Angular momentum in GR:
The role of generalized BRS symmetry

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Based on 2103.17103 with David Nichols
1912.03164 with Roberto Oliveri & Ali Seraj
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Aim. Define J_i : so(3) in GR close to null infinity

Also $\begin{cases} K_i & : \text{center-of-mass} \\ G_i = K_i + u P_i & \text{comoving center-of-mass} \end{cases}$

Aim. Define so(3,1), define its representation by charges, and so(3) ⊂ so(3,1).

We need to specify 3 items:

I) Supertranslation frame (local definition)
or supertranslation-independent formulation
(non-local definition)

Role of generalized BRS symmetry → II) Fix 1-parameter α ambiguity

III) Define so(3) ⊂ so(3,1) with $P_\mu \neq 0$
Center-of-mass frame (non-local definition)

I) Supertranslations

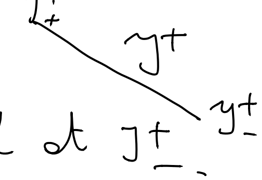
Asymptotic Symmetry Group:

$$\supset SO(3,1) \ltimes \mathrm{Vect}(S^2) \qquad \text{BMS}$$

$$\supset \mathrm{Diff}(S^2) \ltimes \mathrm{Vect}(S^2) \qquad \text{Generalized BMS}$$

Generators of BMS are determined by functions over S^2 (no functions of u)

One can fix the BMS group at spatial infinity, at $u \xrightarrow{\text{fixed}} -\infty$ at $r \to \infty$. ($\mathcal{I}^\pm_-$)



once boundary conditions are fixed at $\mathcal{I}^\pm_-$.

Bondi gauge: $x^\mu = (u, r, x^A)$ $\qquad x^A = (\theta, \phi)$ or $(z, \bar{z})$

$$g_{rA} = 0 = g_{rr} \quad , \begin{cases} \partial_r \det(g_{AB}/r^2) = 0 \\ \det g_{AB} \text{ fixed at order } r^2. \end{cases}$$

$$g_{AB} = r^2 \gamma_{AB} + r C_{AB} + \dots$$

$$g_{uu} = -1 + \frac{2Gm}{c^2 r} + \dots$$

$$g_{uA} = \frac{1}{2} D^B C_{AB} + \frac{1}{r}\left(\frac{2G}{3c^2} N_A - \frac{c}{16} \partial_A (C_{BC} C^{BC})\right) + \dots$$

Boundary condition as $u \to \pm\infty$:

$$\begin{cases} \gamma_{AB} = \bar{\gamma}_{AB}(x^C) \\ C_{AB} = \left(u + \frac{C^\pm}{c}\right) N_{AB}^{vac} - 2 D_A D_B C^\pm + \gamma_{AB} D^C D_C C^\pm + \mathcal{O}(u^{-1}) \end{cases}$$

: $\quad \delta_T C^\pm = T$

Fix supertranslation frame at $\mathcal{I}^+_- \iff$ Fix $C^-(x^A)$ (all harmonics)

II. The α-ambiguity.

Definition: For any infinitesimal $R^A(\theta,\phi) \in \underset{SO(3,1)}{\underset{\cup}{\mathrm{Diff}(S^2)}}$

$$\mathcal{J}_R^{(\alpha)} = \frac{1}{2}\int_S \underbrace{d\Omega}_{= \frac{1}{4\pi}\sqrt{\det \gamma_{AB}}\, d^2x} R^A\left(N_A - \frac{\alpha c^3}{4G}\left[C_{AB} D_C C^{BC} + \frac{1}{4}\partial_A (C_{BC} C^{BC})\right]\right)$$

Angular momenta: $R^A = -\varepsilon^{AB}\partial_B n_i$, $n_i = \frac{x_i}{r}$

$$\mathcal{J}_i^{(\alpha)} = -\frac{1}{2}\int_S d\Omega\, \varepsilon^{AD}\partial_D n_i \left(N_A - \frac{\alpha c^3}{4G} C_{AB} D_C C^{BC}\right)$$

Comoving center-of-mass $R^A = \gamma^{AB}\partial_B n_i$

$$G_i^{(\alpha)} = \frac{1}{2}\int_S d\Omega\, \gamma^{AB}\partial_B n_i \left(N_A - \frac{\alpha c^3}{4G}\left[C_{AB} D_C C^{BC} + \frac{1}{4}\partial_A (C_{BC} C^{BC})\right]\right)$$

These Lorentz charges obey:

(i) $\mathcal{J}_i^{(\alpha)} = G_i^{(\alpha)} = 0$ for Minkowski

(ii) $\mathcal{J}_i^{(\alpha)} = \delta_{iz} J$; $G_i^{(\alpha)} = 0$ for Kerr in CM frame

(iii) locally constructed from tensors at $\mathcal{I}^+$

(iv) obey the BMS algebra at $\mathcal{I}^+_-$

$$\{P_T, P_{T'}\} = 0$$

$$\{J_R^{(\alpha)}, P_T\} = P_{D_R(T)} \qquad D_R(T) \equiv \left(R^A\partial_A - \frac{1}{2}D_A R^A\right)T$$

$$\{J_R^{(\alpha)}, J_{R'}^{(\alpha)}\} = J^{(\alpha)}_{[R,R']} \qquad [R,R'] = R^A\partial_A R'^A - R'^A\partial_A R^A$$

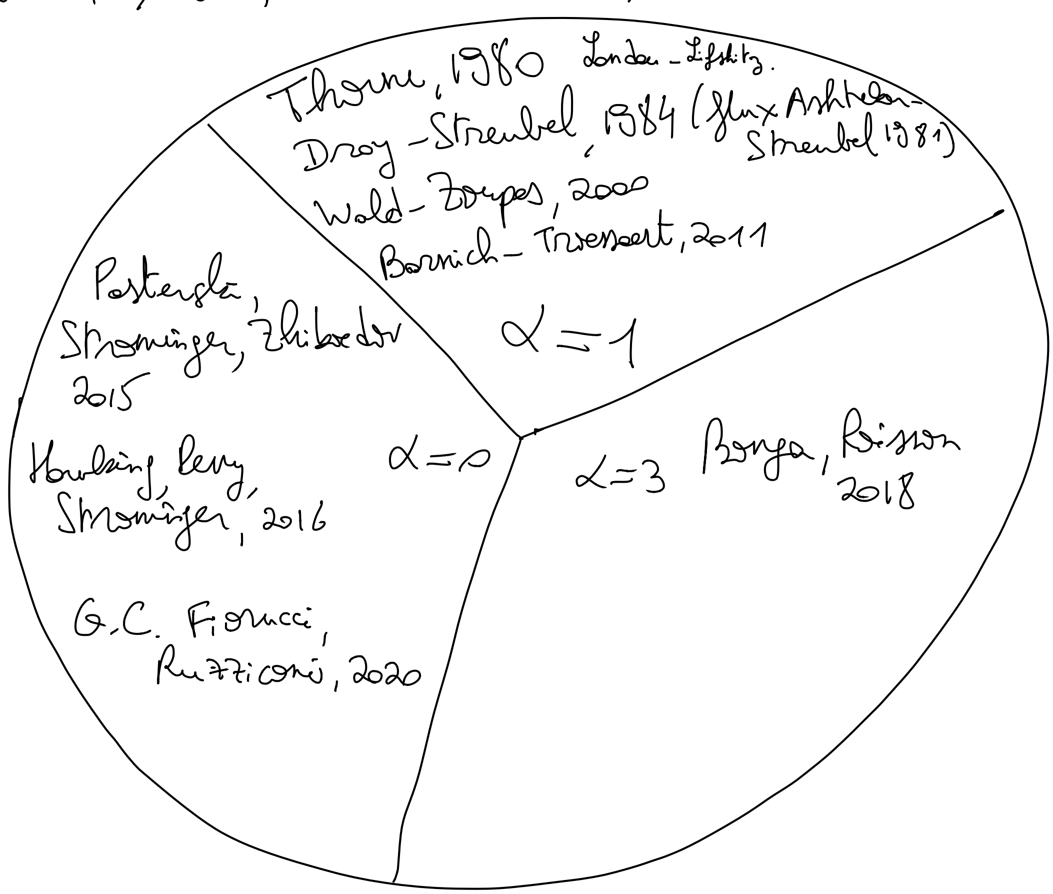
Here $P_T \equiv \frac{1}{c}\int d\Omega\, T \left(m + \frac{1}{8} C^{AB} N_{AB}^{vac} \right)$

(v) satisfy flux-balance laws at $\mathcal{I}^+$

$$\begin{cases} \dot{P}_T = \text{--} \\ \dot{J}_R = \ldots \end{cases}$$

(vi) These flux-balance laws are consistent with the leading and subleading soft graviton theorems.

Match with literature:



Different values of α lead to a 0.1%
effect in $J_i(u)$ at intermediate u
during binary black hole mergers [Nichols, Elhashash 2021]

Proposals:

$\boxed{\alpha = 1}$ $\boxed{\text{Definition valid for any foliation of } \mathcal{J}^+\text{, not just Bondi cuts.}}$

[Ashtekar, Magnon, 1982][Ashtekar, 2014]

$$\mathring{J}_z^{(\alpha=1)} = \frac{1}{16\pi G}\int_S d^2\Omega\, N_{ab}\left(\mathcal{L}_{\partial_\phi} D_c - D_c \mathcal{L}_{\partial_\phi}\right) l_d\, q^{ac} q^{bd}$$

Here: $\begin{cases} D_a : \text{intrinsic } D \text{ on } \mathcal{J}^+ \\ q_{ab} \text{ intrinsic degenerate metric} \\ l_a \text{ any vector } l_a n^a = -1 \\ q^{ab} \sim q^{ab} + V^{(a} n^{b)} \text{ but } \mathring{J}_z \text{ inv.} \end{cases}$

No such intrinsic definition is known
for $\alpha \neq 1$.

Open questions : $\left\{ \begin{array}{l} \text{Uniqueness proof ?} \\ \text{Definition of } K_i \text{ ? or } \mathcal{Y}_R \text{ ?} \end{array} \right.$

Flux in PN expansion :

$$\overset{\circ}{\mathcal{Y}}_i = -\frac{G}{c^5}\left(\frac{2}{5}\varepsilon_{ijk} \overset{(2)}{I}_{jl} \overset{(3)}{I}_{kl}\right) + O\left(\frac{G}{c^7}\right)$$

[Thorne, 1980]

$$\overset{\circ}{G}_i = P_i - \frac{G}{c^7}\left[\frac{1}{21}\left(\overset{(3)}{I}_{jk} \overset{(3)}{I}_{ijk} - \overset{(2)}{I}_{jk} \overset{(4)}{I}_{ijk}\right)\right] + O\left(\frac{1}{c^9}\right)$$

All PN $\{$ [G.C., Oliveri, Seraj, 2019]

3.5PN $\left\{ \begin{array}{l} \text{[Epstein-Wagoner, 1975]} \\ \text{[Kozameh, Nieva, Quiroga, 2017]} \end{array} \right.$

Note : distinct expansion found in [Blanchet, Faye, 2018]
1811.08966

$$G_i^{(BF)} = P_i - \frac{2G}{21c^7} \overset{(3)}{I}_{ijk} \overset{(3)}{I}_{jk} + O(c^{-9})$$

$$G_i^{(BF)} - G_i = -\frac{G}{21c^7} \int dt \left(\overset{(3)}{I}_{ijk} \overset{(3)}{I}_{jk} + \overset{(2)}{I}_{jk} \overset{(4)}{I}_{ijk}\right)$$

$$= -\frac{G}{21c^7} \overset{(2)}{I}_{jk} \overset{(3)}{I}_{ijk}$$

cannot be absorbed by an α shift.

$\rightarrow$ Violates (iii)

Note : The canonical Lagrangian/Hamiltonian methods give

$$\underset{\text{uniquely}}{\delta H_\xi} = \delta(H_\xi) + \Xi_\xi[\delta g; g]$$

Wald–Zoupas prescription

$\boxed{\alpha = 0}$ The generalized BMS charges represent the generalized BMS algebra at spatial infinity ($\mathcal{I}^\pm_-$).

The generalized BMS algebra is

$$\{P_T, P_{T'}\} = 0$$

$$\{J_R^{(2)}, P_T\} = P_{\mathcal{D}_R(T)} \qquad \mathcal{D}_R(T) \equiv (R^A\partial_A - \tfrac{1}{2}\mathcal{D}_A R^A)T$$

$$\{J_R^{(2)}, J_{R'}^{(2)}\} = J^{(2)}_{[R,R']} \qquad [R,R'] = R^A\partial_A R'^A - R'^A\partial_A R^A$$

for $\alpha = 0$. Here $J_R^{(0)} \equiv \frac{1}{2}\int_S d\Omega\, R^A N_A$.

For $\alpha \neq 0$: anomaly in $\{J_R^{(2)}, P_T\}$
$\{J_R^{(\alpha)}, J_{R'}^{(\alpha)}\}$

for $R^A \in Diff(S^2)$ but $D_A R_B + D_B R_A \neq \gamma_{AB} D_C R^C$
(R^A not $SO(3,1)$)

[G.C., Fiorucci, Ruzziconi, 2000]

Drawback:

- So far, charges defined using a Bondi foliation only.

Remark: All Diff(S^2) charges are physical and non-trivial for scattering.

n-pt classical scattering in the limit $|R_i - R_j| \to \infty \ \forall i,j$.

$$M = \sum_{i=1}^{n} \frac{M_{(i)}}{\gamma_{(i)}^3 \left(1 - \frac{\vec{v}_{(i)}}{c} \cdot \vec{n}\right)^3}$$

$$N_A = \frac{2M}{c} \partial_A C^- + \partial_A \left(M \left(u + \frac{C^-}{c}\right)\right) - \sum_{i=1}^{n} \frac{3 J_{(i)} \sin^2 \theta_{(i)} \partial_A \phi_{(i)}}{c^2 \gamma_{(i)}^2 \left(1 - \frac{\vec{v}_{(i)}}{c} \cdot \vec{n}\right)^2}$$

Argument: All conserved charges at spatial infinity should be represented. In the quantum theory, $\{ , \} \to \frac{1}{i\hbar} [,]$. This implies $\alpha = 0$.

Additional property:

- All BMS vacua have identically zero BMS charges. $P_T = 0 = J_R^{(0)}$ for $R_{\mu\nu\alpha\beta} = 0$.

For $\alpha \neq 0$:

$$J_R^{(\alpha)} = \frac{-\alpha}{32\pi G} \int_S d^2\Omega \,(D_A D_B + \gamma_{AB}) D_C R^C D^A C D^B C$$

after using $C_{AB} = -2 D_A D_B C + \gamma_{AB} D_C D^C C$

$= 0$ for $R \in SO(3,1)$

$\neq 0$ otherwise

[G.C. Long, 2016]

- Implication for the quantum theory: the "correct" BMS charges are 0 for vacua. "Soft hair has no BMS charge"

This is an update of [G.C. Long, 2016] [Hawking, Perry, Strominger, 2016] which used $\alpha = 1$.

III Identifying the center-of-mass frame

$$SO(3) \subset SO(3,1)$$

$$\frac{d}{du} P_T = -\frac{c^2}{8G} \int_S d\Omega \, T \left(N_{AB} N^{AB} \right)$$

$\to$ COM depends upon u.

Identify

$$R_i^{(int)A} = \gamma R_i^A + (1-\gamma) v_i \frac{(\vec{v} \cdot \vec{R}^A)}{v^2} + \gamma \, \epsilon_{ijk} v_j K_k^A$$

$$v_i = \frac{P_i}{P_0}, \quad \gamma = \sqrt{1 - v^2/c^2}, \quad v = \sqrt{\vec{v} \cdot \vec{v}}$$

[Ashtekar, De Lorenzo, Khera, 2013]

This defines the $SO(3) \subset SO(3,1)$ for any u.

Comment: Supertranslation-independent definition.

First define the radiative multipole moments completed with $l=0,1$ modes:

$$C_{AB} = -2 D_A D_B C + \gamma_{AB} D^2 C + \epsilon_{C(A} D^C D_{B)} \psi$$

"O_L" "V_L" non-local

$$J_R^{(inv)} = J_R^{(0)} - P_{D_R C} \qquad D_R C \equiv R^A \partial_A C - \tfrac{1}{2} D_A R^A C.$$

$$= \frac{1}{2} \int_S d\Omega \, R^A \left(N_A - \frac{2m}{c} \partial_A C - \frac{1}{c} \partial_A (m C) \right)$$

See also [Chen et al, 2102.03235]

Algebra: at $\mathcal{I}^+_-$

$$\{ J_R^{(inv)}, P_T \} = 0$$

$$\{ J_R^{(inv)}, J_{R'}^{(inv)} \} = J^{inv}_{[R,R']}$$

$$J_i^{(F)} \equiv J^{(inv)}_{R_i^{(int)}} \quad \text{obeys } so(3) \text{ algebra at } \mathcal{I}^+_-.$$

$$\{ J_i^{(F)}, P_T \} = 0$$

$$\{ J_i^{(F)}, J_j^{(F)} \} = \epsilon_{ijk} J_k^{(F)}.$$

[G.C. & Nichols 2021]

Comment: Alternative rewriting

$$J_i^{(2)} = \int_{-\infty}^{u} du\, \mathring{J}_i^{(2)}$$

$$\mathring{J}_i^{(2)} = \frac{c^3}{32\pi G} \int d^2\Omega\, \varepsilon_{ijk} \Big[X^i\, \mathring{f}_{ab}\, \partial_j f_{ab} - 2 f_{ia} \mathring{f}_{ja} + (\alpha - 1) \frac{d}{du} \left(X^i f_{ja}\, \partial_b f_{ab} \right) \Big]$$

where $h_{ij}^{TT} = \dfrac{f_{ij}(u, X^A)}{r} + O(r^{-2})$ [G.C. Olivei, Seraj, 2019]

When $\alpha = 1$, it reduces to the Landau-Lifshitz expression.

See Maggiore, Eq. (2.61).

In PM formalism

$$f_{ij} = G f_{ij}^{(1)}(X^A) + G^2 f_{ij}^{(2)}(u, X^A) + \dots$$

$\Rightarrow$ α-ambiguity appears in J_i at $\underset{2PM}{O(G^2)}$.

See $\alpha = 1$ analysis of [Damour, 2020]

Thank you.