Gravitational Memory in Numerical Relativity

• Computing Memory Effects,
• Correcting Waveforms,
• And the importance of BMS Frames

Keefe Mitman

Conference on Gravitational Scattering, Inspiral and Radiation (virtual)
Galileo Galilei Institute, May 6, 2021
What we are currently seeing...

GW150914

![Graph showing strain versus time for GW150914 event with and without memory effects.](image)

Keefe Mitman — Introduction, Computing Memory Effects, Correcting Waveforms, BMS Frames, Detectability
What we are currently seeing...

GW150914

strain $h_+$ ($10^{-21}$)

Time (s)

Without Memory

Keefe Mitman — Introduction, Computing Memory Effects, Correcting Waveforms, BMS Frames, Detectability
What we should be seeing...

GW150914

strain $h_+ \times 10^{-21}$

Time (s)

With Memory

Without Memory

With Memory

Memory
What is gravitational memory?

Early Work:
- Braginsky and Thorne (1987).

Ordinary (linear) Memory:
Massive objects traveling to asymptotic infinity as $t \rightarrow \infty$

$$\Delta h_{ij} = \Delta \sum_{A=1}^{N} \frac{4M_A}{r \sqrt{1 - v_A^2}} \left( \frac{v_A^i v_A^j}{1 - v_A \cos (\theta_A)} \right)_{TT}$$
What is gravitational memory?

**Early Work:**
- Braginsky and Thorne (1987).

**Ordinary (linear) Memory:**
Massive objects traveling to asymptotic infinity as $t \to \infty$

$$\Delta h_{ij} = \Delta \sum_{A=1}^{N} \frac{4M_A}{r} \sqrt{1 - v_A^2} \left( \frac{v_A^i v_A^j}{1 - v_A \cos \left( \theta_A \right)} \right)^{TT}.$$ 

**Null (non-linear) Memory:**
Null radiation traveling to asymptotic infinity as $r, t \to \infty$ at fixed $u \equiv t - r$

$$\Delta h_{ij} = \frac{4}{r} \int \frac{dE}{d\Omega'} \left( \frac{\xi^i \xi^j}{1 - \cos \left( \theta' \right)} \right)^{TT} d\Omega'.$$

where $\xi'$ is a unit vector from the source to $d\Omega'$.
The BMS Group

BMS Group = Lorentz Group $\rtimes$ supertranslations
BMS Group = Lorentz Group $\rtimes$ supertranslations

Extended BMS Group = $\text{LCKV}(S^2) \rtimes$ supertranslations

Generalized BMS Group = $\text{Diff}(S^2) \rtimes$ supertranslations
The BMS Balance Laws

Bondi-Sachs Metric:

\[ ds^2 = -\left(1 - \frac{2m}{r} \right) du^2 - 2 \left(1 + \mathcal{O}\left(\frac{1}{r^2}\right)\right) dudr \]

\[ + r^2 \left(q_{AB} + \frac{1}{r} C_{AB} + \mathcal{O}\left(\frac{1}{r^2}\right)\right) (d\theta^A - \mathcal{U}^A du) (d\theta^B - \mathcal{U}^B du) \]

where

\[ \mathcal{U}^A = -\frac{1}{2r^2} D_B C^{AB} + \frac{1}{r^3} \left[ -\frac{2}{3} N^A + \frac{1}{16} D^A \left(C_{BC} C^{BC}\right) + \frac{1}{2} C^{AB} D^C C_{BC}\right] + \mathcal{O}\left(\frac{1}{r^4}\right) \]
The BMS Balance Laws

Bondi-Sachs Metric:

\[ ds^2 = -\left(1 - \frac{2m}{r}\right)\,dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}\,dr^2 + r^2\,(d\theta^2 + \sin^2\theta\,d\phi^2) \]

where

\[ \mathcal{U}^A = -\frac{1}{2r^2} D_{BC} A^B + \frac{r^3}{2} \left[ \frac{3}{16} N_{BC} + \frac{1}{2} \left( C_{BC} C_{AC} \right) \right] + \mathcal{O}\left( \frac{1}{r^4} \right) \]

- Identify BMS Charges
- Apply Einstein’s field equations to obtain evolution equations
- Understand them to be balance laws
The BMS Balance Laws

\[ \text{Re} \left[ h \right] (u, \theta, \phi) = \left( \begin{array}{c} \text{Bondi Mass Charge} \\ + \\ \text{Energy Flux} \end{array} \right) \]

\[ \text{Im} \left[ h \right] (u, \theta, \phi) = \frac{d}{du} \left( \begin{array}{c} \text{Angular Momentum Charge} \\ + \\ \text{Angular Momentum Flux} \end{array} \right) \]
The BMS Balance Laws (for memory)

\[ \Delta \text{Re} [h] (\theta, \phi) = \Delta \left( \text{Bondi Mass Charge} + \text{Energy Flux} \right) \rightarrow \text{Ordinary Electric Memory} \]

\[ \Delta \text{Im} [h] (\theta, \phi) = \Delta \frac{d}{du} \left( \text{Angular Momentum Charge} + \text{Angular Momentum Flux} \right) \rightarrow \text{Null Magnetic Memory} \]
What we know about memory

Gravitational memory = the relative displacement of (initially comoving) observers induced by the passage of gravitational radiation

$$\Delta \xi^\alpha(u_1, u_0) = \Delta K^\alpha_\beta(u_1, u_0) \xi^\beta(u_0)$$

$$+ (u_1 - u_0) \Delta H^\alpha_\beta \xi^\beta(u_0) + \cdots$$

$$\Delta K^\alpha_\beta \rightarrow \text{usual displacement memory}$$

$$\Delta H^\alpha_\beta \rightarrow \text{sub-leading displacement memory}$$

- spin memory (magnetic)
- center-of-mass memory (electric)
Charge-Flux breakdown of the strain

\[ h = (J_m + J_\mathcal{E}) + (J_\mathcal{N} + J_\mathcal{J}) \]  
[from the BMS balance laws]

where

\[ J_m \equiv \frac{1}{2} \bar{\delta}^2 \mathcal{D}^{-1} m \]  
\[ J_\mathcal{E} \equiv \frac{1}{2} \bar{\delta}^2 \mathcal{D}^{-1} \left[ \frac{1}{4} \int_{-\infty}^{u} |\check{h}|^2 \, du \right] \]  
\[ J_\mathcal{N} \equiv \frac{1}{2} i \bar{\delta}^2 \mathcal{D}^{-1} D^{-2} \text{Im} \left[ \bar{\delta} \left( \partial_u \hat{N} \right) \right] \]  
\[ J_\mathcal{J} \equiv \frac{1}{2} i \bar{\delta}^2 \mathcal{D}^{-1} D^{-2} \text{Im} \left[ \frac{1}{8} \bar{\delta} \left( 3h\check{\delta}\check{h} - 3h\check{\delta}\check{h} + \hat{h}\check{\delta}\check{h} - \check{h}\check{\delta}\check{h} \right) \right] \]

Note: \( D^2 \) is the Laplacian on \( S^2, \mathcal{D} \equiv \frac{1}{8} D^2 (D^2 + 2), \delta \eta = - (\sin(\theta)^s) \left\{ \frac{\partial}{\partial \theta} + i \csc(\theta) \frac{\partial}{\partial \phi} \right\} \left[ (\sin(\theta))^{-s} \eta \right], \)

\[ m = - \text{Re} \left[ \Psi_2 + \frac{1}{4} \check{h}\check{h} \right], \text{ and } \hat{N} = 2\Psi_1 - \frac{1}{4} \hat{h}\check{h} + u \check{m} + \frac{1}{8} \bar{\delta} \left( h\check{h} \right). \]
We can use these to study contributions to the strain!

mass ratio $q = 1$, non-spinning
We can use these to study contributions to the strain!

mass ratio $q = 1$, non-spinning
We can use these to study contributions to the strain!

mass ratio $q = 1$, non-spinning
We can use these to study contributions to the strain!

mass ratio $q = 1$, non-spinning

But where is the memory?
Waveform Extraction (extrapolation)

1. Obtain metric data on finite-radius world tubes $\Gamma$ from a BBH evolution

2. Interpolate between points on the various world tubes

3. Extrapolate to $r \to \infty$
Waveform Extraction (extrapolation)

1. Obtain metric data on finite-radius world tubes $\Gamma$ from a BBH evolution

2. Interpolate between points on the various world tubes

3. Extrapolate to $r \to \infty$

Never solve Einstein’s equations!
Cauchy-characteristic Extraction (CCE)

1. Obtain metric data on a finite-radius world tube $\Gamma$ from a BBH evolution

2. Choose initial data for the null hyper surface $\Sigma_u$

3. Evolve $\Sigma_u$ forward in time

4. Get better waveforms! (and Weyl scalars!)
Dominant Displacement Memory Mode

mass ratio $q = 1$, non-spinning

Keefe Mitman — Introduction, Computing Memory Effects, Correcting Waveforms, BMS Frames, Detectability
mass ratio $q = 1$, non-spinning
Dominant Displacement Memory Mode

mass ratio $q = 1$, non-spinning

Can we “fix” these extrapolated waveforms?

this is what extrapolation looks like
Correcting the Extrapolated Waveforms

\[ J_m \equiv \frac{1}{2} \bar{\delta}^2 \mathcal{D}^{-1} m \]

\[ J_\mathcal{E} \equiv \frac{1}{2} \bar{\delta}^2 \mathcal{D}^{-1} \left[ \frac{1}{4} \int_{-\infty}^{\mu} |\dot{h}|^2 du \right] \]

\[ J_\hat{N} \equiv \frac{1}{2} i \bar{\delta}^2 \mathcal{D}^{-1} D^{-2} \text{Im} \left[ \bar{\delta} \left( \partial_u \hat{N} \right) \right] \]

\[ J_\mathcal{J} \equiv \frac{1}{2} i \bar{\delta}^2 \mathcal{D}^{-1} D^{-2} \text{Im} \left[ \frac{1}{8} \delta \left( 3\dot{h}\bar{\delta}h - 3\dot{h}\bar{\delta}h + \dot{\bar{h}}\delta h - \bar{\delta} \dot{h}h \right) \right] \]

Mass Charge

Energy Flux

Angular Momentum Charge

Angular Momentum Flux

Keefe Mitman — Introduction, Computing Memory Effects, Correcting Waveforms, BMS Frames, Detectability
Correcting the Extrapolated Waveforms

\[ J_m \equiv \frac{1}{2} \delta^2 \mathcal{D}^{-1} m \]

\[ J_{\mathcal{E}} \equiv \frac{1}{2} \delta^2 \mathcal{D}^{-1} \left[ \frac{1}{4} \int_{-\infty}^{u} |\dot{h}|^2 du \right] \]

\[ J_{\hat{N}} \equiv \frac{1}{2} i \delta^2 \mathcal{D}^{-1} D^{-2} \text{Im} \left[ \delta \left( \partial_u \hat{N} \right) \right] \]

\[ J_{\mathcal{J}} \equiv \frac{1}{2} i \delta^2 \mathcal{D}^{-1} D^{-2} \text{Im} \left[ \frac{1}{8} \delta \left( 3h \ddot{\delta} h - 3h \ddot{\delta} h + \dot{h} \ddot{\delta} h - \ddot{h} \ddot{\delta} h \right) \right] \]

Extrapolation missing the flux terms?
• Just compute these and add them on!
Correcting Waveforms without Memory

mass ratio $q = 1$, non-spinning

$\frac{R}{M} \text{Re}[h_{(2,0)}]$
Correcting Waveforms without Memory

mass ratio $q = 1$, non-spinning

$R/M \text{Im}[h_{(3,0)}]$ vs. $(u - u_{peak})/M$ for EXT and CCE.

$|h_{(3,0)}^{CCE} - h_{(3,0)}^{EXT}|$
Quantifying this Improvement

\[ h = (J_m + J_\mathcal{G}) + (J_\hat{N} + J_\mathcal{J}) \]

can also serve as a consistency check!

Waveforms should satisfy the constraint

\[ h - J = 0, \quad \text{where} \quad J \equiv (J_m + J_\mathcal{G}) + (J_\hat{N} + J_\mathcal{J}) \]
Violation of BMS Balance Laws by NR Waveforms

BMS Balance Law Violation (by system)

Legend:
- CCE
- EXT
- EXT w/ Memory

Graph showing violations of BMS balance laws for various waveforms.

Keefe Mitman — Introduction, Computing Memory Effects, Correcting Waveforms, BMS Frames, Detectability
The Importance of BMS Frames

mass ratio $q = 1$, non-spinning

These waveforms are in different "BMS frames"
What is a BMS Frame?

• LIGO assumes their waveforms are in the center-of-mass frame

• So, map waveforms to the center-of-mass frame using the Poincaré center-of-mass charge:

\[
\vec{G} \equiv \frac{1}{\gamma M_B} \frac{1}{4\pi} \int_{S^2} \text{Re} \left[ (\tilde{\delta} r) \left( \hat{N} + u \delta m \right) \right] \, d\Omega
\]
What is a BMS Frame?

- LIGO assumes their waveforms are in the center-of-mass frame

- So, map waveforms to the center-of-mass frame using the Poincaré center-of-mass charge:

\[
\vec{G} \equiv \frac{1}{\gamma M_B} \frac{1}{4\pi} \int_{S^2} \text{Re} \left[ (\tilde{\delta}\vec{r}) \left( \hat{N} + u\delta m \right) \right] d\Omega
\]

What about the supertranslation freedom?
Fixing the Supertranslation Freedom

Fix the supertranslation freedom with a supertranslation charge

- Extend the Bondi four-momentum via the Moreschi supermomentum: \( \Psi^M \equiv \Psi_2 + \sigma \dot{\sigma} + \delta^2 \sigma \)

When this function only has a temporal component, call the BMS frame the “nice section” or the “super rest frame”
Fixing the Supertranslation Freedom

Fix the supertranslation freedom with a supertranslation charge

- Extend the Bondi four-momentum via the Moreschi supermomentum: $\Psi^M \equiv \Psi_2 + \delta \dot{\sigma} + \delta^2 \sigma$

When this function only has a temporal component, call the BMS frame the “nice section” or the “super rest frame”

Note, $\Psi^M$ can never be made exactly zero:

$$\Delta \Psi^M \equiv \int_{u_1}^{u_2} \dot{\Psi}^M(u) \, du = \int_{u_1}^{u_2} \left[ \left( \dot{\Psi}_2 + [\delta \ddot{\sigma} + \delta^2 \dot{\sigma}] \right) + |\dot{\sigma}|^2 \right] \, du = \int_{u_1}^{u_2} |\dot{\sigma}|^2 \, du$$

Vanishes due to the Bianchi identities
Choices for the super rest frame

1. Mapping to the super rest frame using $\mathcal{I}^-$ data
   - Equivalent to mapping to the “PN BMS frame”
     - i.e., what PN waveforms are in
   - What LIGO and other detectors expect

2. Mapping to the super rest frame using $\mathcal{I}^+$ data
   - Essential for performing quasinormal mode (QNM) analyses
Benefits of mapping $\mathcal{F}^-$ to the super rest frame

Errors before and after mapping to PN BMS Frame

- No Alignment
- Time/Phase Alignment
- BMS Frame Alignment

CCE
EXT

Errors before and after mapping to PN BMS Frame

- q1_nospin
- q1_aligned.chi0_2
- q1_aligned.chi0_4
- q1_aligned.chi0_6
- q1_antialigned.chi0_2
- q1_antialigned.chi0_4
- q1_antialigned.chi0_6
- q1_precessing
- q1_superkick
- q4_nospin
- q4_aligned.chi0_4
- q4_antialigned.chi0_4
- q4_precessing

Keefe Mitman — Introduction, Computing Memory Effects, Correcting Waveforms, BMS Frames, Detectability
Combining NR and PN Waveforms

\[ h^{\text{hybrid}} = h^{\text{PN}} + f \left( \frac{u - u_1}{u_2 - u_1} \right) (h^{\text{NR}} - h^{\text{PN}}), \quad \text{where} \quad f(x) = \begin{cases} 0 & x \leq 0, \\ \left( 1 + \exp \left[ \frac{1}{x-1} + \frac{1}{x} \right] \right)^{-1} & 0 < x < 1, \\ 1 & x \geq 1. \end{cases} \]
Benefits of mapping $\mathcal{I}^+$ to the super rest frame

Quasinormal Mode Mismatch

$M(NR_{(2,2)}, h_{(2,2)}^{QNM})$

- EXT
- CCE
- CCE (super rest frame)

Number of Overtones

$10^0$ $10^{-1}$ $10^{-2}$ $10^{-3}$ $10^{-4}$ $10^{-5}$ $10^{-6}$
Example Memory Values

\[ \times 10^{-21} \]

- strain
- Displacement Memory
- Spin Memory \((2 \times 10^{-3} \text{ s})\)

<table>
<thead>
<tr>
<th>Label</th>
<th>Displacement</th>
<th>Spin</th>
<th>Strain</th>
</tr>
</thead>
<tbody>
<tr>
<td>q1_nospin</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q1_aligned.chi0.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q1_aligned.chi0.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q1_aligned.chi0.6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q1_antialigned.chi0.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q1_antialigned.chi0.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q1_antialigned.chi0.6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q1_superkick</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q4_nospin</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q4_aligned.chi0.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>q4_antialigned.chi0.4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Detectability of Gravitational Memory

It won’t be easy!

- Can we even detect memory with an interferometer?

- LIGO SNRs for an idealized GW150914 event:
  - Displacement: $\sim 2$
  - sub-leading: $\sim 0.05$

- Will rely on “stacking” events
  - Need to break a memory sign degeneracy
  - Need $\sim \mathcal{O}(2000)$ events
Summary

- Can now produce waveforms with memory
- Waveforms without memory can be corrected
- Waveform accuracy can be tested using the BMS balance laws
- Fixing the BMS frame (via the super rest frame) is critical for modeling and analysis
- Detectability estimates improve with NR waveforms

“Fixing the BMS Frame of Numerical Relativity Waveforms” (on arXiv later this week), K. Mitman, et al.