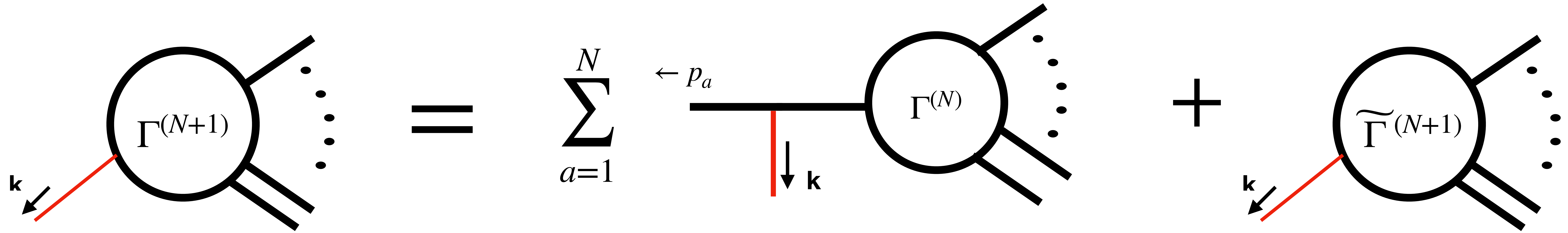
Non-universal i.e. theory dependent piece

Not known whether fully factorizable for a generic theory !!

Q. For a simple theory of scalars minimally coupled to gravity , can one expect soft factorization should hold to arbitrary order in soft momenta expansion even for tree level amplitudes?

Demanding gauge invariance what information about $(sub)^n$-leading soft factor can be extracted?

Hamada, Shiu -2018; Li, Lin, Zhang- 2018



$$\Gamma^{(N+1)} = \varepsilon^{\mu\nu}\left[\sum_a \frac{p_{a\mu}p_{a\nu}}{p_a \,.\, k}\Gamma^{(N)}(p_a + k) + \widetilde{\Gamma}^{(N+1)}_{\mu\nu}(k, \{p_a\})\right]$$

$$= \sum_a \frac{\varepsilon^{\mu\nu}p_{a\mu}p_{a\nu}}{p_a \,.\, k} + \varepsilon_{\mu\nu}\sum_{r=1}^{\infty}\left[\sum_a \frac{p_a^{\mu}p_a^{\nu}}{p_a \,.\, k}\frac{1}{r!}k_{\alpha_1}k_{\alpha_2}\cdots k_{\alpha_r}\frac{\partial}{\partial p_{a\alpha_1}}\cdots\frac{\partial}{\partial p_{a\alpha_r}}\Gamma^{(N)}(p_a) \;+\; k_{\alpha_1}\cdots k_{\alpha_{r-1}}\mathbf{R}^{(r)\mu\nu,\alpha_1\cdots\alpha_{r-1}}(\{p_a\})\right]$$

Demand gauge invariance: $k^\mu\,\Gamma^{(N+1)}_{\mu\nu} = 0 = k^\nu\,\Gamma^{(N+1)}_{\mu\nu}$

$\Rightarrow$ Up to sub-subleading order one recovers the result for minimally coupled scalar hard particles.

$\Rightarrow$ At $(sub)^n$-leading order for $n \geq 3$ one finds :

$$\Delta_{(n)}\Gamma^{(N+1)} = \frac{1}{n!}\left[\sum_{a=1}^{N} \frac{\varepsilon_{\mu\nu}k_\rho k_\sigma}{p_a\,.\,k}\left(p_a^\mu\frac{\partial}{\partial p_{a\rho}} - p_a^\rho\frac{\partial}{\partial p_{a\mu}}\right)\left(p_a^\nu\frac{\partial}{\partial p_{a\sigma}} - p_a^\sigma\frac{\partial}{\partial p_{a\nu}}\right)\left(k^\alpha\frac{\partial}{\partial p_a^\alpha}\right)^{n-2}\right]\Gamma^{(N)}$$

$$+k_{\alpha_1}\cdots k_{\alpha_{n-1}}\mathbf{R}^{(r)\mu\nu,\alpha_1\cdots\alpha_{n-1}}(\{p_a\})$$

May not be expressible as some operator operating on $\Gamma^{(N)}$

R is antisymmetric under $\mu \leftrightarrow \alpha_i$ and $\nu \leftrightarrow \alpha_i$ exchange.

Soft Graviton Theorem in D=4

D=4 : Gravitational S-matrix is infrared divergent as well as one assumption of the general covariantized prescription breaks down !

Soft theorem is the relation between two S-matrices. Even if they are individually IR divergent, can we remove same IR divergent piece from both the S-matrices? And write down soft theorem in terms of infrared finite S-matrices?

self interaction of gravitons

IR finite

$$\Gamma^{(N+1)} = exp\{-K_{phase}\}exp\{K_{gr}\}\ \Gamma_G^{(N+1)}$$

$$\Gamma^{(N)} = exp\{K_{gr}\}\ \Gamma_G^{(N)}$$

Reduced to leading Eikonal factor in IR limit

We able to separate out the IR divergent piece and the IR finite piece using Grammer-Yennie technique originally developed for QED in 1973.

After cancelling the common IR divergent pieces from both side of the soft theorem relation:

$$\Gamma_G^{(N+1)} = (S_{gr}^{(0)} + S_{gr}^{(1)})\ \Gamma_G^{(N)}$$

Strategy

Split the graviton propagator with momentum ℓ flowing from particle-a to particle-b into two parts: call K-graviton and G-graviton.

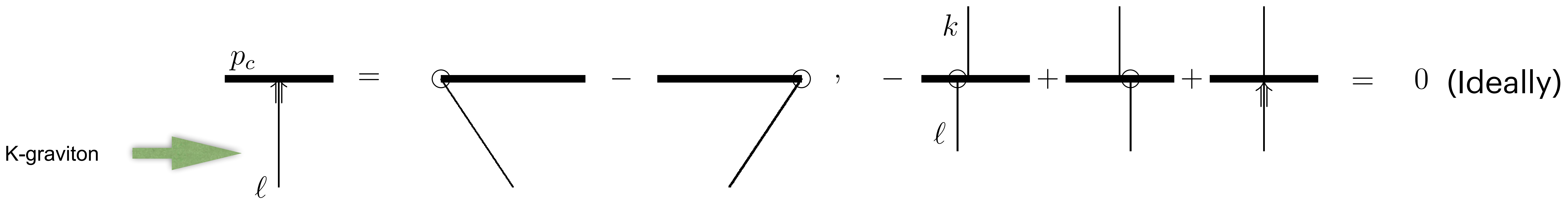
$$-\frac{i}{\ell^2 - i\epsilon}\,\frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}\right) \quad = -\frac{i}{\ell^2 - i\epsilon}\,\frac{1}{2}\left[K^{\mu\nu,\rho\sigma}_{(a,b)} + G^{\mu\nu,\rho\sigma}_{(ab)}\right]$$

such that :

⁕ Loop diagram computed with K-graviton propagator contains the full IR divergent factor

⇒ Loop diagrams with G-graviton propagators are IR finite.

⁕ For a virtual K-graviton insertion the following property should hold in off-shell :



K-G decomposition Result

$$G^{\mu\nu,\rho\sigma}_{(ab)}(\ell, p_a, p_b) \;=\; (\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}) \;-\; K^{\mu\nu,\rho\sigma}_{(ab)}(\ell, p_a, p_b)\,,$$

$$K^{\mu\nu,\rho\sigma}_{(ab)}(\ell, p_a, p_b) \;=\; \mathcal{C}(\ell, p_a, p_b)\,\left[(p_a+\ell)^{\mu}\ell^{\nu} + (p_a+\ell)^{\nu}\ell^{\mu}\right]\left[(p_b-\ell)^{\rho}\ell^{\sigma} + (p_b-\ell)^{\sigma}\ell^{\rho}\right]$$

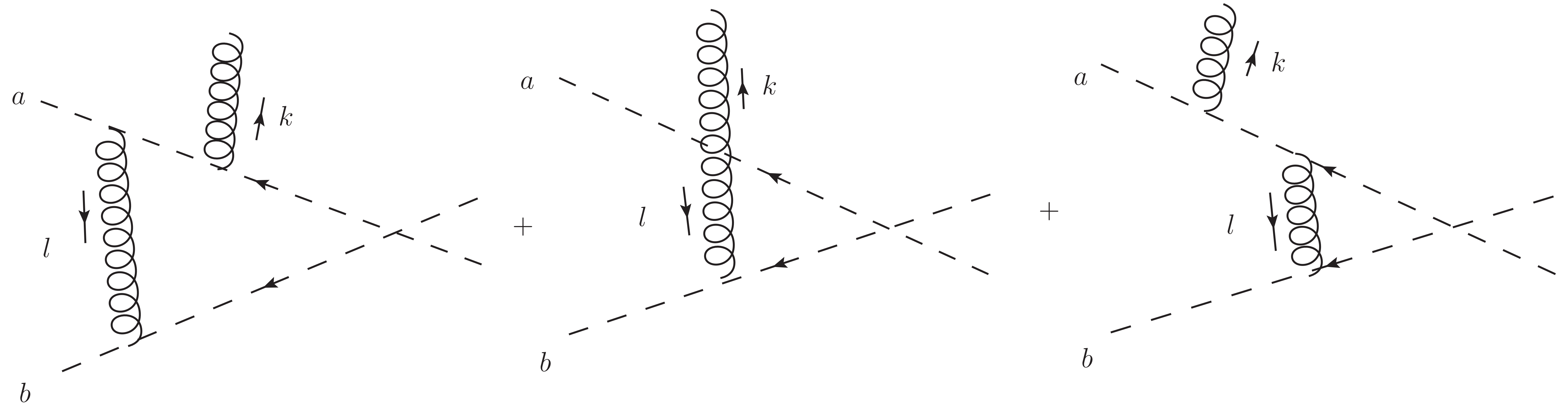
Where,

$$\mathcal{C}(\ell, p_a, p_b) \;=\; \frac{(-1)}{\{p_a.(p_a+\ell) - i\epsilon\}\,\{p_b.(p_b-\ell) - i\epsilon\}\{\,\ell.(\ell+2p_a) - i\epsilon\}\,\{\ell.(\ell-2p_b) - i\epsilon\}} \times \left[2(p_a.p_b)^2 - p_a^2 p_b^2 - \ell^2(p_a.p_b) - 2(p_a.p_b)(p_a.\ell) + 2(p_a.p_b)(p_b.\ell)\right].$$

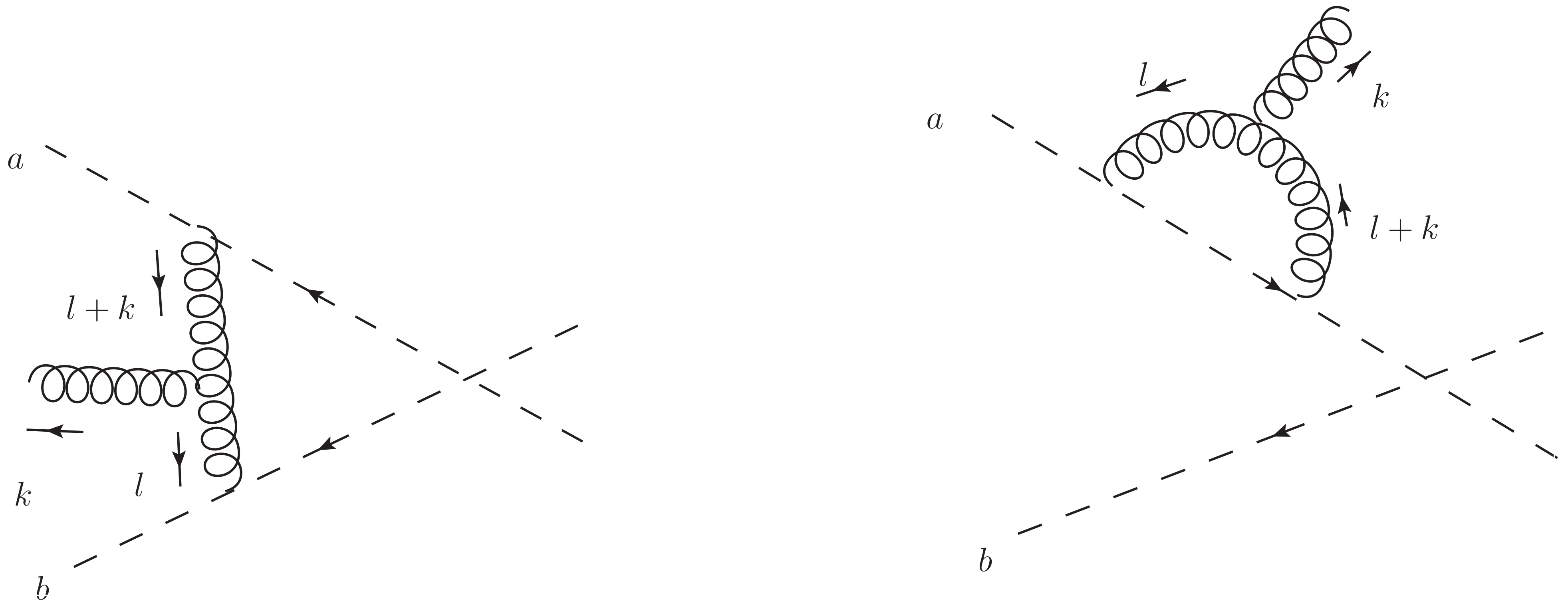
⁕ Disclaimer: The full exponentialization of the Eikonal factor analysing all loop orders has not been proved yet rigorously with this construction, which existed for QED case.

⁕ We used this prescription at one loop order and derived sub-leading soft graviton theorem which turns out to be at order $\ln\omega$ in small ω expansion.

For scalar coupled to gravity at one loop order we need to analyze the following set of diagrams:



⇒ here internal propagators are G-gravitons, so are IR finite.



⇒ here internal propagators are full-graviton propagators, so contains IR divergent piece, needs to regulate.

+ self-energy kind of diagrams

All the diagrams contribute to $\ln \omega$ in the integration region where loop momentum is larger than ω. In the integration region where loop momentum is smaller than ω the last two kind of diagrams contribute to $\ln \omega$ also.

Coefficient of $\ln\omega$ in subleading soft graviton theorem

$c=1\,,\,8\pi G=1$

$$S^{(ln)}\big(\varepsilon,k;\{p_a\}\big)=$$

$$-\frac{i}{8\pi}\sum_a\frac{\varepsilon_{\mu\nu}p_a^\nu k_\rho}{p_a.k}\sum_{\substack{b\neq a\\ \eta_a\eta_b=1}}\frac{p_b.p_a}{\{(p_b.p_a)^2-m_a^2m_b^2\}^{3/2}}\left(p_b^\rho p_a^\mu-p_b^\mu p_a^\rho\right)\left\{2(p_b.p_a)^2-3m_a^2m_b^2\right\}$$

$$-\frac{i}{4\pi}\sum_{b,\,\eta_b=1}p_b\cdot k\sum_a\frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{p_a.k}$$

$$-\frac{1}{8\pi^2}\sum_a\frac{\varepsilon_{\mu\rho}p_a^\rho k_\nu}{p_a.k}\left\{p_a^\mu\frac{\partial}{\partial p_{a\nu}}-p_a^\nu\frac{\partial}{\partial p_{a\mu}}\right\}\sum_{b\neq a}\frac{\{(p_a.p_b)^2-\frac{1}{2}p_a^2p_b^2\}}{\sqrt{(p_a.p_b)^2-p_a^2p_b^2}}\ln\left(\frac{p_a.p_b+\sqrt{(p_a.p_b)^2-p_a^2p_b^2}}{p_a.p_b-\sqrt{(p_a.p_b)^2-p_a^2p_b^2}}\right)$$

$$-\frac{1}{8\pi^2}\sum_a\frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{p_a.k}\sum_b p_b.k\,\ln\frac{m_b^2}{(p_b.\hat{k})^2}$$

where $\eta_a=+1$ if particle-a is outgoing and $\eta_a=-1$ if particle-a is ingoing.

Re-writing the subleading soft graviton factor:

K_{gr} approximated in the integration region $\omega << |\ell^\mu| << |p_a|$

$$S_{gr}^{(1)} = K_{phase}^{reg} \sum_{a=1}^{N} \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a \, . \, k} + \sum_{a=1}^{N} \frac{\varepsilon_{\mu\nu} p_a^\mu k_\rho}{p_a \, . \, k} \left(p_a^\nu \frac{\partial}{\partial p_{a\rho}} - p_a^\rho \frac{\partial}{\partial p_{a\nu}} \right) K_{gr}^{reg}$$

where,

$$K_{\rm gr}^{\rm reg} \equiv \frac{i}{2} \sum_{\substack{a,b \\ b \neq a}} \left\{ (p_a . p_b)^2 - \frac{1}{2} p_a^2 p_b^2 \right\} \int_{\rm reg} \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - i\epsilon} \frac{1}{(p_a . \ell - i\epsilon)(p_b . \ell + i\epsilon)}$$

$$= \frac{i}{2} \sum_{\substack{a,b \\ b \neq a}} \frac{1}{4\pi} \ln \omega^{-1} \frac{\left\{ (p_a . p_b)^2 - \frac{1}{2} p_a^2 p_b^2 \right\}}{\sqrt{(p_a . p_b)^2 - p_a^2 p_b^2}} \left\{ \delta_{\eta_a \eta_b, 1} - \frac{i}{2\pi} \ln \left(\frac{p_a . p_b + \sqrt{(p_a . p_b)^2 - p_a^2 p_b^2}}{p_a . p_b - \sqrt{(p_a . p_b)^2 - p_a^2 p_b^2}} \right) \right\}$$

and

$$K_{phase}^{reg} = -\frac{i}{4\pi} \ln(\omega) \left[\sum_{b,\, \eta_b = 1} p_b \, . \, k - \frac{i}{2\pi} \sum_{b=1}^{N} p_b \, . \, k \, \ln\left(\frac{p_b^2}{(p_b \, . \, \mathbf{n})^2} \right) \right]$$

Key Observation

If we naively assume the validity of D>4 soft theorem for D=4 as well and keep the loop momentum in the regulated range of integration. We find :

$$\Gamma^{(N+1)} = \left[\sum_{a=1}^{N} \frac{\varepsilon_{\mu\nu} p_a^{\mu} p_a^{\nu}}{p_a . k} + \sum_{a=1}^{N} \frac{\varepsilon_{\mu\nu} p_a^{\mu} k_{\rho}}{p_a . k} \left(p_a^{\nu} \frac{\partial}{\partial p_{a\rho}} - p_a^{\rho} \frac{\partial}{\partial p_{a\nu}} \right) \right] \Gamma^{(N)}$$

In the regulated range of integration :

$$\Gamma^{(N+1)} = exp\{-K_{phase}^{reg}\} exp\{K_{gr}^{reg}\}\ \Gamma_G^{(N+1)} \qquad \Gamma^{(N)} = exp\{K_{gr}^{reg}\}\ \Gamma_G^{(N)}$$

$\mathcal{O}(\omega \ln \omega)$ $\mathcal{O}(\ln \omega)$

If we substitute the above expressions in the naive soft theorem relation and commute through the differential operators and expand in power of ω we correctly recover the $\ln \omega$ soft factor.

Conjecture at Sub-subleading order $\mathcal{O}(\omega(\ln\omega)^2)$ term

$$
\begin{aligned}
& S_{gr}^{(2)}\left(\varepsilon, k, \{p_a\}\right) \\
= \quad & \frac{1}{2}\left\{K_{phase}^{reg}\right\}^2 \sum_{a=1}^{N} \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} \\
& + \left\{K_{phase}^{reg}\right\} \sum_{a=1}^{N} \frac{\varepsilon_{\mu\nu} p_a^\mu k_\rho}{p_a.k}\left[\left(p_a^\nu \frac{\partial}{\partial p_{a\rho}} - p_a^\rho \frac{\partial}{\partial p_{a\nu}}\right) K_{gr}^{reg}\right] \\
& + \frac{1}{2} \sum_{a=1}^{N} \frac{\varepsilon_{\mu\nu} k_\rho k_\sigma}{p_a.k} \times \left[\left(p_a^\mu \frac{\partial}{\partial p_{a\rho}} - p_a^\rho \frac{\partial}{\partial p_{a\mu}}\right) K_{gr}^{reg}\right]\left[\left(p_a^\nu \frac{\partial}{\partial p_{a\sigma}} - p_a^\sigma \frac{\partial}{\partial p_{a\nu}}\right) K_{gr}^{reg}\right] \\
& + \mathcal{O}(\omega \ln\omega)
\end{aligned}
$$

This can be proved analysing two loop amplitudes using the same prescription developed for one loop.

Generalising this idea to $(sub)^n-$leading order

$$
\begin{aligned}
S_{gr}^{(n)} =\ & \frac{1}{n!}\left\{K_{phase}^{reg}\right\}^{n} \sum_{a=1}^{N} \frac{\varepsilon_{\mu\nu} p_a^{\mu} p_a^{\nu}}{p_a \cdot k} \\
& + \frac{1}{(n-1)!}\left\{K_{phase}^{reg}\right\}^{n-1} \sum_{a=1}^{N} \frac{\varepsilon_{\mu\nu} p_a^{\mu} k_{\rho}}{p_a \cdot k}\left[\left(p_a^{\nu}\frac{\partial}{\partial p_{a\rho}} - p_a^{\rho}\frac{\partial}{\partial p_{a\nu}}\right) K_{gr}^{reg}\right] \\
& + \sum_{r=2}^{n} \frac{1}{(n-r)!}\left\{K_{phase}^{reg}\right\}^{n-r} \sum_{a=1}^{N} \frac{\varepsilon_{\mu\nu} k_{\rho} k_{\sigma}}{p_a \cdot k}\left[\left(p_a^{\mu}\frac{\partial}{\partial p_{a\rho}} - p_a^{\rho}\frac{\partial}{\partial p_{a\mu}}\right) K_{gr}^{reg}\right]\left[\left(p_a^{\nu}\frac{\partial}{\partial p_{a\sigma}} - p_a^{\sigma}\frac{\partial}{\partial p_{a\nu}}\right) K_{gr}^{reg}\right] \frac{1}{r!}\left[k^{\alpha}\frac{\partial}{\partial p_a^{\alpha}} K_{gr}^{reg}\right]^{r-2} \\
& + \sum_{r=3}^{n} \frac{1}{(n-r)!}\left\{K_{phase}^{reg}\right\}^{n-r} (\ln\omega)^{r} \sum_{a=1}^{N} \varepsilon_{\mu\nu} k_{\alpha_1} k_{\alpha_2} \cdots k_{\alpha_{r-1}} \mathbf{R}^{(r)\mu\nu,\alpha_1\cdots\alpha_{n-1}}(\{p_a\}) \\
& + \mathcal{O}(\omega^{n-1}(\ln\omega)^{n-1})
\end{aligned}
$$

We expect this part should also be universal and only depends on the momenta of scattered objects.

To derive this result and fix $R^{(n)}$, we have to analyze n-loop amplitude.

Soft expansion in $\omega \to 0$ limit →

Leading non-analytic term (first term column)

	Leading non-analytic term					
0-loop (tree)	ω^{-1}	ω^{0}	ω	$\dots$		
1-loop	$\ln\omega$	ω^{0}	$\omega\ln\omega$ Spin dependent	ω	$\omega^{2}\ln\omega$	$\dots$
2-loop	$\omega(\ln\omega)^{2}$	$\omega\ln\omega$	ω	$\omega^{2}(\ln\omega)^{2}$	$\omega^{2}\ln\omega$	$\dots$
n-loop	$\omega^{n-1}(\ln\omega)^{n}$	$\omega^{n-1}(\ln\omega)^{n-1}$	$\dots$			

Exact at that loop order, does not receive contribution from higher loops. Fixed from minimal coupling.

Classical Limit of Soft Theorem

CLASSICAL LIMIT OF SOFT THEOREM IN D>4

Laddha , Sen

Large number of soft gravitons need to emit to produce classical radiation

↓

Energy of scattered objects need to be large in unit of M_{pl}

⇒replace orbital angular momenta by classical angular momenta

Total radiation energy has to be less than energy of each finite energy particles

↓

Large impact parameter or probe scatterer limit

⇒drop the contact terms in multiple soft factor.

Though later we shall show this can be relaxed if we consider flux of finite energy gravitational radiation as hard particle.

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With radiation wavelength larger than characteristic length scale

up to a undetermined phase:

$$\varepsilon^{\alpha\beta}\widetilde{e}_{\alpha\beta}(\omega, \vec{x}) = \mathcal{N} S_{gr}(\varepsilon, \omega\hat{x})$$

$$\mathcal{N} = \left(\frac{\omega}{2\pi i|\vec{x}|}\right)^{\frac{D-2}{2}} \frac{1}{2\omega}\, e^{i\omega|\vec{x}|}$$

Let us naively assume the result of classical limit of the soft theorem is also valid in D=4.

$$\varepsilon^{\alpha\beta}(k) \int_{-\infty}^{\infty} dt\; e^{i\omega t}\; e_{\alpha\beta}(t, \overrightarrow{x}) \equiv \varepsilon^{\alpha\beta}\widetilde{e}_{\alpha\beta}(\omega, \overrightarrow{x}) \simeq \frac{1}{4\pi i R} e^{i\omega R + i\psi}\; S_{gr}(\varepsilon, k)$$

where $$S_{\rm gr} = \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a . k} + i \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho \mathbf{J}_a^{\rho\mu}}{p_a . k}$$ classical angular momenta

where

ψ is some undetermined phase,

$e_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta} h^\rho_\rho$ is the trace reversed metric fluctuation,

$\overrightarrow{x} = R\hat{n}$ with R being the distance of the detector from the scattering centre and $\hat{n}$ being the unit vector along the detector from the scattering centre,

$k^\mu = \omega(1, \hat{n})$

But in D=4 the trajectory of the particle-a takes form: $r_a^\mu(\sigma) = \eta_a \frac{1}{m_a} p_a^\mu \sigma + c_a^\mu \ln|\sigma| + \cdots$

Effect of long range interaction

In large $|\sigma|$ the classical angular momentum diverges:

$$\mathbf{J}_a^{\mu\nu} \simeq r_a^\mu(\sigma) p_a^\nu - r_a^\nu(\sigma) p_a^\mu + \text{spin} = (c_a^\mu p_a^\nu - c_a^\nu p_a^\mu) \ln|\sigma| + \cdots$$

So naive substitution of classical angular momenta makes subleading soft factor divergent !

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So naive substitution of classical angular momenta makes subleading soft factor divergent !

- But we expect the radiative mode of low frequency gravitational waveform should be finite from physical ground.

- And physically if we are interested to determine gravitational waveform with frequency ω , then we expect the cut off for $|\sigma|$ to be in order of ω^{-1} .

- Also the emitted graviton's trajectory will receive logarithmic correction due to long range force of scattered object (back scattering effect).

Prescription: Find corrected trajectory by solving geodesic equation and then replace $\ln|\sigma| \to \ln \omega^{-1}$ in the classical soft factor.

$$c_a^\alpha = \eta_a \frac{1}{4\pi} \sum_{\substack{b\neq a \\ \eta_a\eta_b=1}} m_b \frac{1}{\{(V_b.V_a)^2 - 1\}^{3/2}} \left\{-\frac{1}{2}V_a^\alpha + \frac{1}{2}V_b^\alpha\left(2(V_b.V_a)^3 - 3V_b.V_a\right)\right\}$$

Observation:

$$ic_a^\mu \ln \omega^{-1} = \frac{\partial K_{gr}^{cl}}{\partial p_{a\mu}}$$

From now on we are restoring $(8\pi G)$ factors

$$K_{gr}^{cl} = -\frac{i}{2}(8\pi G)\sum_{\substack{b,c \\ c\neq b}} \int_\omega^{L^{-1}} \frac{d^4\ell}{(2\pi)^4} G_r(\ell) \frac{1}{p_b.\ell + i\epsilon} \frac{1}{p_c.\ell - i\epsilon} \left\{(p_b.p_c)^2 - \frac{1}{2}p_b^2 p_c^2\right\}$$

$$= -\frac{i}{2}(2G) \sum_{\substack{b,c \\ c\neq b \\ \eta_b\eta_c=1}} \ln\left\{L(\omega + i\epsilon\eta_b)\right\} \frac{(p_b.p_c)^2 - \frac{1}{2}p_b^2 p_c^2}{\sqrt{(p_b.p_c)^2 - p_b^2 p_c^2}}$$

* We get K_{gr}^{cl} from K_{gr}^{reg} if we replace Feynman propagator by Retarded propagator $G_r(\ell)$ for graviton.

Emitted soft graviton's trajectory will also receives logarithmic correction due to long range gravitational force by other scattered objects, which generate time delay.

So if we combine this time delay along with backscattering affect, the gravitational waveform at $\mathcal{O}(\ln \omega)$ turns out:

$$\Delta\tilde{e}^{\mu\nu}(\omega, \vec{x} = R\hat{n}) = (-i)\frac{2G}{R}\, exp\Big\{i\omega R - 2iG\ln R \sum_{b,\eta_b=1} p_b \,.\, k\Big\}$$

$$\times \left[K^{cl}_{phase}\sum_{a=1}^{N}\frac{p_a^\mu p_a^\nu}{p_a \,.\, k} + \sum_{a=1}^{N}\frac{p_a^\mu k_\rho}{p_a \,.\, k}\left(p_a^\nu \frac{\partial}{\partial p_{a\rho}} - p_a^\rho \frac{\partial}{\partial p_{a\nu}}\right) K^{cl}_{gr}\right]$$

$$k = \omega(1,\hat{n})$$

Where, $K^{cl}_{phase} = -\,2iG\;\ln\omega \sum_{b,\eta_b=1} p_b \,.\, k$ is gettable if we evaluate K^{reg}_{phase} replacing Feynman propagator by retarded propagator for the graviton.

Hence boldly generalising this observations, just by replacing the classical counterparts of K_{gr} and K_{phase} we can predict the higher order gravitational waveforms from higher order soft factors.

Conjecture on spin dependent gravitational waveform

Extending our observations up to sub-subleading order we find:

$$
\begin{aligned}
&\widetilde{e}^{\mu\nu}(\omega,\vec{x}) \\
&= (-i)\,\frac{2G}{R}\,e^{i\omega R}\,\exp\Big[-2iG\ln\{(\omega+i\epsilon)R\}\sum_{b=1}^{N}p_b.k\Big] \\
&\times\Bigg[\sum_{a=1}^{M+N}\frac{p_a^{\mu}p_a^{\nu}}{p_a.k}+\sum_{a=1}^{M+N}\frac{p_a^{(\mu}k_\rho}{p_a.k}\Bigg\{\Bigg(p_a^{\nu)}\frac{\partial}{\partial p_{a\rho}}-p_a^{\rho}\frac{\partial}{\partial p_{a\nu)}}\Bigg)K_{gr}^{cl}-i\Big(r_a^{\rho}p_a^{\nu)}-r_a^{\nu)}p_a^{\rho}+\Sigma_a^{\rho\nu)}\Big)\Bigg\} \\
&+\frac{1}{2}\sum_{a=1}^{M+N}\frac{k_\rho k_\sigma}{p_a.k}\Bigg\{\Bigg(p_a^{\mu}\frac{\partial}{\partial p_{a\rho}}-p_a^{\rho}\frac{\partial}{\partial p_{a\mu}}\Bigg)K_{gr}^{cl}-i\Big(r_a^{\rho}p_a^{\mu}-r_a^{\mu}p_a^{\rho}+\Sigma_a^{\rho\mu}\Big)\Bigg\} \\
&\times\Bigg\{\Bigg(p_a^{\nu}\frac{\partial}{\partial p_{a\sigma}}-p_a^{\sigma}\frac{\partial}{\partial p_{a\nu}}\Bigg)K_{gr}^{cl}-i\Big(r_a^{\sigma}p_a^{\nu}-r_a^{\nu}p_a^{\sigma}+\Sigma_a^{\sigma\nu}\Big)\Bigg\}\Bigg]
\end{aligned}
$$

Expanding the above expression in $\omega\to 0$ limit, we found the order $\omega\ln\omega$ waveform at order $\mathcal{O}(G^2)$ which depends on spin of scattered objects.

$$\Delta^{(\omega\ln\omega)}_{(G^2)}\,\widetilde{e}^{\mu\nu}(\omega,\vec{x}) = (-i)\,\frac{2G}{R}\,\exp\left\{i\omega R - 2iG\ln R\sum_{b=1}^{N}p_b.k\right\}\Bigg[-2G\ln\{\omega+i\epsilon\}\sum_{b=1}^{N}p_b.k$$

$$\times\sum_{a=1}^{M+N}\frac{p_a^{(\mu}k_\rho}{p_a.k}\left(r_a^\rho p_a^{\nu)} - r_a^{\nu)}p_a^\rho + \Sigma_a^{\rho\nu)}\right)$$

$$-\frac{i}{2}\sum_{a=1}^{M+N}\frac{k_\rho k_\sigma}{p_a.k}\left\{\left(p_a^\mu\frac{\partial}{\partial p_{a\rho}} - p_a^\rho\frac{\partial}{\partial p_{a\mu}}\right)K_{gr}^{cl}\times\left(r_a^\sigma p_a^\nu - r_a^\nu p_a^\sigma + \Sigma_a^{\sigma\nu}\right)\right.$$

$$\left.+\left(p_a^\nu\frac{\partial}{\partial p_{a\sigma}} - p_a^\sigma\frac{\partial}{\partial p_{a\nu}}\right)K_{gr}^{cl}\times\left(r_a^\rho p_a^\mu - r_a^\mu p_a^\rho + \Sigma_a^{\rho\mu}\right)\right\}\Bigg]$$

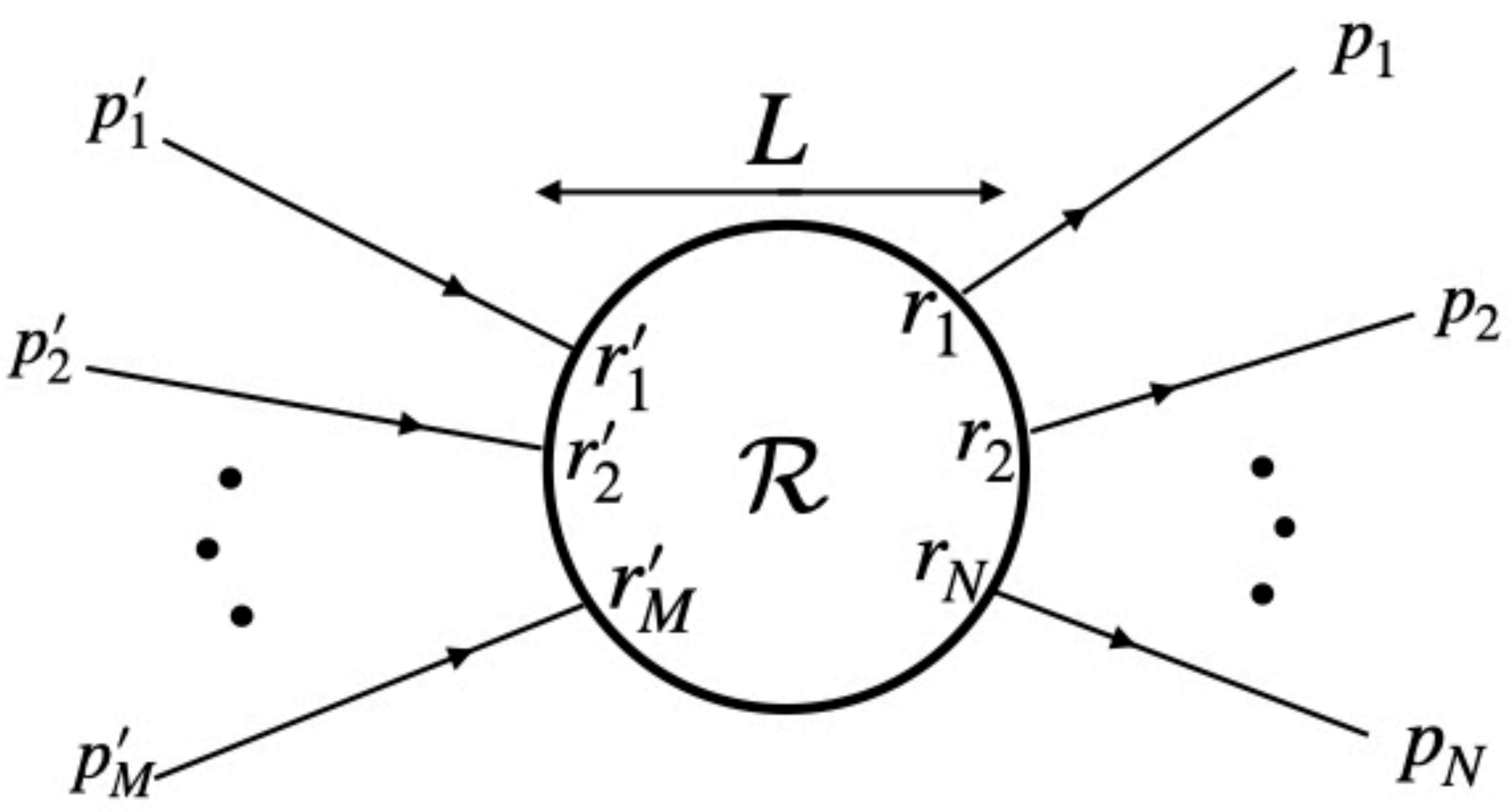
But this is not the full $\mathcal{O}(\omega\ln\omega)$ contribution, it receives correction at order G^3.

A systematic study

of

Classical Soft Graviton Theorem

Set up: M number of objects coming in, undergoes complicated interaction within the region $\mathscr{R}$ and disperse to N number of final objects.



Region $\mathscr{R}$ is chosen to be sufficiently large so that all non-trivial interactions take place inside region $\mathscr{R}$ and outside only long-range gravitational interaction exists.

Goal: Determine gravitational waveform in retarded time u for $|u| >> L$.

$\Rightarrow$ Determine gravitational waveform with frequency ω for $\omega << L^{-1}$.

} Related by Fourier transform

Define deviation of the metric from Minkowski metric as,

$$h_{\mu\nu} = \frac{1}{2}(g_{\mu\nu} - \eta_{\mu\nu}), \quad e_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\,\eta^{\rho\sigma}h_{\rho\sigma}$$

Linearised Einstein equation in de Donder gauge:

$$\eta^{\alpha\mu}\,\eta^{\beta\nu}\,\eta^{\rho\sigma}\partial_\rho\partial_\sigma e_{\alpha\beta} = -8\,\pi\,G\,T^{\mu\nu}(x), \qquad T^{\mu\nu} \equiv T^{X\mu\nu} + T^{h\mu\nu}$$

— where $T^{h\mu\nu}$ is the gravitational energy momentum tensor defined as:

$$T^{h\mu\nu} = \frac{1}{8\pi G}\left[-\sqrt{-g}\left\{R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}\right\} - \eta^{\alpha\mu}\eta^{\beta\nu}\eta^{\rho\sigma}\partial_\rho\partial_\sigma e_{\alpha\beta}\right]$$

Radiative mode of gravitational waveform:

$$\widetilde{e}^{\mu\nu}(\omega, \vec{x} = R\hat{n}) = \int dx^0 e^{i\omega x^0}\, e^{\mu\nu}(x^0, \vec{x}) \quad \simeq \frac{2G}{R}\, e^{i\omega R}\, \widehat{T}^{\mu\nu}(k)$$

where $\widehat{T}^{\mu\nu}(k) = \int d^4y\; e^{-ik.y}\; T^{\mu\nu}(y)$ **with** $k^\mu = \omega(1,\hat{n})$.

The matter energy-momentum tensor outside the region $\mathscr{R}$ has the following derivative expansion:

$$T^{X\alpha\beta}(x) = \sum_{a=1}^{M+N} \int_0^\infty d\sigma \left[m_a \frac{dX_a^\alpha(\sigma)}{d\sigma} \frac{dX_a^\beta(\sigma)}{d\sigma} \delta^{(4)}\big(x - X_a(\sigma)\big) + \frac{dX_a^{(\alpha}(\sigma)}{d\sigma} \Sigma_a^{\beta)\gamma}(\sigma)\, \partial_\gamma \delta^{(4)}\big(x - X_a(\sigma)\big) + \cdots \right]$$

Goldberger, Rothstein ; ...

- Above we considered all the incoming particles as some extra outgoing particles under proper identifications of $X_a(\sigma)$, $\Sigma_a(\sigma)$, ... under $\sigma \to -\sigma$.

- The " ... " terms carries the information about multiple moments of the compact objects as well as tidal response, which involves two or more derivative on the delta function.

- More derivative on delta function generates more power of soft momenta in the Fourier transform, So if we are interested to determine the leading non-analytic piece at each iterative order, we can only keep the first term.

Geodesic equation and Spin evolution outside region $\mathscr{R}$:

$$\frac{d^2X^\mu}{d\sigma^2} + \Gamma^\mu_{\alpha\beta}\frac{dX^\alpha}{d\sigma}\frac{dX^\beta}{d\sigma} = -\frac{1}{m}\left[\frac{d^2\Sigma^\mu{}_\nu}{d\sigma^2} + \frac{1}{2}R^\mu{}_{\nu\rho\sigma}\Sigma^{\rho\sigma}\right]\frac{dX^\nu}{d\sigma} \quad + \quad \cdots$$

$$\frac{D\Sigma^{\mu\nu}}{d\sigma} - \frac{dX^\mu}{d\sigma}\frac{dX_\rho}{d\sigma}\frac{D\Sigma^{\nu\rho}}{d\sigma} + \frac{dX^\nu}{d\sigma}\frac{dX_\rho}{d\sigma}\frac{D\Sigma^{\mu\rho}}{d\sigma} = 0$$

* The right hand side of the geodesic equation involves Riemann tensor or derivative over Spin along the trajectory, which turns out to be insignificant if we are interested to determine the leading non-analytic term at each iterative order or even the leading spin dependent non-analytic piece.

Boundary conditions:

$$X^\mu_a(\sigma=0) = r^\mu_a \, , \quad \left.\frac{dX^\mu_a(\sigma)}{d\sigma}\right|_{\sigma\to\infty} = v^\mu_a \quad , \qquad \left.\Sigma^{\mu\nu}_a(\sigma)\right|_{\sigma\to\infty} = \Sigma^{\mu\nu}_a$$

Strategy

In $c = 1$ unit the dimensionless parameters are: $GM\omega$, $G\Sigma\omega^2$, $GMr\omega^2$, ...

So we develop an iterative procedure considering G as an iterative parameter and solve iteratively

Einstein equation to get corrected metric

And

Geodesic equation to get corrected trajectory

* Finally at each order in G expansion of $\widehat{T}^{\mu\nu}(k)$ extract the non-analytic terms in $\omega \to 0$ limit.

Start at zeroth iterative order with: $e_{\mu\nu}(x) = 0$ and $X_a^\mu(\sigma) = r_a^\mu + v_a^\mu \sigma$.

Consider corrected asymptotic trajectory: $X_a^\mu(\sigma) = r_a^\mu + v_a^\mu \sigma + Y_a^\mu(\sigma)$

$$Y_a^\mu(\sigma) = \Delta_{(1)} Y_a^\mu(\sigma) + \Delta_{(2)} Y_a^\mu(\sigma) + \Delta_{(3)} Y_a^\mu(\sigma) + \cdots \qquad \Delta_{(r)} Y_a(\sigma) \sim G^r$$

$$\widehat{T}^{\mu\nu}(k) = \Delta_{(0)} \widehat{T}^{\mu\nu}(k) + \Delta_{(1)} \widehat{T}^{\mu\nu}(k) + \Delta_{(2)} \widehat{T}^{\mu\nu}(k) + \cdots \qquad \Delta_{(r)} \widehat{T}^{\mu\nu}(k) \sim G^r$$

$$e_{\mu\nu}(x) = \Delta_{(0)} e_{\mu\nu}(x) + \Delta_{(1)} e_{\mu\nu}(x) + \Delta_{(2)} e_{\mu\nu}(x) + \Delta_{(3)} e_{\mu\nu}(x) + \cdots \qquad \Delta_{(r)} e_{\mu\nu}(x) \sim G^{r+1}$$

$$\boxed{\Delta_{(r)} e_{\mu\nu}(x) = -8\pi G \int \frac{d^4\ell}{(2\pi)^4} G_r(\ell)\, e^{i\ell.x}\, \Delta_{(r)} \widehat{T}_{\mu\nu}(\ell)}$$

Matter energy-momentum tensor:

$$\begin{aligned}\widehat{T}^{X\mu\nu}(k) &= \int d^4x\; e^{-ik.x}\; T^{X\mu\nu}(x)\\ &= \sum_{a=1}^{M+N} m_a \int_0^\infty d\sigma\; e^{-ik\cdot X_a(\sigma)}\; \frac{dX_a^\mu(\sigma)}{d\sigma}\; \frac{dX_a^\nu(\sigma)}{d\sigma}\\ &= \sum_{a=1}^{M+N} m_a e^{-ik.r_a} \int_0^\infty d\sigma\; e^{-ik\cdot v_a\sigma} \sum_{w=0}^{\infty}\frac{1}{w!}\Big\{-ik\cdot\sum_{s=1}^{\infty}\Delta_{(s)}Y_a(\sigma)\Big\}^w\\ &\quad\times\Big\{v_a^\mu + \sum_{t=1}^{\infty}\frac{d\Delta_{(t)}Y_a^\mu(\sigma)}{d\sigma}\Big\}\Big\{v_a^\nu + \sum_{u=1}^{\infty}\frac{d\Delta_{(u)}Y_a^\nu(\sigma)}{d\sigma}\Big\}\\ &\equiv \sum_{r=0}^{\infty}\Delta_{(r)}\widehat{T}^{X\mu\nu}(k)\end{aligned}$$

Gravitational energy-momentum tensor:

$$\Delta_{(r)}T^{h\mu\nu}(x) \sim \frac{1}{8\pi G}\Big[\partial\partial(\Delta_{(0)}e)^{r+1} + \partial\partial\{(\Delta_{(0)}e)^{r-1}\Delta_{(1)}e\} + \cdots + \partial\partial\{(\Delta_{(0)}e)(\Delta_{(r-1)}e)\}\Big] + \mathcal{O}(\partial\partial\partial\partial)$$

Goal : Extract $\mathcal{O}(\omega^{r-1}(\ln\omega)^r)$ coefficient from the analysis of $\Delta_{(r)}\widehat{T}^{X\mu\nu}(k)$ and $\Delta_{(r)}\widehat{T}^{h\mu\nu}(k)$, which is the leading non-analytic term in $\omega\to 0$ limit.

Important Observations

- If we are interested to evaluate the leading non-analytic part of gravitational waveform at n'th iterative order $\mathcal{O}(G^{n+1})$, which goes like $\omega^{n-1}(\ln\omega)^n$ in $\omega \to 0$ limit:

It is enough to consider the scattered objects as non-spinning point particles

and Fourier transform of energy-momentum tensor has to be only evaluated outside the region $\mathscr{R}$.

- The leading spin dependent non-analytic term appears at order $\mathcal{O}(G^2\ \omega \ln\omega)$:

It is enough to consider the scattered objects as structureless spinning point particles

and at first iterative order the Fourier transform of energy-momentum tensor has to be evaluated both outside and inside the region $\mathscr{R}$. In the next iterative order only outside.

Gravitational Tail Memories

Fourier transform in ω of the gravitational waveforms produce gravitational memory at large retarded time ($u \to \pm\infty$)

0-loop (tree) $\mathcal{O}(G)$	ω^{-1} $\theta(u)$	ω^{0} $\delta(u)$	ω $\delta'(u)$	$\cdots$		
1-loop $\mathcal{O}(G^2)$	$\ln\omega$ u^{-1}	ω^{0} $\delta(u)$	Spin dependent $\omega\ln\omega$ u^{-2}	ω $\delta'(u)$	$\omega^2\ln\omega$ u^{-3}	$\cdots$
2-loop $\mathcal{O}(G^3)$	$\omega(\ln\omega)^2$ $u^{-2}\ln u$	$\omega\ln\omega$ u^{-2}	ω $\delta'(u)$	$\omega^2(\ln\omega)^2$ $u^{-3}\ln u$	$\omega^2\ln\omega$ u^{-3}	$\cdots$
n-loop $\mathcal{O}(G^{n+1})$	$\omega^{n-1}(\ln\omega)^n$ $u^{-n}(\ln u)^{n-1}$	$\omega^{n-1}(\ln\omega)^{n-1}$ $u^{-n}(\ln u)^{n-2}$	$\cdots$			

Results of this column are exact at that order in G expansion.

Gravitational DC Memory

$$\Delta_{(0)}e^{\mu\nu}(t,R,\hat{n})\Big|_{(t-R)\to\infty} - \Delta_{(0)}e^{\mu\nu}(t,R,\hat{n})\Big|_{(t-R)\to-\infty} = \frac{2G}{R}\left[-\sum_{a=1}^{N}\frac{p_a^\mu p_a^\nu}{p_a.\mathbf{n}} + \sum_{a=1}^{M}\frac{p_a'^\mu p_a'^\nu}{p_a'.\mathbf{n}}\right]$$

Leading gravitational tail Memory

$$\begin{aligned}\Delta_{(1)}e^{\mu\nu}(t,R,\hat{n}) &= \frac{2G}{R}\,\frac{1}{u}\left\{2G\sum_{b=1}^{N}p_b\cdot\mathbf{n}\right\}\left(\sum_{a=1}^{N}\frac{p_a^\mu p_a^\nu}{p_a.\mathbf{n}} - \sum_{a=1}^{M}\frac{p_a'^\mu p_a'^\nu}{p_a'.\mathbf{n}}\right)\\ &-\frac{4G^2}{R}\,\frac{1}{u}\sum_{a=1}^{N}\sum_{\substack{b=1\\ b\neq a}}^{N}\frac{p_a.p_b}{[(p_a.p_b)^2-p_a^2p_b^2]^{3/2}}\left\{\frac{3}{2}p_a^2p_b^2-(p_a.p_b)^2\right\}\\ &\times\frac{p_a^\mu\mathbf{n}_\rho}{p_a.\mathbf{n}}\left\{p_a^\nu p_b^\rho - p_a^\rho p_b^\nu\right\} \qquad \text{for } u\to+\infty\end{aligned}$$

$$u = t - R + 2G\ln R\sum_{b=1}^{N}p_b\cdot\mathbf{n}$$

$$\begin{aligned}\Delta_{(1)}e^{\mu\nu}(t,R,\hat{n}) &= \frac{4G^2}{R}\,\frac{1}{u}\sum_{a=1}^{M}\sum_{\substack{b=1\\ b\neq a}}^{M}\frac{p_a'.p_b'}{[(p_a'.p_b')^2-p_a'^2p_b'^2]^{3/2}}\left\{\frac{3}{2}p_a'^2p_b'^2-(p_a'.p_b')^2\right\}\\ &\times\frac{p_a'^\mu\mathbf{n}_\rho}{p_a'.\mathbf{n}}\left\{p_a'^\nu p_b'^\rho - p_a'^\rho p_b'^\nu\right\} \qquad \text{for } u\to-\infty\end{aligned}$$

Sub-leading gravitational tail Memory

$$
\begin{aligned}
&\Delta_{(2)} e^{\mu\nu}(t, R, \hat{n}) \\
= \quad & \frac{2G}{R}\,\frac{\ln|u|}{u^2}\,\Big\{2G\sum_{b=1}^{N} p_b.\mathbf{n}\Big\}^2 \left(\sum_{a=1}^{N}\frac{p_a^\mu p_a^\nu}{p_a.\mathbf{n}} - \sum_{a=1}^{M}\frac{p_a'^\mu p_a'^\nu}{p'_a.\mathbf{n}}\right) \\
& -\frac{4G}{R}\,\frac{\ln|u|}{u^2}\,\Big\{2G\sum_{c=1}^{N} p_c.\mathbf{n}\Big\}\,\sum_{a=1}^{N}\left[(2G)\sum_{\substack{b=1\\ b\neq a}}^{N}\frac{p_a.p_b}{[(p_a.p_b)^2 - p_a^2 p_b^2]^{3/2}}\Big\{\frac{3}{2}p_a^2 p_b^2 - (p_a.p_b)^2\Big\}\,\frac{p_a^\mu \mathbf{n}_\rho}{p_a.\mathbf{n}}\Big\{p_a^\nu p_b^\rho - p_a^\rho p_b^\nu\Big\}\right] \\
& -\frac{2G}{R}\,\frac{\ln|u|}{u^2}\,\Big\{2G\sum_{c=1}^{N} p_c.\mathbf{n}\Big\}\,\sum_{a=1}^{M}\left[(2G)\sum_{\substack{b=1\\ b\neq a}}^{M}\frac{p'_a.p'_b}{[(p'_a.p'_b)^2 - p_a'^2 p_b'^2]^{3/2}}\Big\{\frac{3}{2}p_a'^2 p_b'^2 - (p'_a.p'_b)^2\Big\}\,\frac{p_a'^\mu \mathbf{n}_\rho}{p'_a.\mathbf{n}}\Big\{p_a'^\nu p_b'^\rho - p_a'^\rho p_b'^\nu\Big\}\right] \\
& + \frac{2G}{R}\,\frac{\ln|u|}{u^2}\sum_{a=1}^{N}\frac{\mathbf{n}_\rho \mathbf{n}_\sigma}{p_a.\mathbf{n}}\left[(2G)\sum_{\substack{b=1\\ b\neq a}}^{N}\frac{p_a.p_b}{[(p_a.p_b)^2 - p_a^2 p_b^2]^{3/2}}\Big\{\frac{3}{2}p_a^2 p_b^2 - (p_a.p_b)^2\Big\}\,\Big\{p_a^\mu p_b^\rho - p_a^\rho p_b^\mu\Big\}\right] \\
& \times\left[(2G)\sum_{\substack{c=1\\ c\neq a}}^{N}\frac{p_a.p_c}{[(p_a.p_c)^2 - p_a^2 p_c^2]^{3/2}}\Big\{\frac{3}{2}p_a^2 p_c^2 - (p_a.p_c)^2\Big\}\,\Big\{p_a^\nu p_c^\sigma - p_a^\sigma p_c^\nu\Big\}\right] + \mathcal{O}(u^{-2}) \qquad \text{for } u \to +\infty,
\end{aligned}
$$

$$
u = t - R + 2G\ln R \sum_{b=1}^{N} p_b\cdot\mathbf{n}
$$

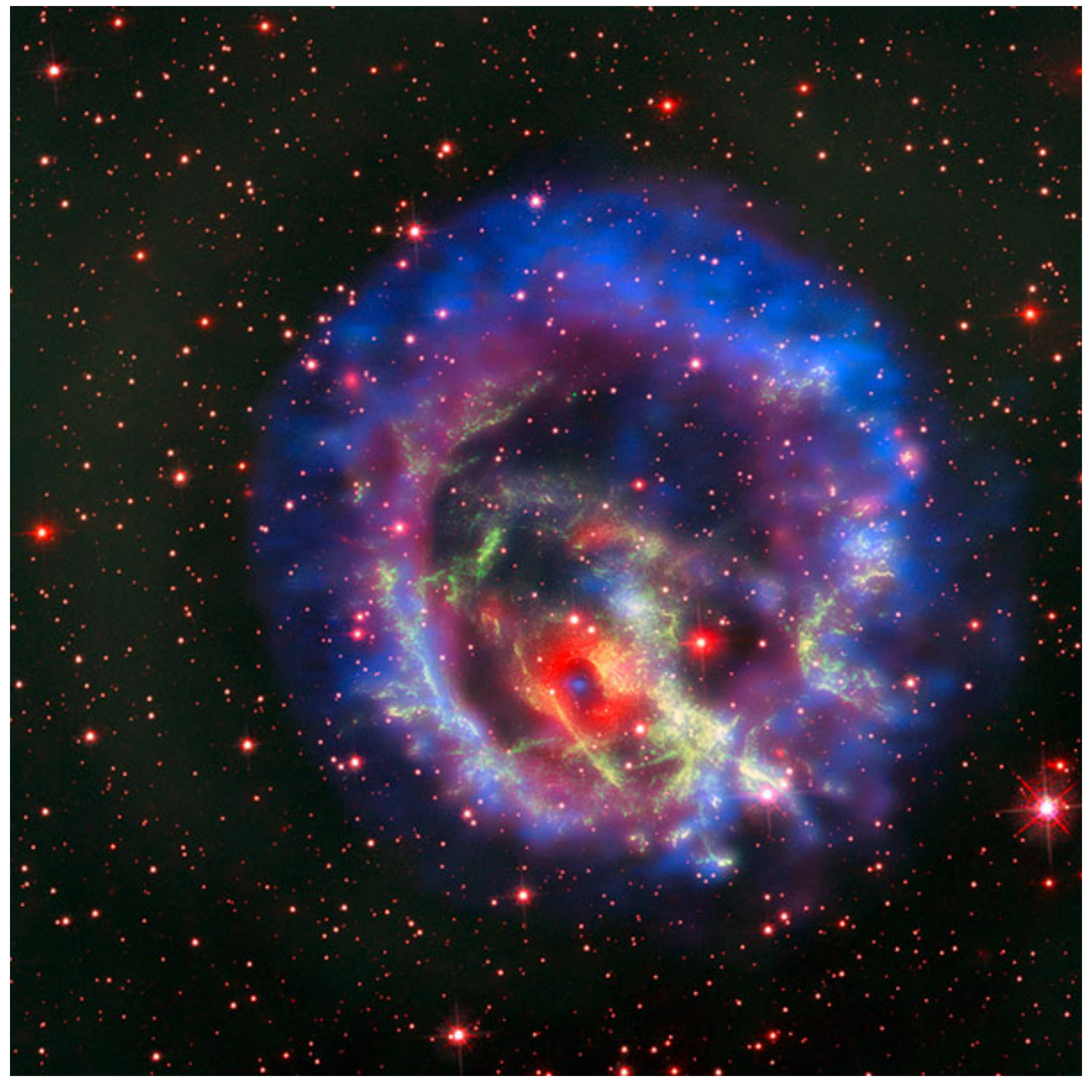
$$
\begin{aligned}
&\Delta_{(2)} e^{\mu\nu}(t, R, \hat{n}) \\
= \quad & \frac{2G}{R}\,\frac{\ln|u|}{u^2}\,\Big\{2G\sum_{c=1}^{N} p_c.\mathbf{n}\Big\}\,\sum_{a=1}^{M}\left[(2G)\sum_{\substack{b=1\\ b\neq a}}^{M}\frac{p'_a.p'_b}{[(p'_a.p'_b)^2 - p_a'^2 p_b'^2]^{3/2}}\right. \\
& \left.\times\Big\{\frac{3}{2}p_a'^2 p_b'^2 - (p'_a.p'_b)^2\Big\}\,\frac{p_a'^\mu \mathbf{n}_\rho}{p'_a.\mathbf{n}}\Big\{p_a'^\nu p_b'^\rho - p_a'^\rho p_b'^\nu\Big\}\right] \\
& + \frac{2G}{R}\,\frac{\ln|u|}{u^2}\sum_{a=1}^{M}\frac{\mathbf{n}_\rho \mathbf{n}_\sigma}{p'_a.\mathbf{n}}\left[(2G)\sum_{\substack{b=1\\ b\neq a}}^{M}\frac{p'_a.p'_b}{[(p'_a.p'_b)^2 - p_a'^2 p_b'^2]^{3/2}}\Big\{\frac{3}{2}p_a'^2 p_b'^2 - (p'_a.p'_b)^2\Big\}\,\Big\{p_a'^\mu p_b'^\rho - p_a'^\rho p_b'^\mu\Big\}\right] \\
& \times\left[(2G)\sum_{\substack{c=1\\ c\neq a}}^{M}\frac{p'_a.p'_c}{[(p'_a.p'_c)^2 - p_a'^2 p_c'^2]^{3/2}}\Big\{\frac{3}{2}p_a'^2 p_c'^2 - (p'_a.p'_c)^2\Big\}\,\Big\{p_a'^\nu p_c'^\sigma - p_a'^\sigma p_c'^\nu\Big\}\right] + \mathcal{O}(u^{-2}) \quad \text{for } u \to -\infty
\end{aligned}
$$

Spin dependent tail memory

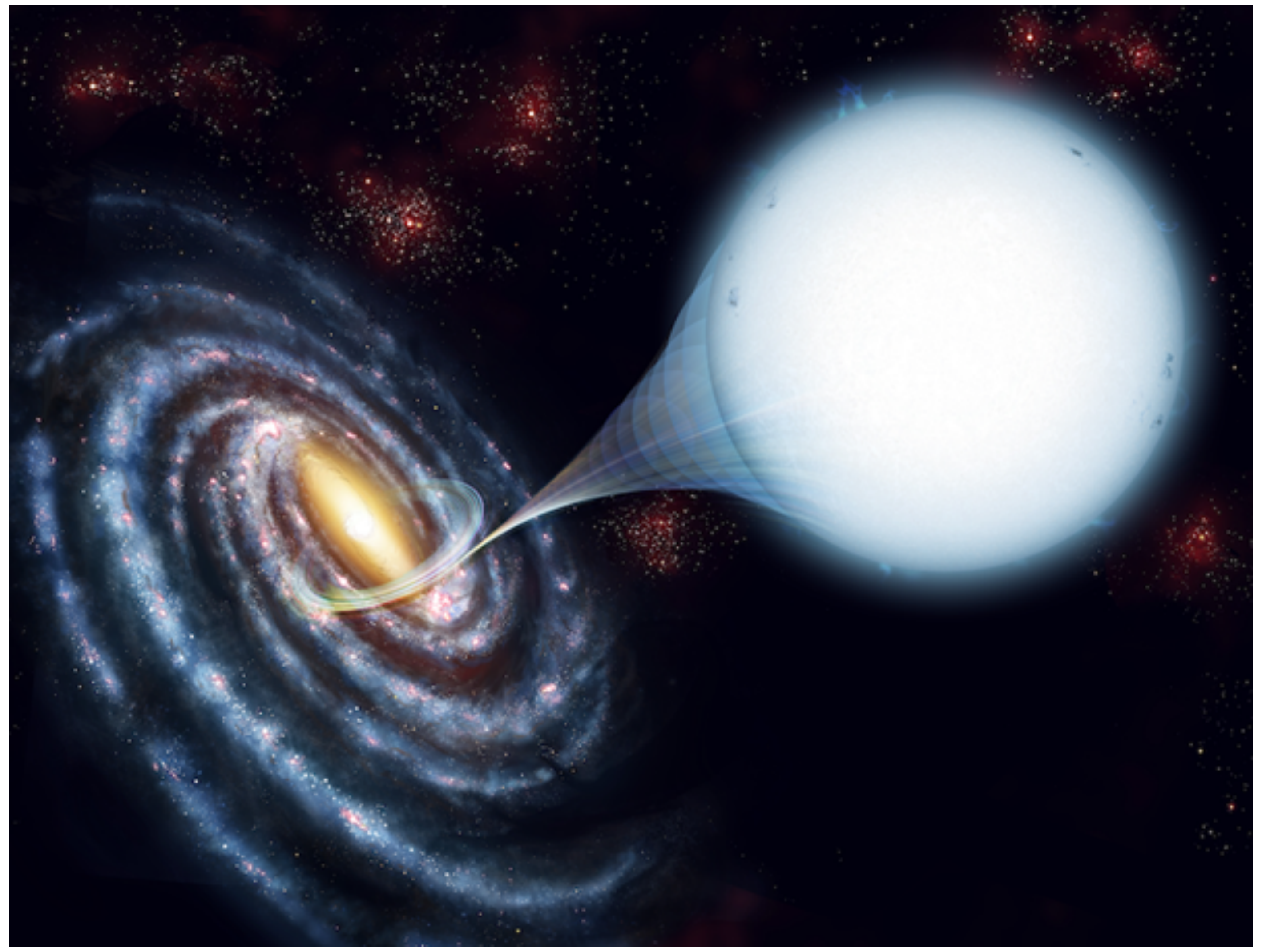
$$
\begin{aligned}
&\Delta^{(1/u^2)}_{(G^2)}\, e^{\mu\nu}(u,\vec{x}=R\hat{n}) \\
&= -\frac{G^2}{R}\,\frac{1}{u^2}\Bigg[4\sum_{b=1}^{N} p_b.\mathbf{n}\Bigg\{\sum_{a=1}^{N}\frac{p_a^{(\mu}\mathbf{n}_\rho}{p_a.\mathbf{n}}\Big(r_a^{\rho}p_a^{\nu)} - r_a^{\nu)}p_a^{\rho} + \Sigma_a^{\rho\nu)}\Big) \\
&\quad - \sum_{a=1}^{M}\frac{p_a^{\prime(\mu}\mathbf{n}_\rho}{p'_a.\mathbf{n}}\Big(r_a^{\prime\rho}p_a^{\prime\nu)} - r_a^{\prime\nu)}p_a^{\prime\rho} + \Sigma_a^{\prime\rho\nu)}\Big)\Bigg\} \\
&\quad + \sum_{a=1}^{N}\sum_{b=1}^{N}\frac{p_a.p_b}{[(p_a.p_b)^2 - p_a^2p_b^2]^{3/2}}\{2(p_a.p_b)^2 - 3p_a^2p_b^2\}\frac{\mathbf{n}_\rho\mathbf{n}_\sigma}{p_a.\mathbf{n}}\Big\{(p_a^{\mu}p_b^{\rho} - p_a^{\rho}p_b^{\mu})(r_a^{\sigma}p_a^{\nu} - r_a^{\nu}p_a^{\sigma} + \Sigma_a^{\sigma\nu}) \\
&\quad + (p_a^{\nu}p_b^{\sigma} - p_a^{\sigma}p_b^{\nu})(r_a^{\rho}p_a^{\mu} - r_a^{\mu}p_a^{\rho} + \Sigma_a^{\rho\mu})\Big\}\Bigg], \qquad \text{for } u\to+\infty
\end{aligned}
$$

This is the result written as a conjecture from classical limit of soft theorem. After a tedious computation we are getting some unwanted extra terms, currently we are struggling to fix them!!!

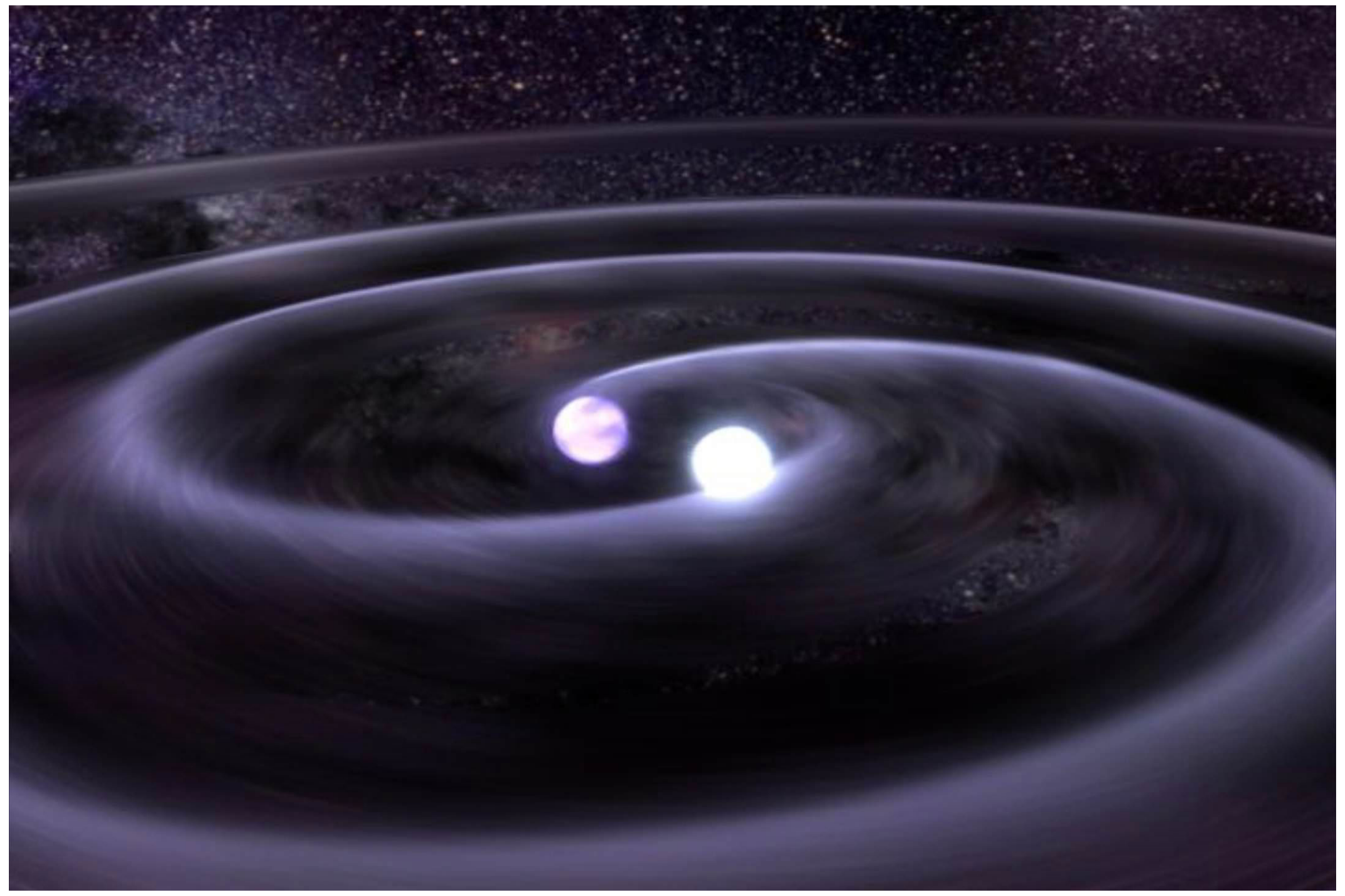
$$
\begin{aligned}
&\Delta^{(1/u^2)}_{(G^2)}\, e^{\mu\nu}(u,\vec{x}=R\hat{n}) \\
&= \frac{G^2}{R}\,\frac{1}{u^2}\sum_{a=1}^{M}\sum_{b=1}^{M}\frac{p'_a.p'_b}{[(p'_a.p'_b)^2 - p_a^{\prime 2}p_b^{\prime 2}]^{3/2}}\{2(p'_a.p'_b)^2 - 3p_a^{\prime 2}p_b^{\prime 2}\}\frac{\mathbf{n}_\rho\mathbf{n}_\sigma}{p'_a.\mathbf{n}} \\
&\quad \Big\{(p_a^{\prime\mu}p_b^{\prime\rho} - p_a^{\prime\rho}p_b^{\prime\mu})(r_a^{\prime\sigma}p_a^{\prime\nu} - r_a^{\prime\nu}p_a^{\prime\sigma} + \Sigma_a^{\prime\sigma\nu}) \;+\; (p_a^{\prime\nu}p_b^{\prime\sigma} - p_a^{\prime\sigma}p_b^{\prime\nu})(r_a^{\prime\rho}p_a^{\prime\mu} - r_a^{\prime\mu}p_a^{\prime\rho} + \Sigma_a^{\prime\rho\mu})\Big\}, \qquad \text{for } u\to-\infty
\end{aligned}
$$



core collapse supernova



Hyper-velocity star



Neutron star merger

For the above mentioned astrophysical scattering events, the gravitational strain due to order u^{-1} tail term turns out: $\frac{\Delta L}{L} \sim 10^{-22}$, which is in the edge of the resolution of current GW detectors.

On the other hand for binary blackhole merger process the order u^{-1} , $u^{-2}\ln u$ tail terms vanishes.

Observation of (non-vanishing or vanishing) gravitational tail memory will be a test of general relativity.

Thank You