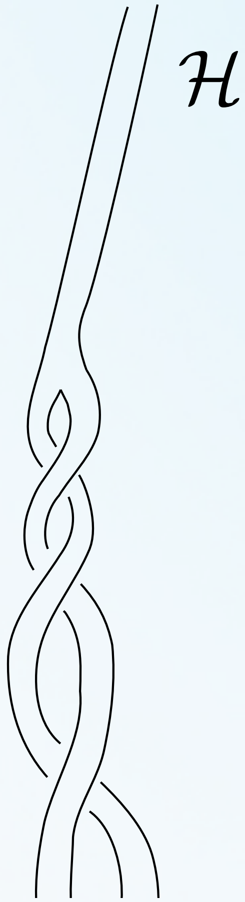
Gravitational Memory in Numerical Relativity

- Computing Memory Effects,
- Correcting Waveforms,
- And the importance of BMS Frames

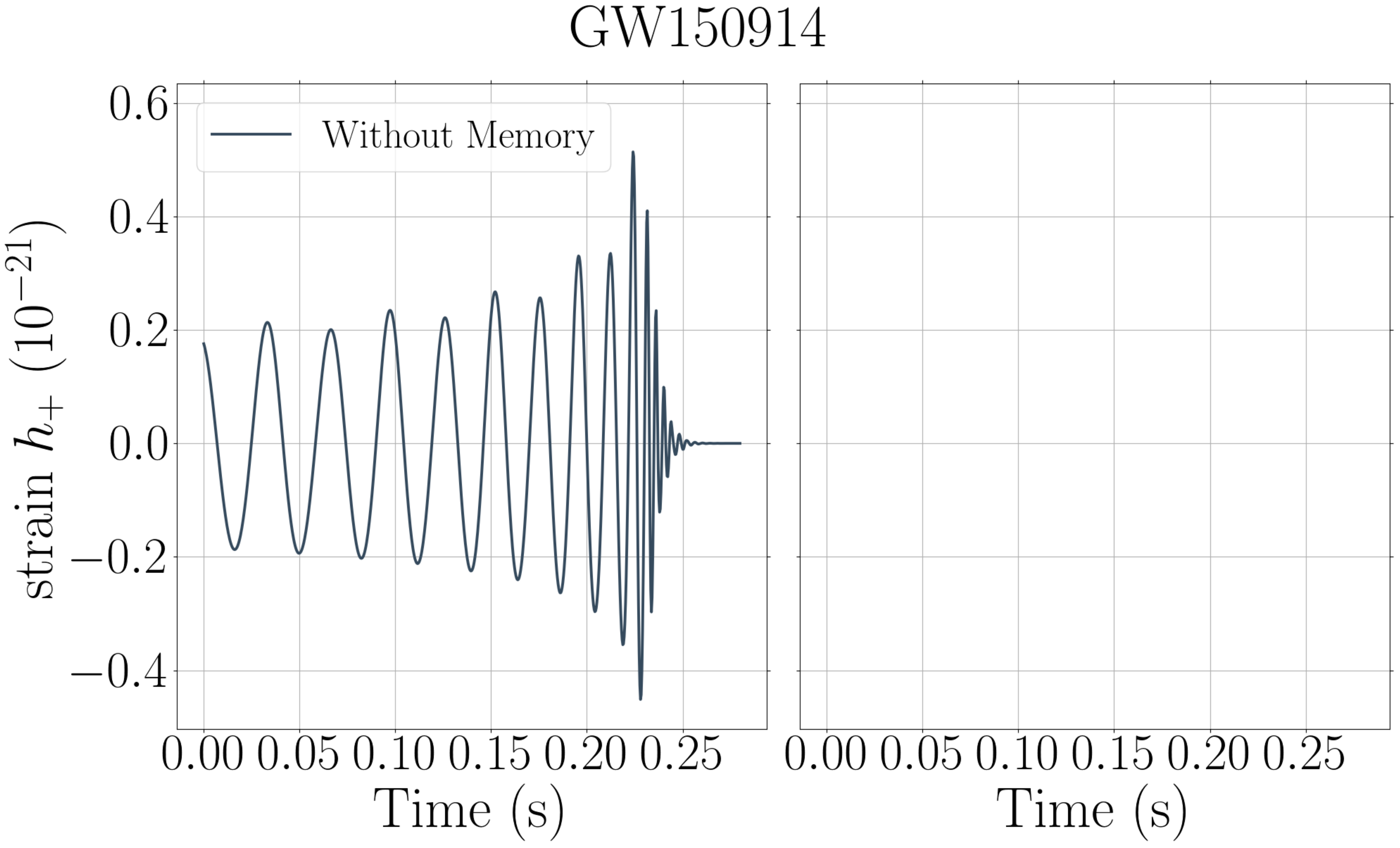


Keefe Mitman

Conference on Gravitational Scattering, Inspiral and Radiation (virtual)
Galileo Galilei Institute, May 6, 2021

Caltech
SIMULATING eXTREME SPACETIMES
Black holes, neutron stars, and beyond...

What we are currently seeing...



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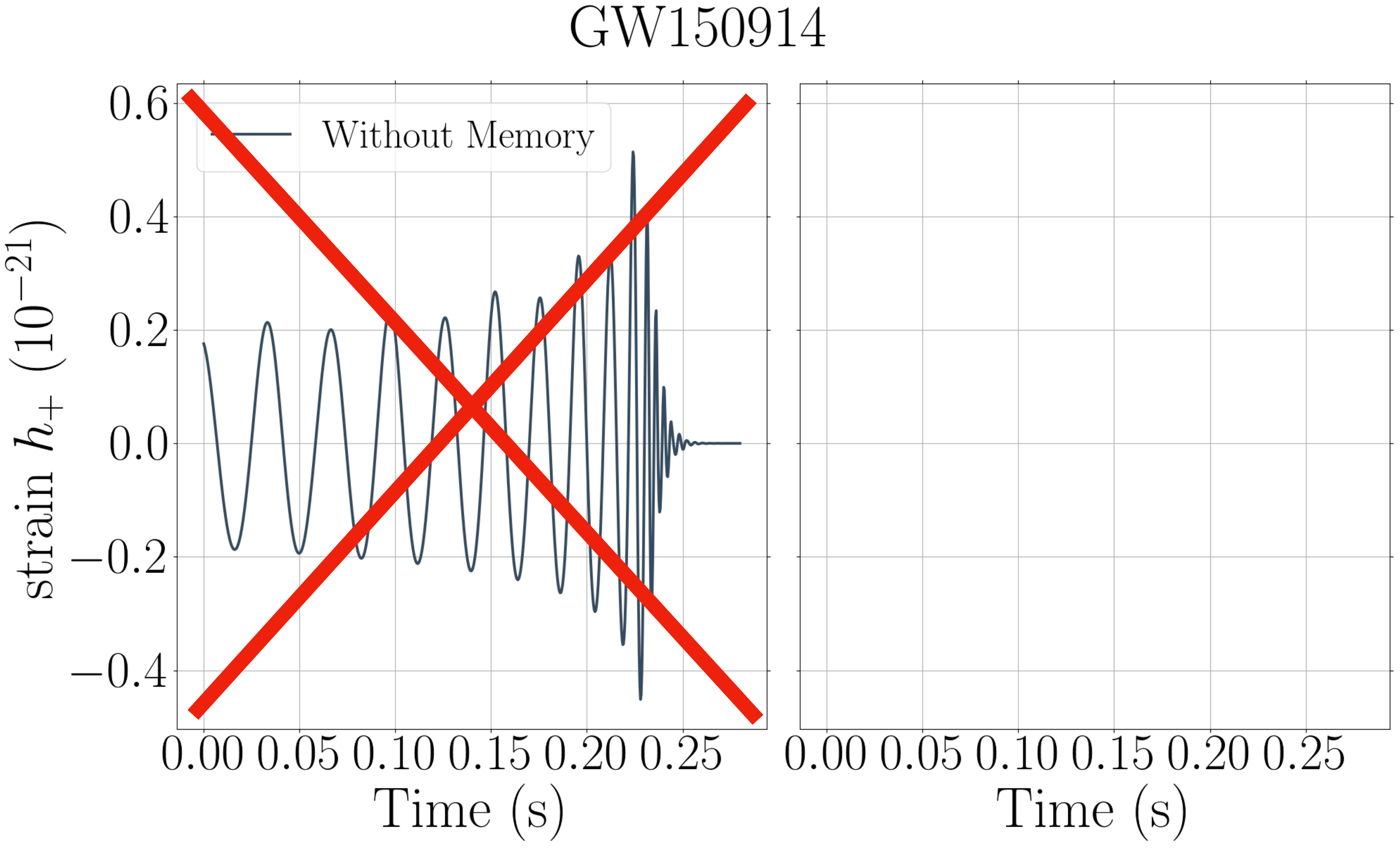

What we should be seeing...

What is gravitational memory?

Early Work:

- Zel'dovich and Polnarev (1974).
- Braginsky and Thorne (1987).
- Christodoulou (1991).
- Thorne (1992).

Ordinary (linear) Memory:

Massive objects traveling to asymptotic infinity as $t \to \infty$

$$\Delta h_{ij} = \Delta \sum_{A=1}^{N} \frac{4M_A}{r\sqrt{1-v_A^2}} \left(\frac{v_A^i v_A^j}{1 - v_A \cos\left(\theta_A\right)} \right)^{\mathrm{TT}} .$$

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Null (non-linear) Memory:

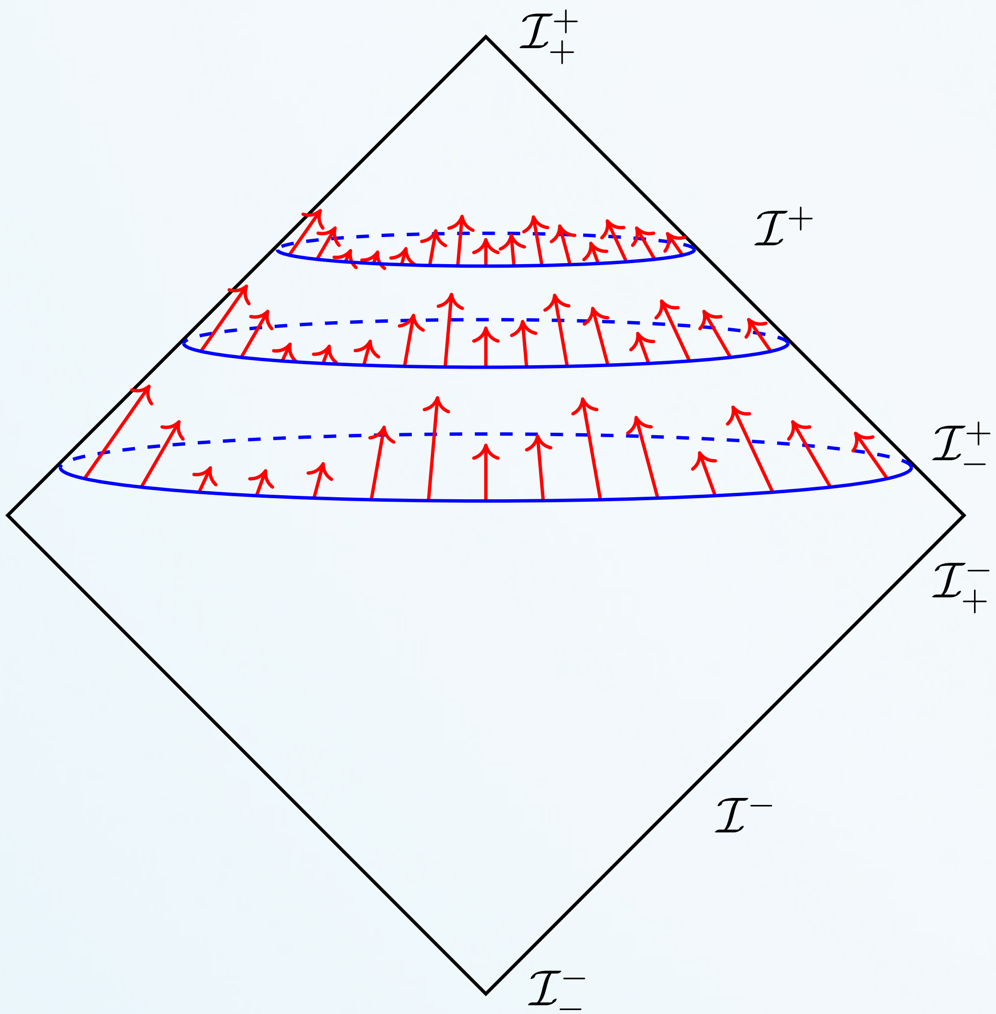
Null radiation traveling to asymptotic infinity as $r, t \to \infty$ at fixed $u \equiv t - r$

$$\Delta h_{ij} = \frac{4}{r} \int \frac{dE}{d\Omega'} \left(\frac{\xi^{i'} \xi^{j'}}{1 - \cos\left(\theta'\right)} \right)^{\mathrm{TT}} d\Omega'$$

where ξ' is a unit vector from the source to $d\Omega'$.

The BMS Group

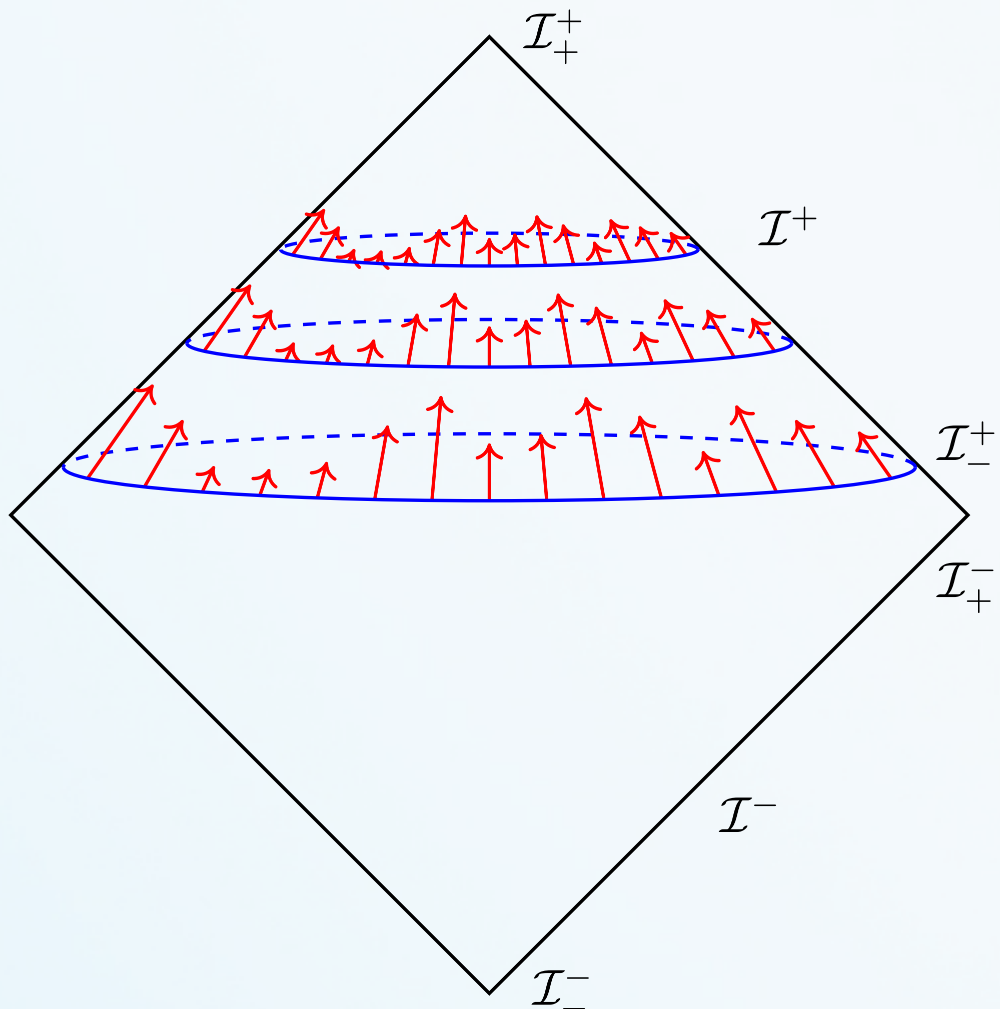
BMS Group = Lorentz Group $\ltimes$ supertranslations



The BMS Group

$$\text{BMS Group} = \text{Lorentz Group} \ltimes \text{supertranslations}$$

$$\text{Extended BMS Group} = \text{LCKV}(S^2) \ltimes \text{supertranslations}$$
$$\text{Generalized BMS Group} = \text{Diff}(S^2) \ltimes \text{supertranslations}$$



The BMS Balance Laws

Bondi-Sachs Metric:

$$ds^2 = -\left(1 - \frac{2m}{r} + \mathcal{O}\left(\frac{1}{r^2}\right)\right) du^2 - 2\left(1 + \mathcal{O}\left(\frac{1}{r^2}\right)\right) dudr$$
$$+ r^2\left(q_{AB} + \frac{1}{r}C_{AB} + \mathcal{O}\left(\frac{1}{r^2}\right)\right)\left(d\theta^A - \mathcal{U}^A du\right)\left(d\theta^B - \mathcal{U}^B du\right)$$

where

$$\mathcal{U}^A = -\frac{1}{2r^2}D_B C^{AB} + \frac{1}{r^3}\left[-\frac{2}{3}N^A + \frac{1}{16}D^A\left(C_{BC}C^{BC}\right) + \frac{1}{2}C^{AB}D^C C_{BC}\right] + \mathcal{O}\left(\frac{1}{r^4}\right)$$

The BMS Balance Laws

Bondi-Sachs Metric:

where

- Identify BMS Charges
- Apply Einstein's field equations
  to obtain evolution equations
- Understand them to be balance laws

The BMS Balance Laws

$$\mathrm{Re}\left[h\right]\left(u,\theta,\phi\right)=\begin{pmatrix}\text{Bondi Mass Charge}\\+\\\text{Energy Flux}\end{pmatrix}$$

$$\mathrm{Im}\left[h\right]\left(u,\theta,\phi\right)=\frac{d}{du}\begin{pmatrix}\text{Angular Momentum Charge}\\+\\\text{Angular Momentum Flux}\end{pmatrix}$$

The BMS Balance Laws (for memory)

$$\Delta \mathrm{Re}\left[h\right](\theta,\phi) = \Delta \begin{pmatrix} \text{Bondi Mass Charge} \\ + \\ \text{Energy Flux} \end{pmatrix}$$

Bondi Mass Charge → ***Ordinary Electric Memory***

Energy Flux → ***Null Electric Memory***

$$\Delta \mathrm{Im}\left[h\right](\theta,\phi) = \Delta \frac{d}{du} \begin{pmatrix} \text{Angular Momentum Charge} \\ + \\ \text{Angular Momentum Flux} \end{pmatrix}$$

Angular Momentum Charge → ***Ordinary Magnetic Memory***

Angular Momentum Flux → ***Null Magnetic Memory***

What we know about memory

Gravitational memory = the relative displacement of (initially comoving) observers induced by the passage of gravitational radiation

$$\Delta\xi^\alpha(u_1,u_0) = \Delta K^\alpha{}_\beta(u_1,u_0)\,\xi^\beta(u_0) + (u_1-u_0)\,\Delta H^\alpha{}_\beta\,\dot{\xi}^\beta(u_0) + \cdots$$

$\Delta K^\alpha{}_\beta \rightarrow$ usual displacement memory } supertranslations

$\Delta H^\alpha{}_\beta \rightarrow$ sub-leading displacement memory } super Lorentz transformations

- spin memory (magnetic)
- center-of-mass memory (electric)

Charge-Flux breakdown of the strain

$$h = \left(J_m + J_{\mathscr{E}}\right) + \left(J_{\hat{N}} + J_{\mathscr{J}}\right) \quad \text{[from the BMS balance laws]}$$

where

$$J_m \equiv \frac{1}{2}\bar{\eth}^2\mathfrak{D}^{-1}m \longrightarrow$$ **_Mass Charge_**

$$J_{\mathscr{E}} \equiv \frac{1}{2}\bar{\eth}^2\mathfrak{D}^{-1}\left[\frac{1}{4}\int_{-\infty}^{u}|\dot{h}|^2 du\right] \longrightarrow$$ **_Energy Flux_**

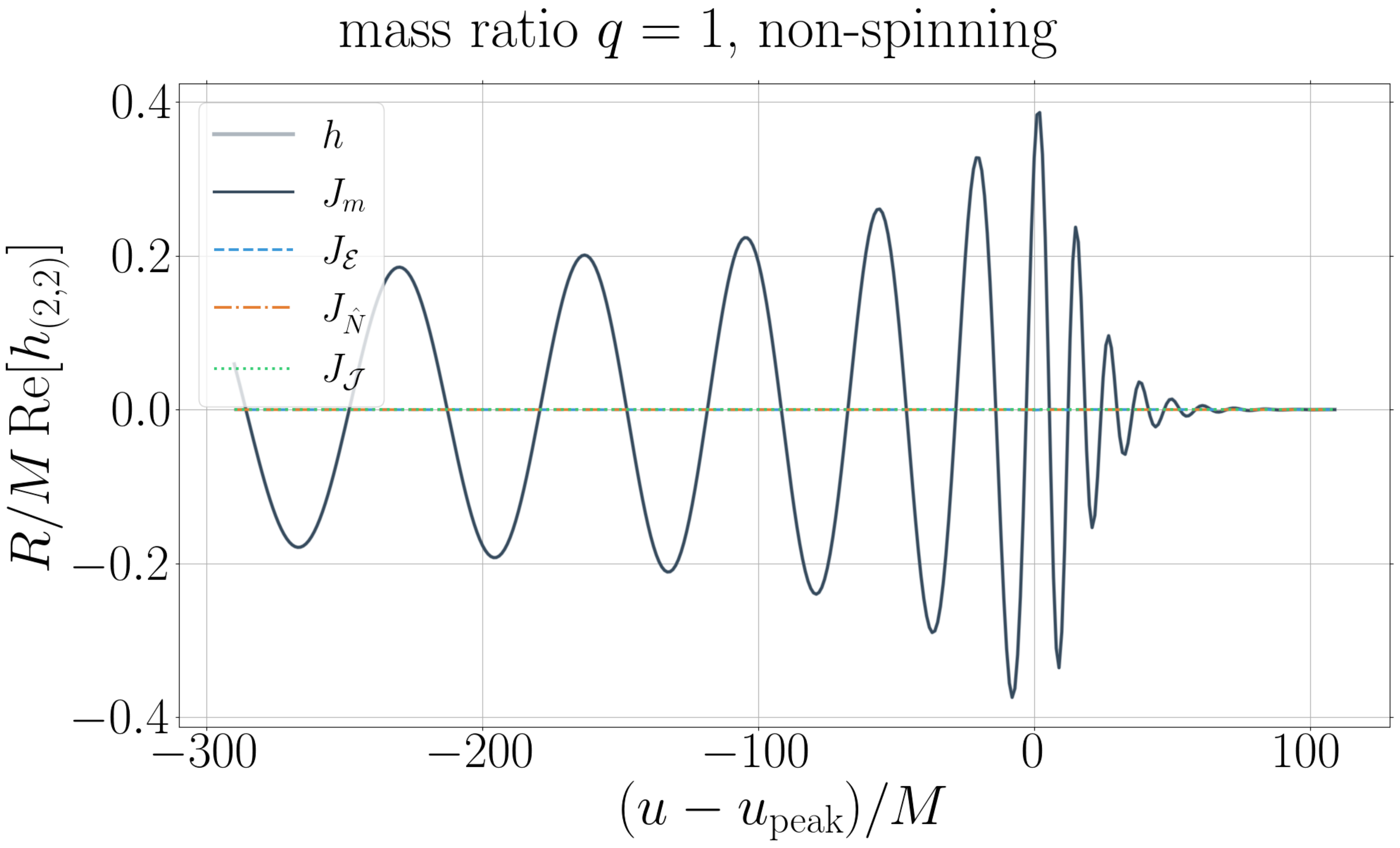
$$J_{\hat{N}} \equiv \frac{1}{2}i\bar{\eth}^2\mathfrak{D}^{-1}D^{-2}\mathrm{Im}\left[\bar{\eth}\left(\partial_u\hat{N}\right)\right] \longrightarrow$$ **_Angular Momentum Charge_**

$$J_{\mathscr{J}} \equiv \frac{1}{2}i\bar{\eth}^2\mathfrak{D}^{-1}D^{-2}\mathrm{Im}\left[\frac{1}{8}\eth\left(3h\bar{\eth}\dot{\bar{h}} - 3\dot{h}\bar{\eth}\bar{h} + \dot{\bar{h}}\bar{\eth}h - \bar{h}\bar{\eth}\dot{h}\right)\right]$$ **_Angular Momentum Flux_**

Note: D^2 is the Laplacian on S^2, $\mathfrak{D} \equiv \frac{1}{8}D^2(D^2+2)$, $\eth\eta = -\left(\sin(\theta)^s\right)\left\{\frac{\partial}{\partial\theta} + i\csc(\theta)\frac{\partial}{\partial\phi}\right\}\left[\left(\sin(\theta)\right)^{-s}\eta\right]$,

$m = -\,\mathrm{Re}\left[\Psi_2 + \frac{1}{4}\dot{h}\bar{h}\right]$, and $\hat{N} = 2\Psi_1 - \frac{1}{4}\bar{h}\eth h + u\eth m + \frac{1}{8}\eth\left(h\bar{h}\right)$.

We can use these to study contributions to the strain!



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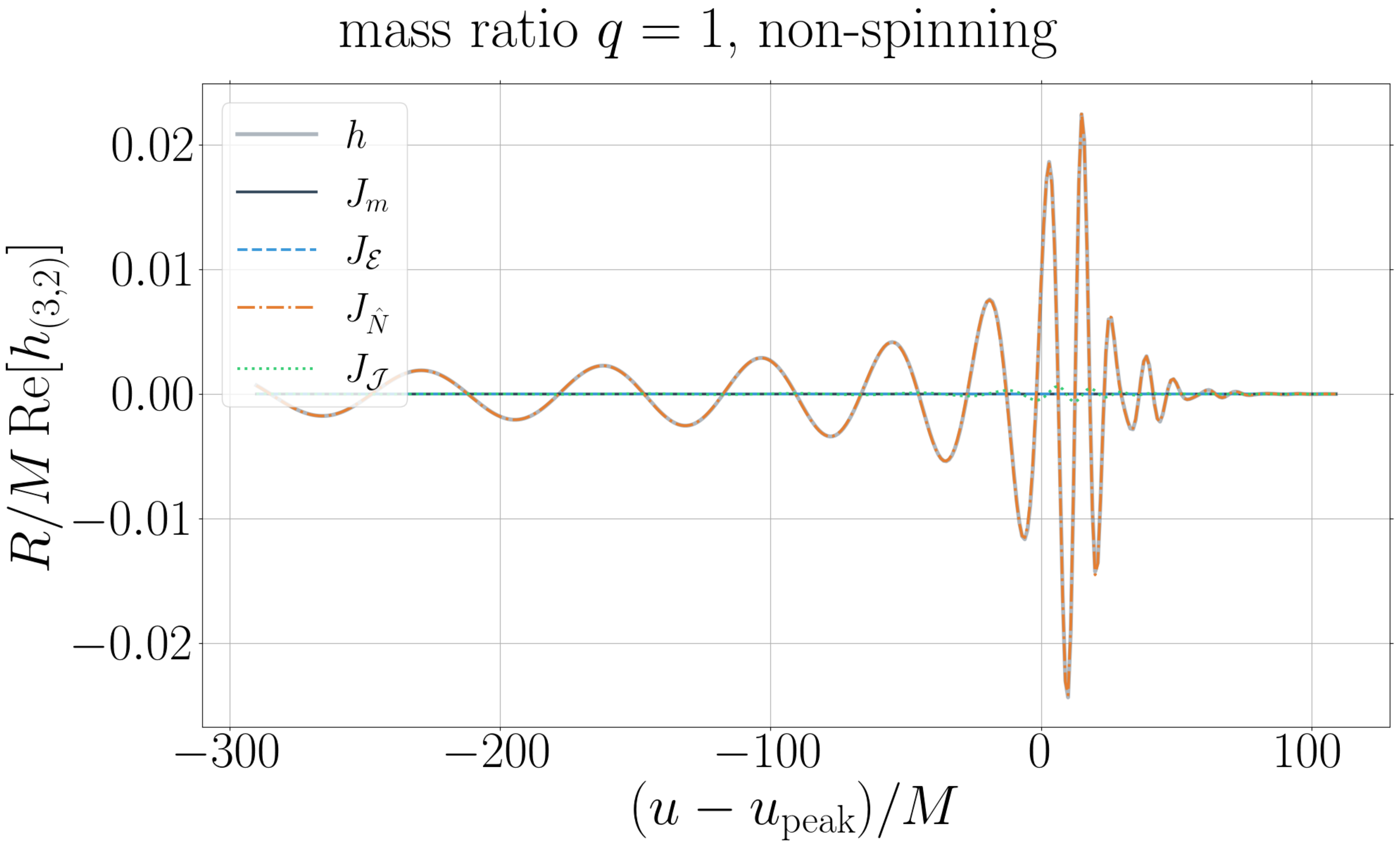

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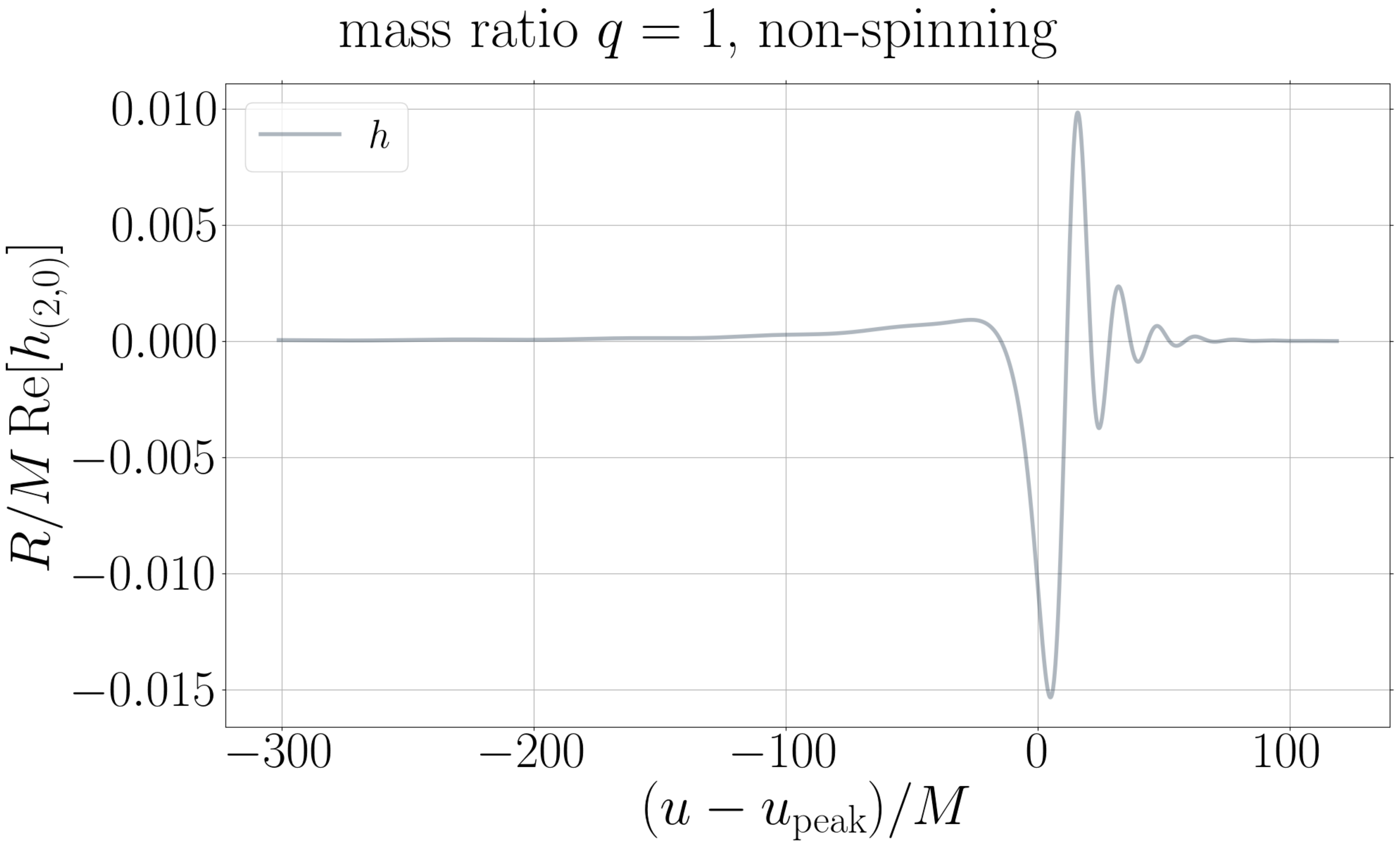

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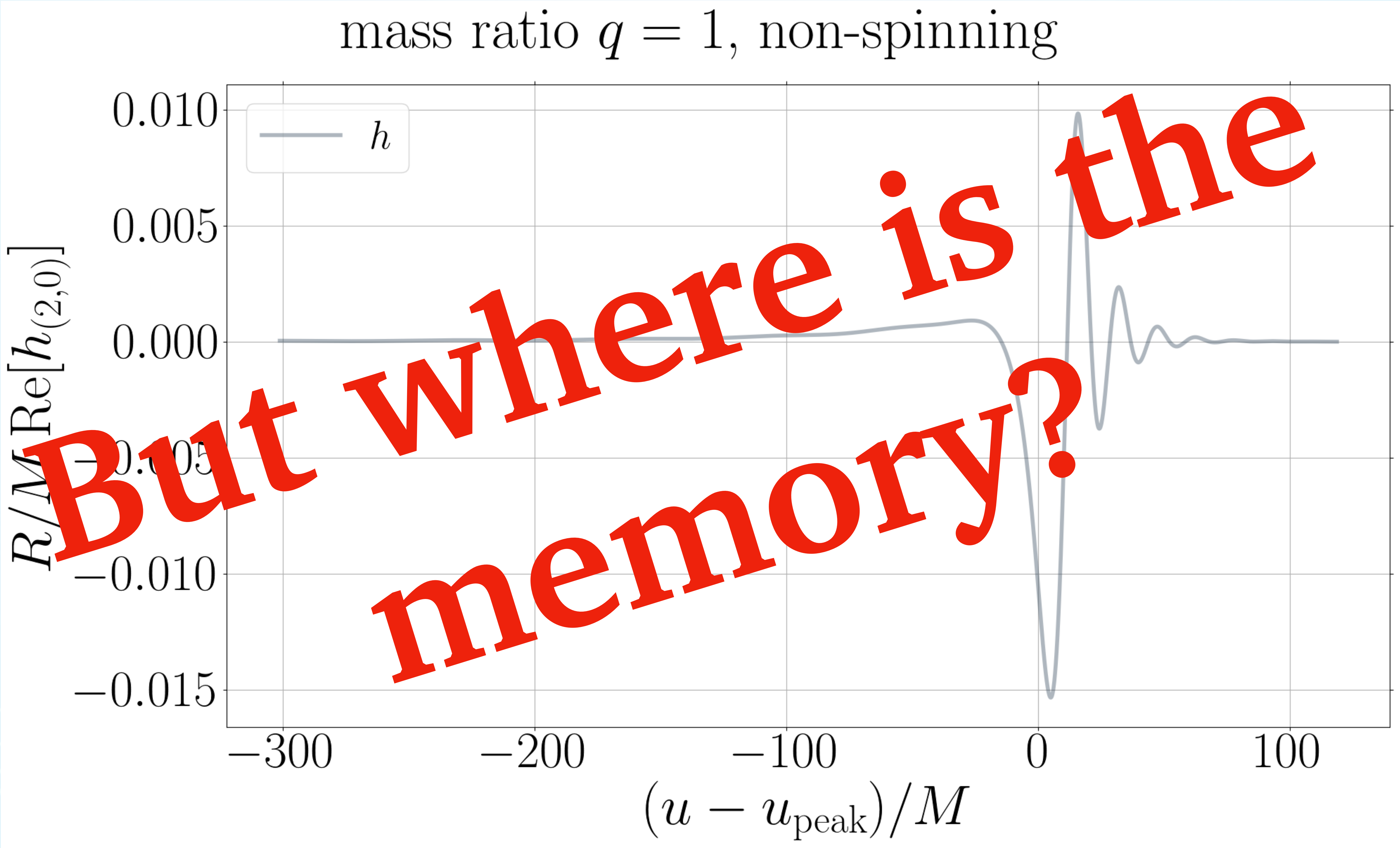
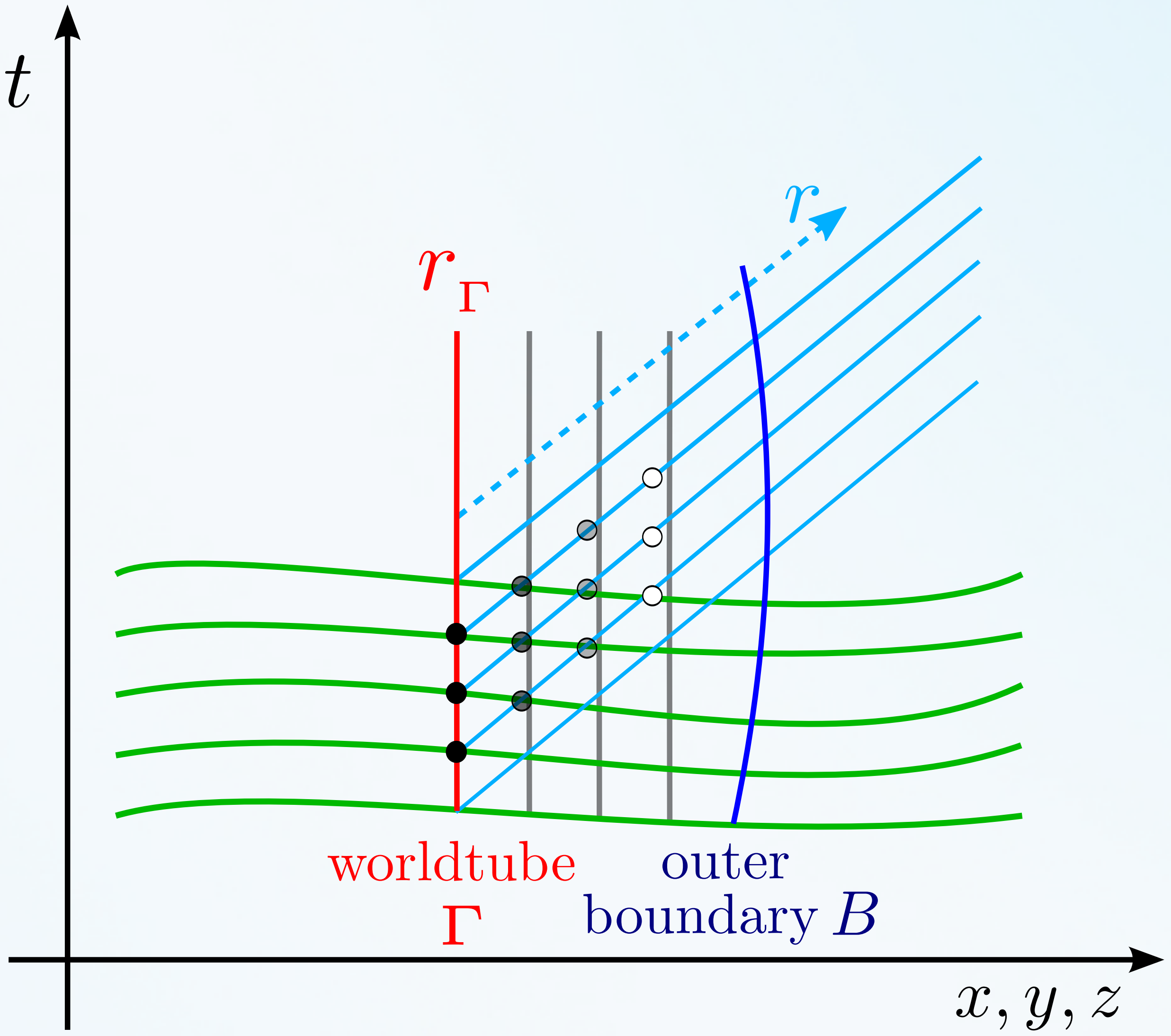


Waveform Extraction (extrapolation)

1. Obtain metric data on finite-radius world tubes Γ from a BBH evolution

2. Interpolate between points on the various world tubes

3. Extrapolate to $r \to \infty$



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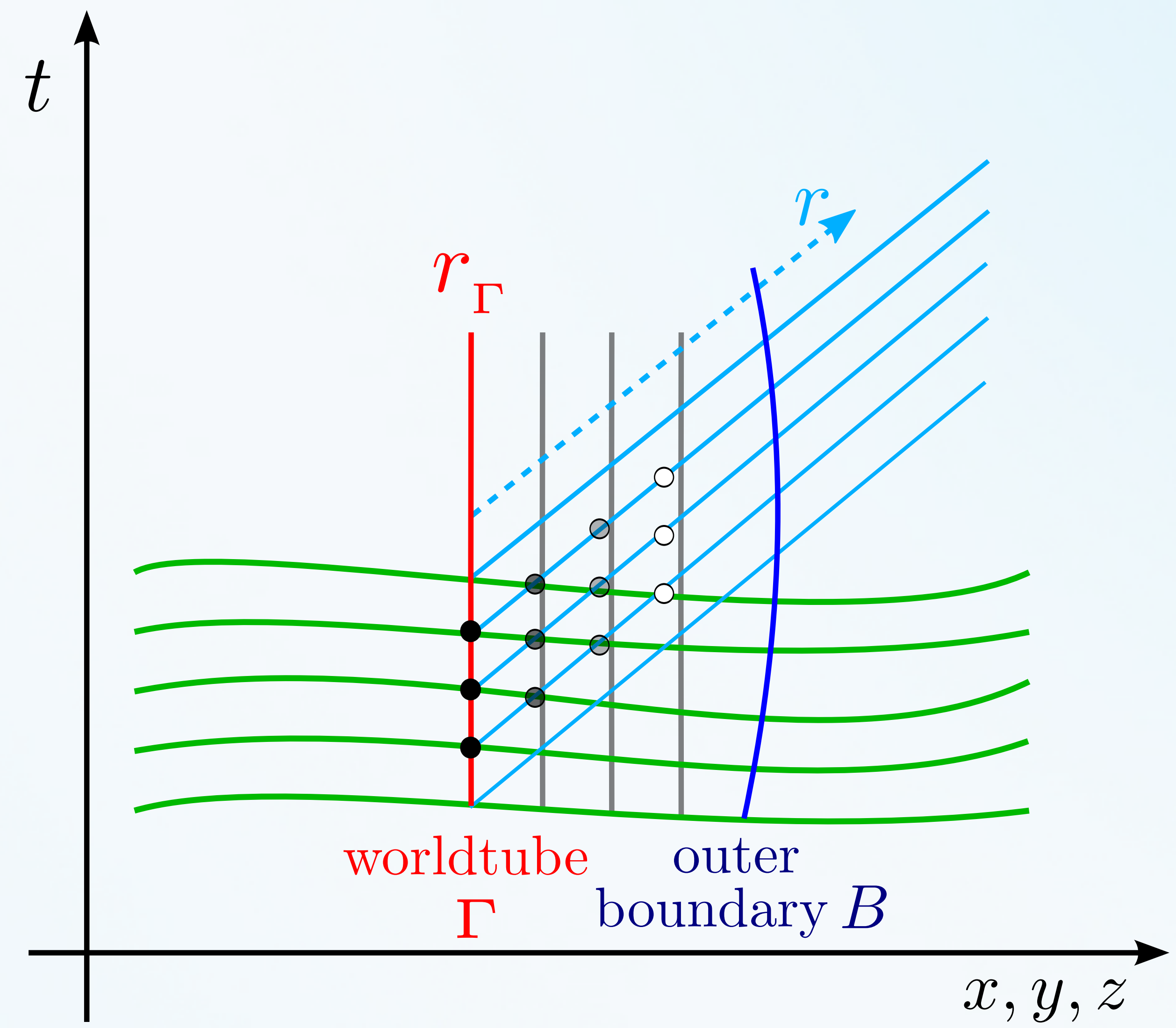


Never solve Einstein's equations!

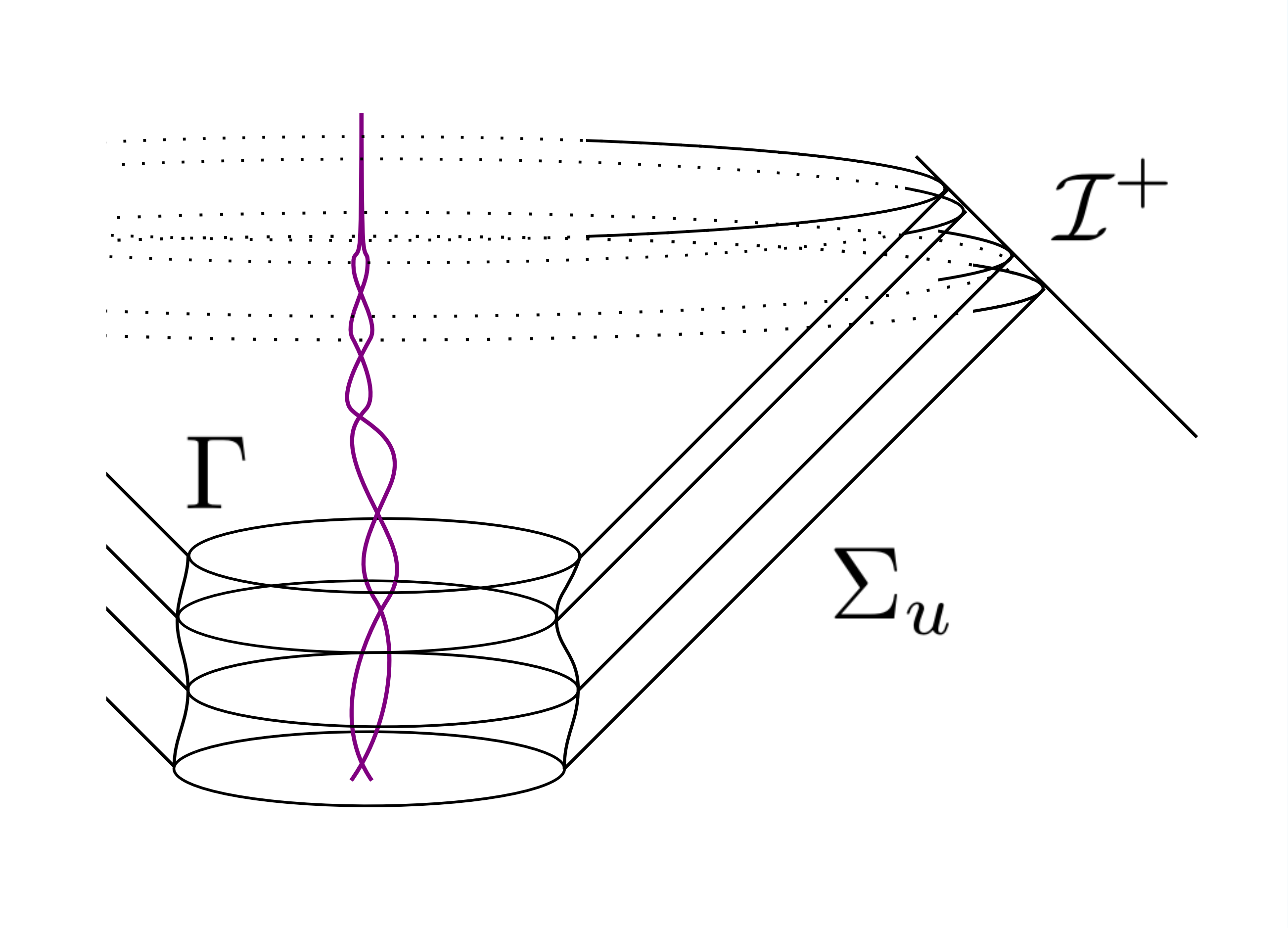
Cauchy-characteristic Extraction (CCE)

1. Obtain metric data on a finite-radius world tube Γ from a BBH evolution

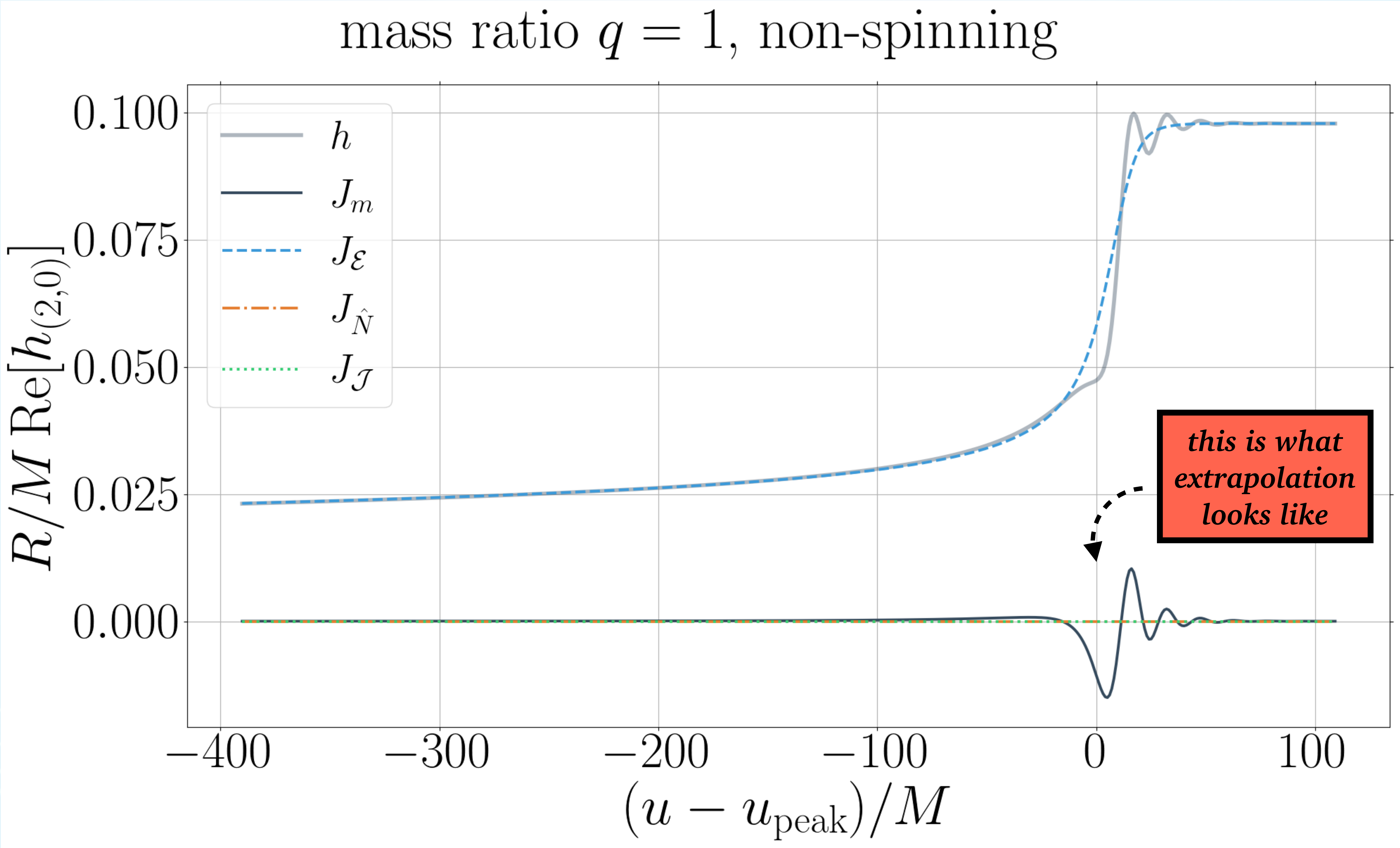
2. Choose initial data for the null hyper surface Σ_u

3. Evolve Σ_u forward in time

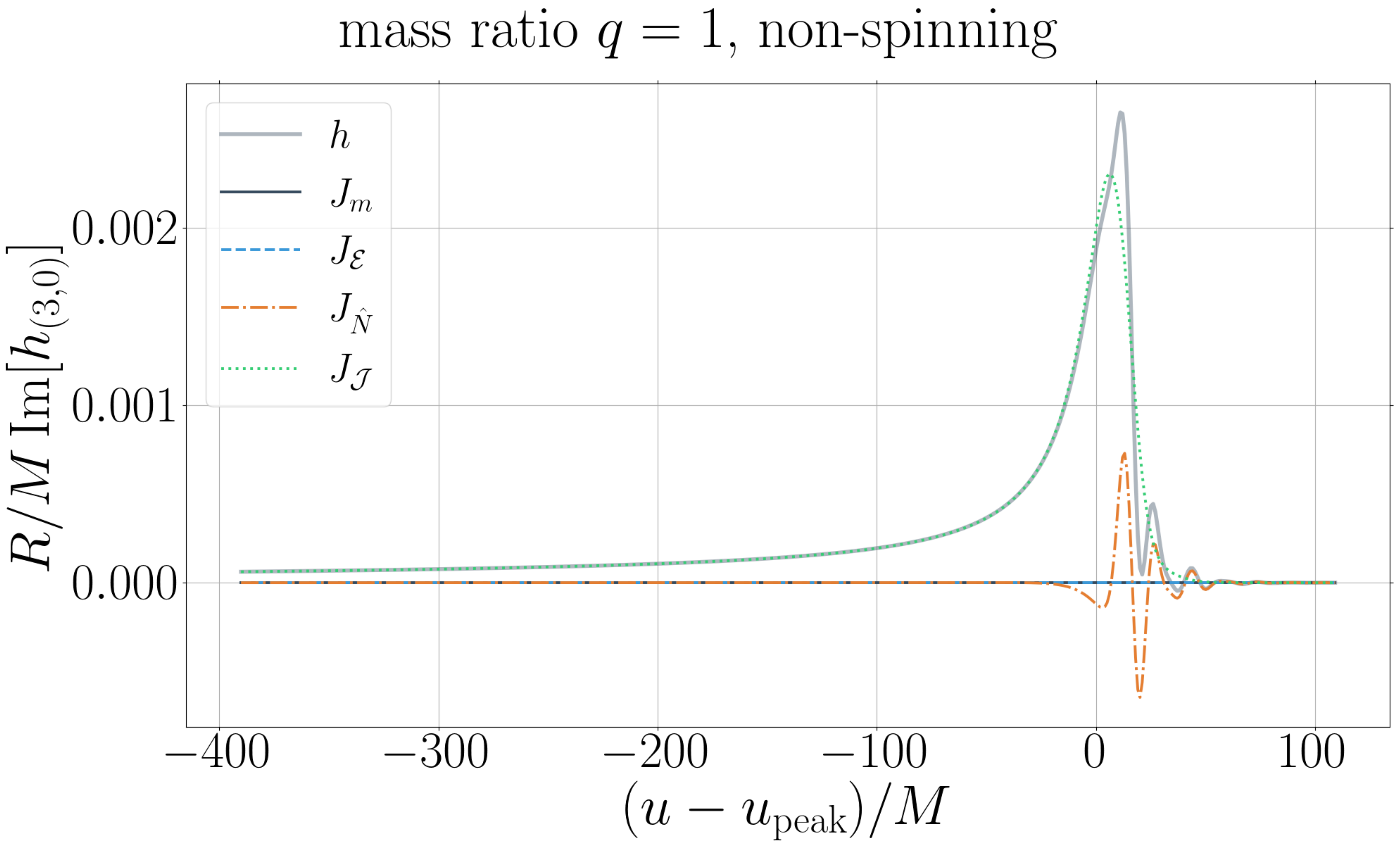
4. Get better waveforms! (and Weyl scalars!)



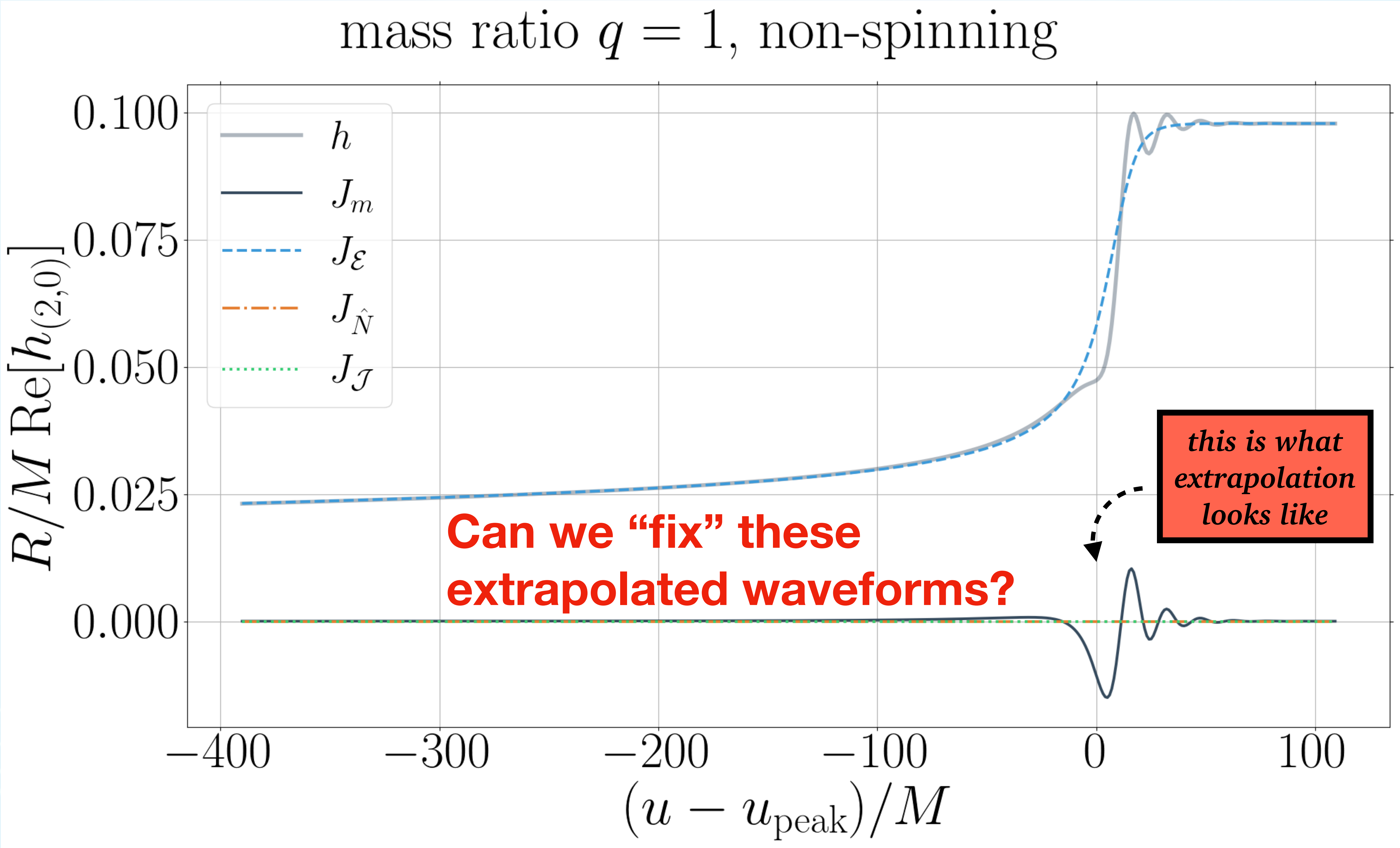
Dominant Displacement Memory Mode



Dominant sub-leading Displacement Memory Mode



Dominant Displacement Memory Mode



Correcting the Extrapolated Waveforms

$$J_m \equiv \frac{1}{2}\bar{\eth}^2\mathfrak{D}^{-1}m$$ → **Mass Charge**

$$J_{\mathcal{E}} \equiv \frac{1}{2}\bar{\eth}^2\mathfrak{D}^{-1}\left[\frac{1}{4}\int_{-\infty}^{u}|\dot{h}|^2\,du\right]$$ → **Energy Flux**

$$J_{\hat{N}} \equiv \frac{1}{2}i\bar{\eth}^2\mathfrak{D}^{-1}D^{-2}\text{Im}\left[\bar{\eth}\left(\partial_u\hat{N}\right)\right]$$ → **Angular Momentum Charge**

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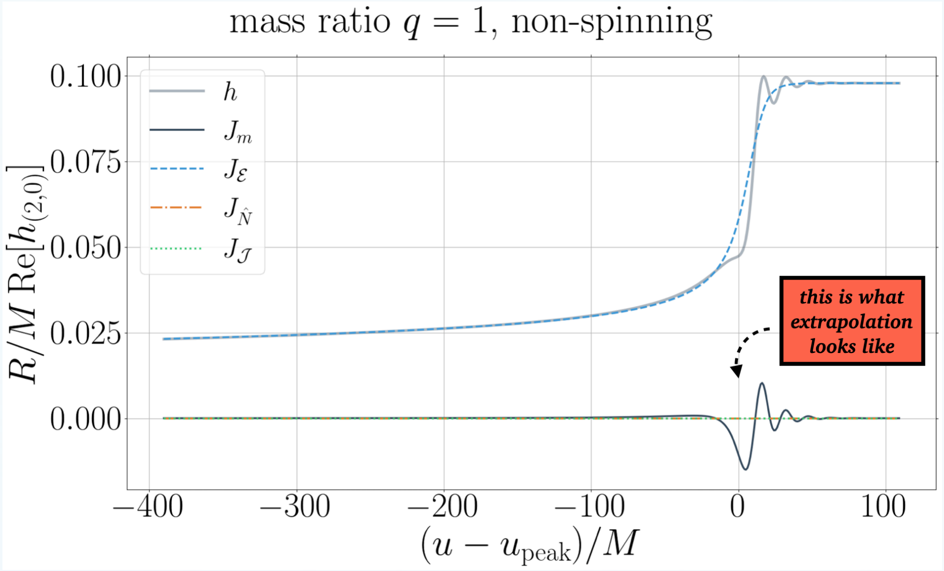

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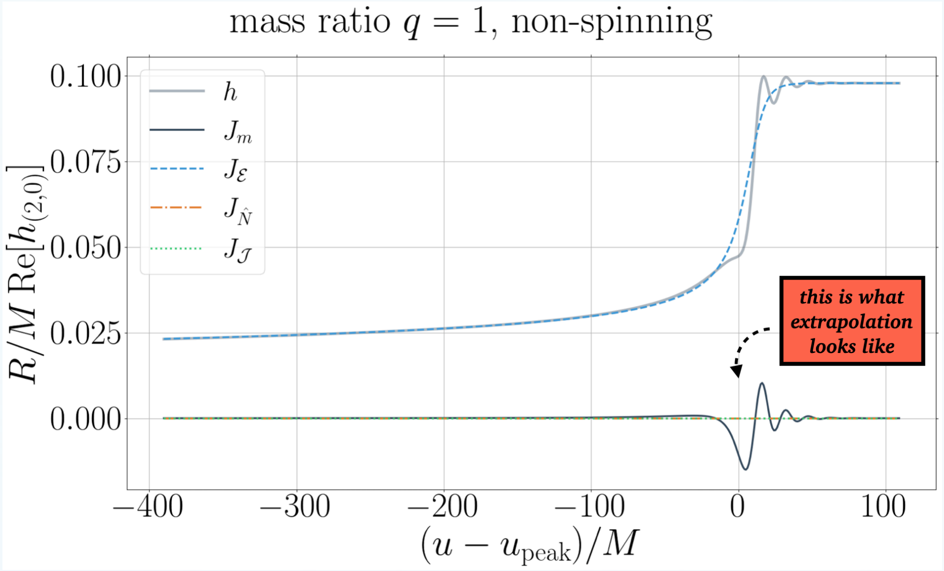
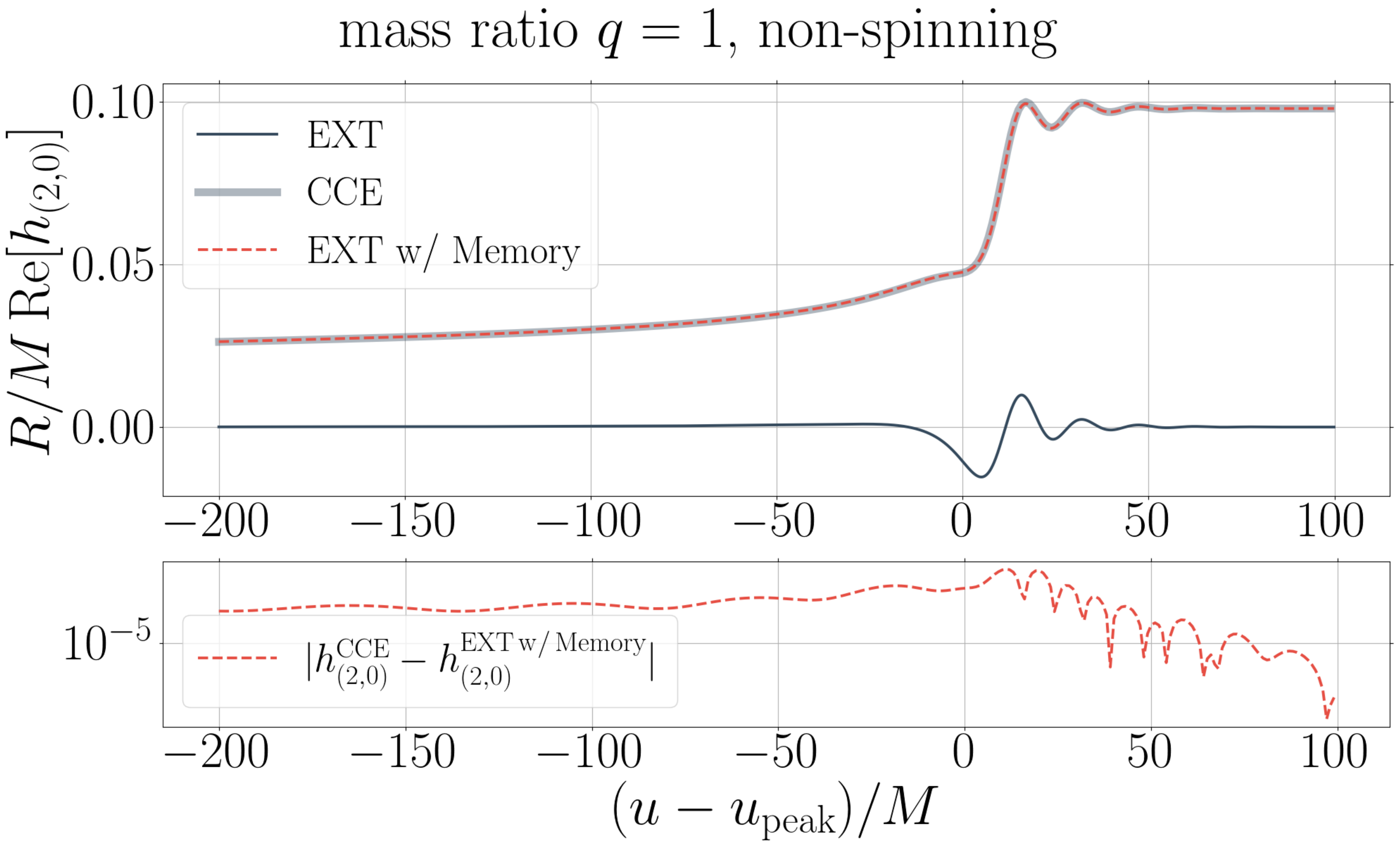
Extrapolation missing the flux terms?

- Just compute these and add them on!



Correcting Waveforms without Memory



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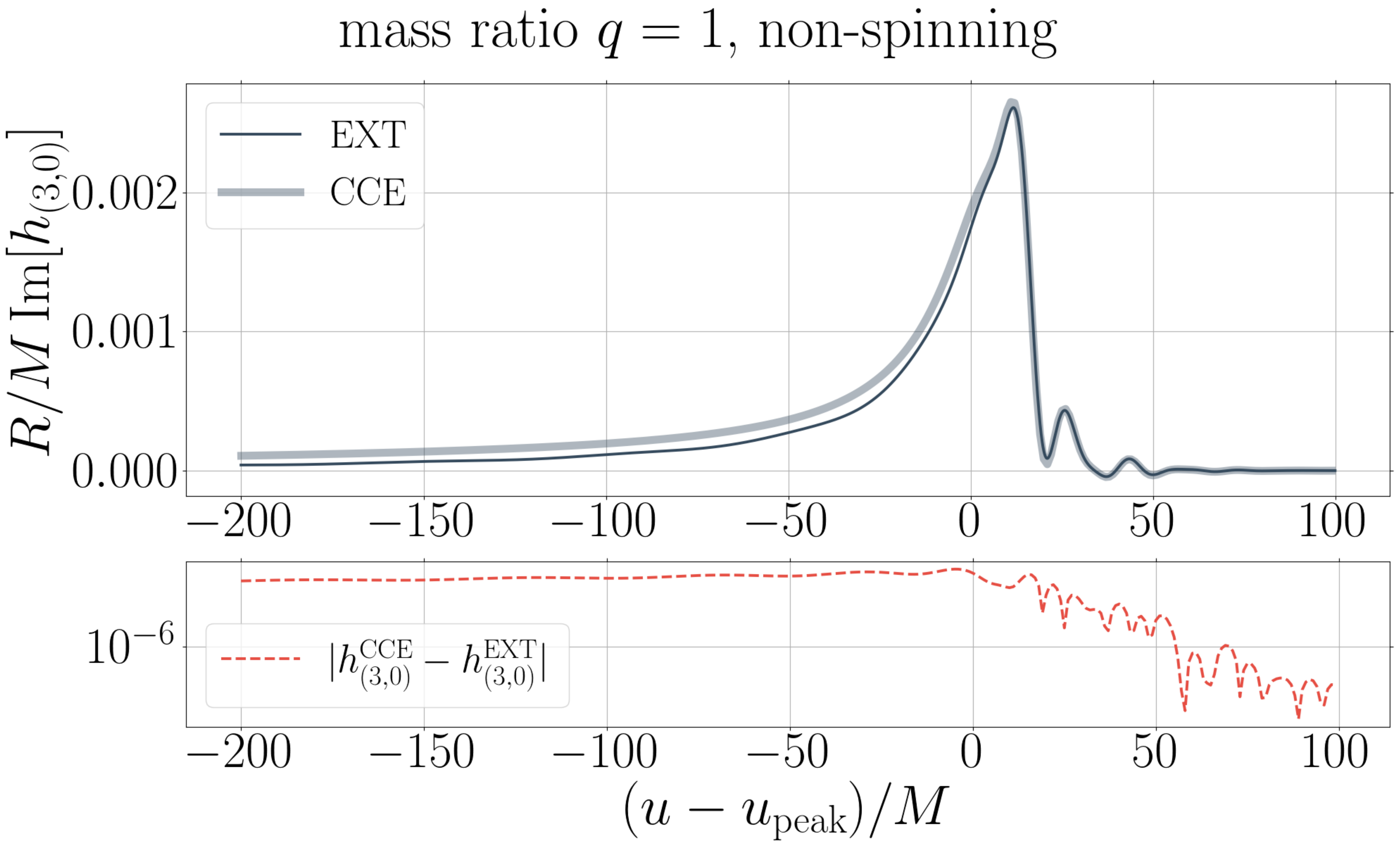


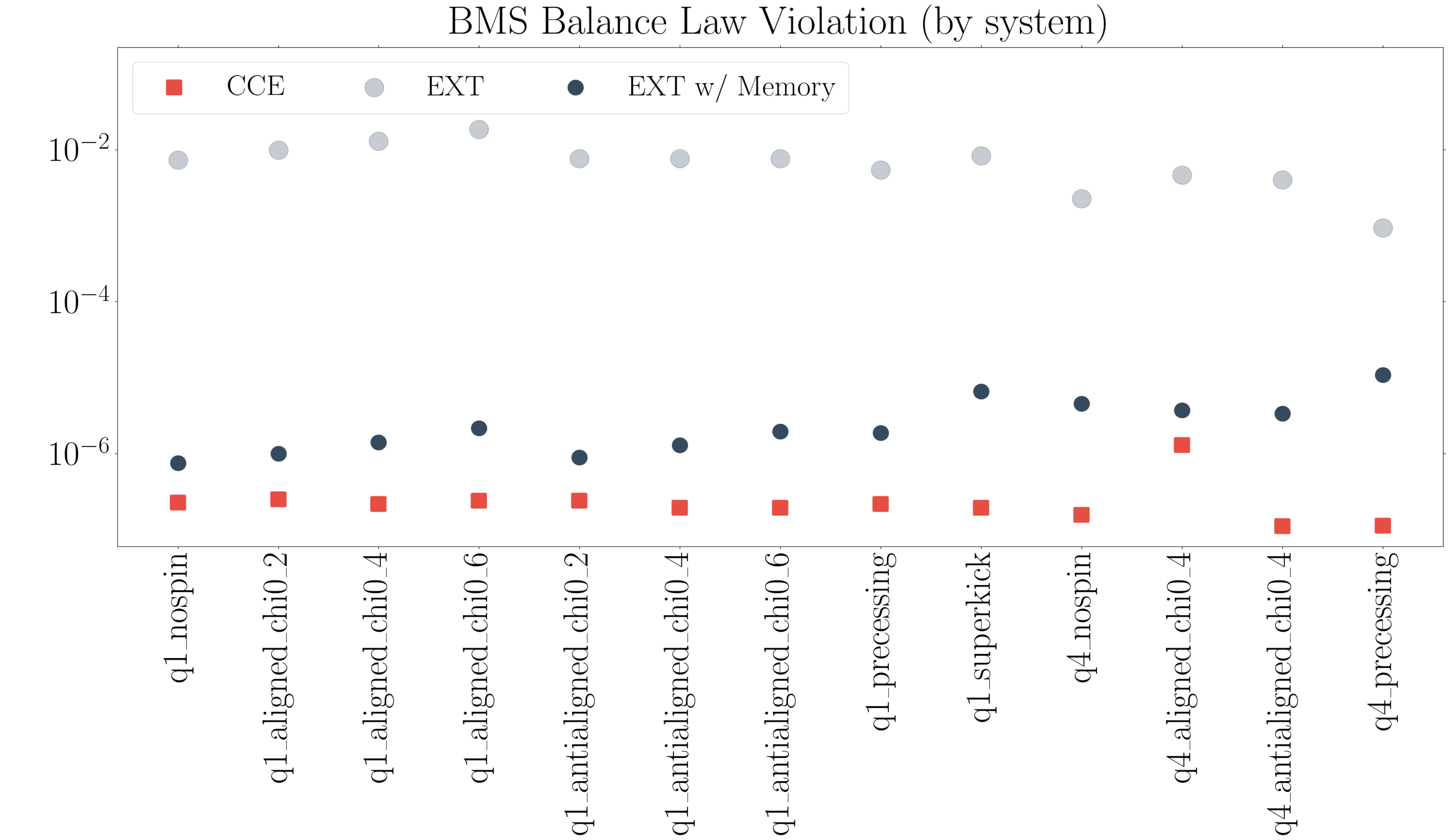
Quantifying this Improvement

$h = \left(J_m + J_{\mathscr{E}}\right) + \left(J_{\hat{N}} + J_{\mathscr{J}}\right)$ can also serve as a consistency check!

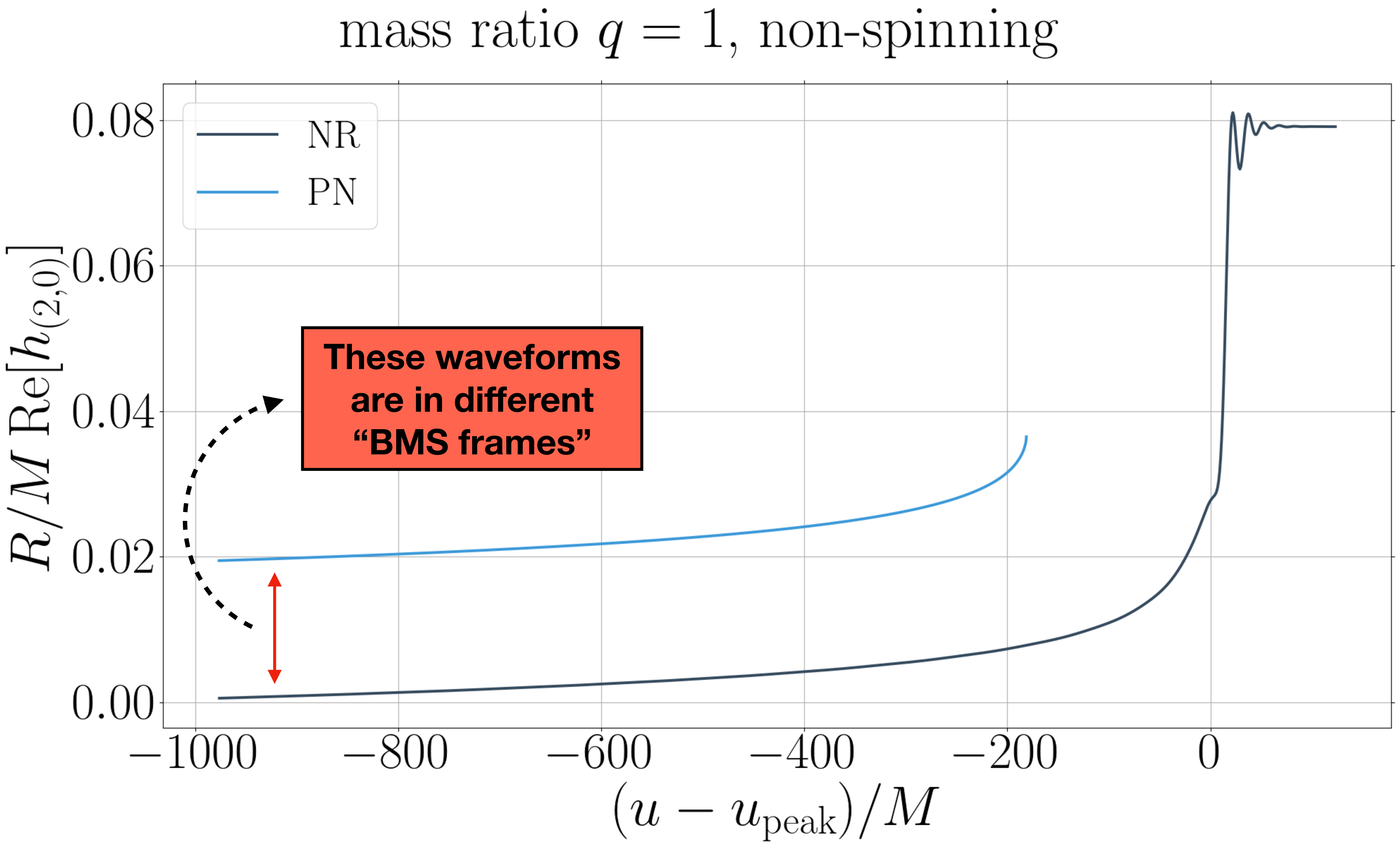
Waveforms should satisfy the constraint

$$h - J = 0, \quad \text{where} \quad J \equiv \left(J_m + J_{\mathscr{E}}\right) + \left(J_{\hat{N}} + J_{\mathscr{J}}\right)$$

Violation of BMS Balance Laws by NR Waveforms



The Importance of BMS Frames



What is a BMS Frame?

- LIGO assumes their waveforms are in the center-of-mass frame

- So, map waveforms to the center-of-mass frame using the Poincaré center-of-mass charge:

$$\vec{G} \equiv \frac{1}{\gamma M_B} \frac{1}{4\pi} \int_{S^2} \mathrm{Re}\left[\left(\bar{\eth} \vec{r} \right) \left(\hat{N} + u \eth m \right) \right] d\Omega$$

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What about the supertranslation freedom?

Fixing the Supertranslation Freedom

Fix the supertranslation freedom with a supertranslation charge

- Extend the Bondi four-momentum via the Moreschi supermomentum: $\Psi^{\mathrm{M}} \equiv \Psi_2 + \sigma\dot{\bar{\sigma}} + \eth^2\bar{\sigma}$

  When this function only has a temporal component, call the BMS frame the “nice section” or the “super rest frame”

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Note, Ψ^{M} can never be made exactly zero:

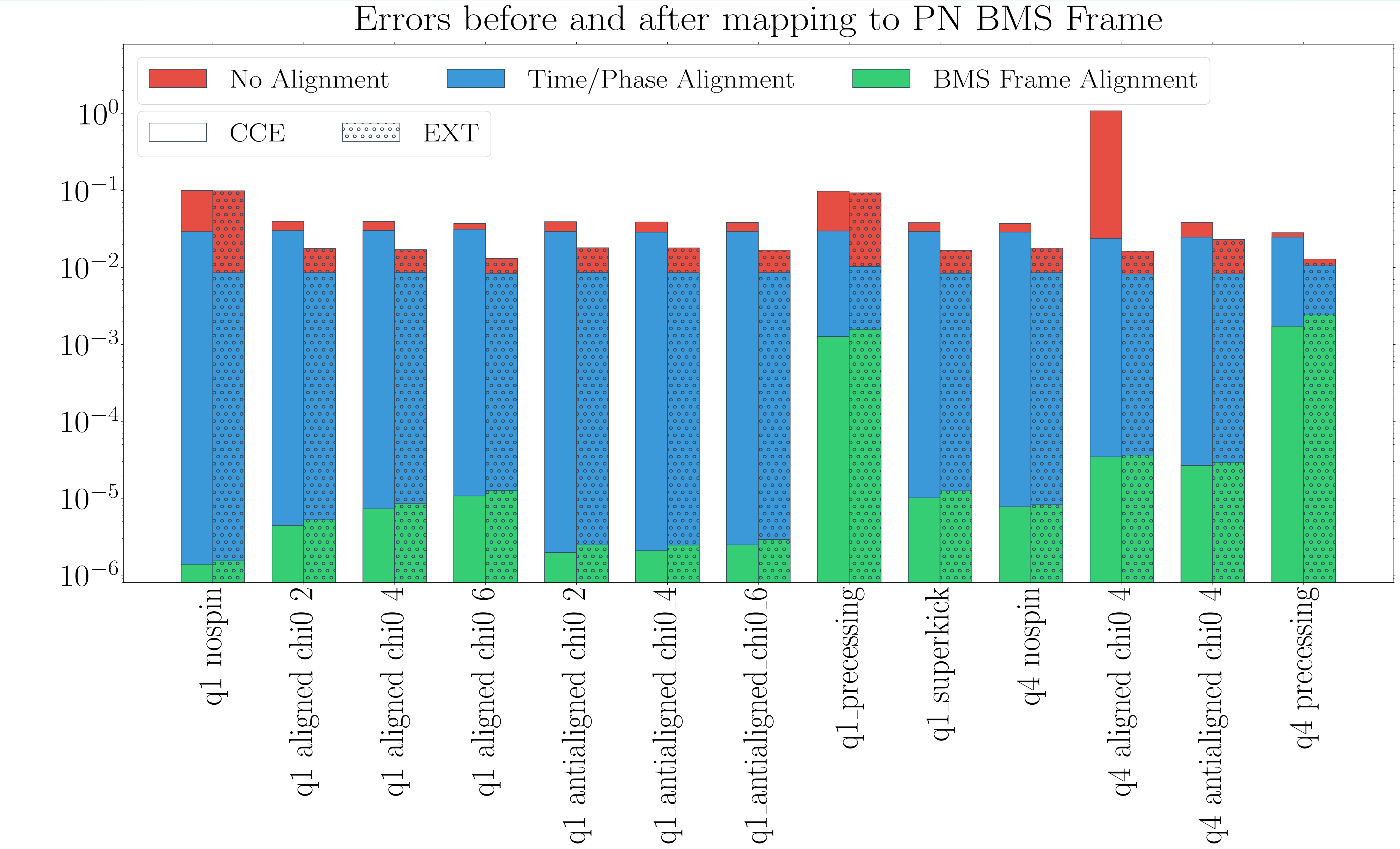
$$\Delta\Psi^{\mathrm{M}} \equiv \int_{u_1}^{u_2} \dot{\Psi}^{\mathrm{M}}(u)\,du = \int_{u_1}^{u_2} \left[\left(\dot{\Psi}_2 + \left[\sigma\ddot{\bar{\sigma}} + \eth^2\dot{\bar{\sigma}}\right]\right) + |\dot{\sigma}|^2\right] du = \int_{u_1}^{u_2} |\dot{\sigma}|^2\,du$$

Vanishes due to the Bianchi identities

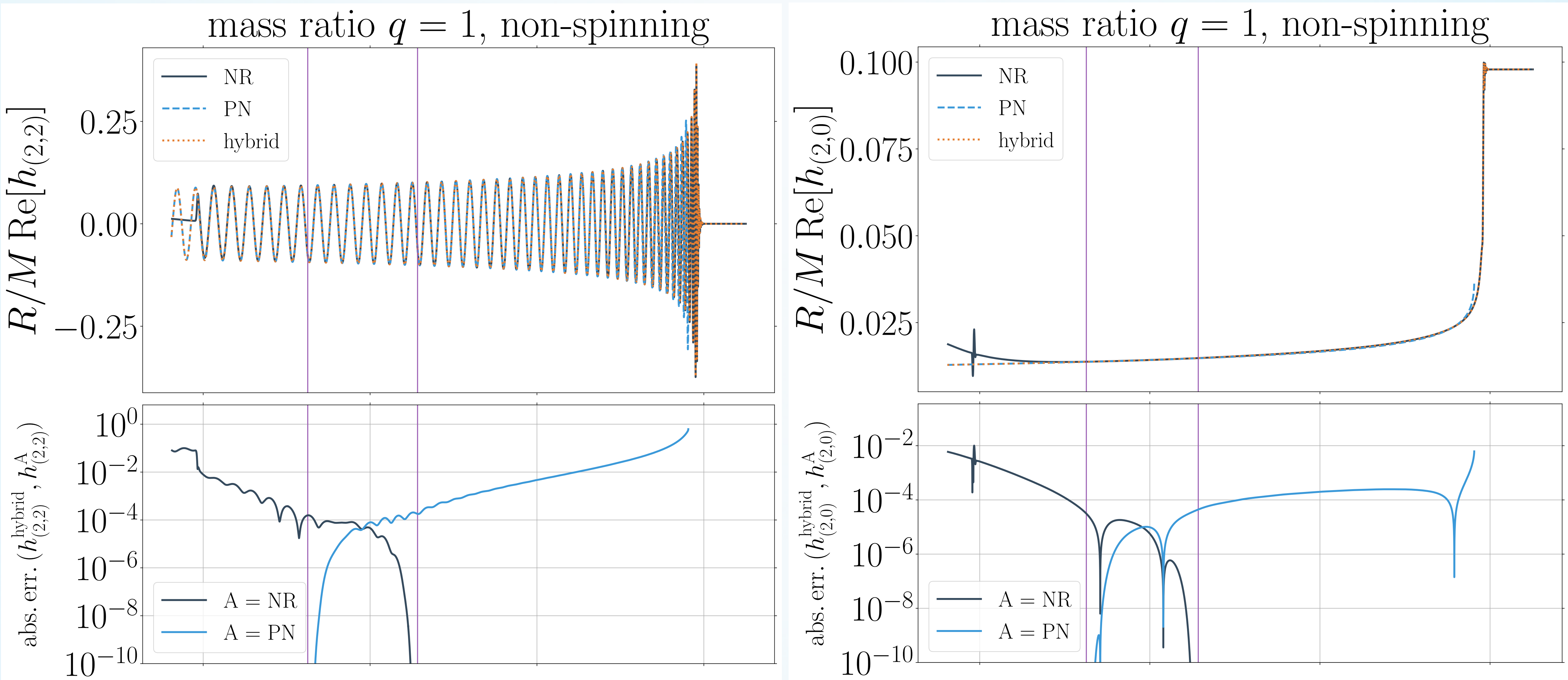
Choices for the super rest frame

1. Mapping to the super rest frame <u>using $\mathscr{I}^-$ data</u>
   - Equivalent to mapping to the "PN BMS frame"
     - i.e., what PN waveforms are in
   - What LIGO and other detectors expect
2. Mapping to the super rest frame <u>using $\mathscr{I}^+$ data</u>
   - **Essential** for performing quasinormal mode (QNM) analyses

Benefits of mapping $\mathscr{I}^{-}$ to the super rest frame

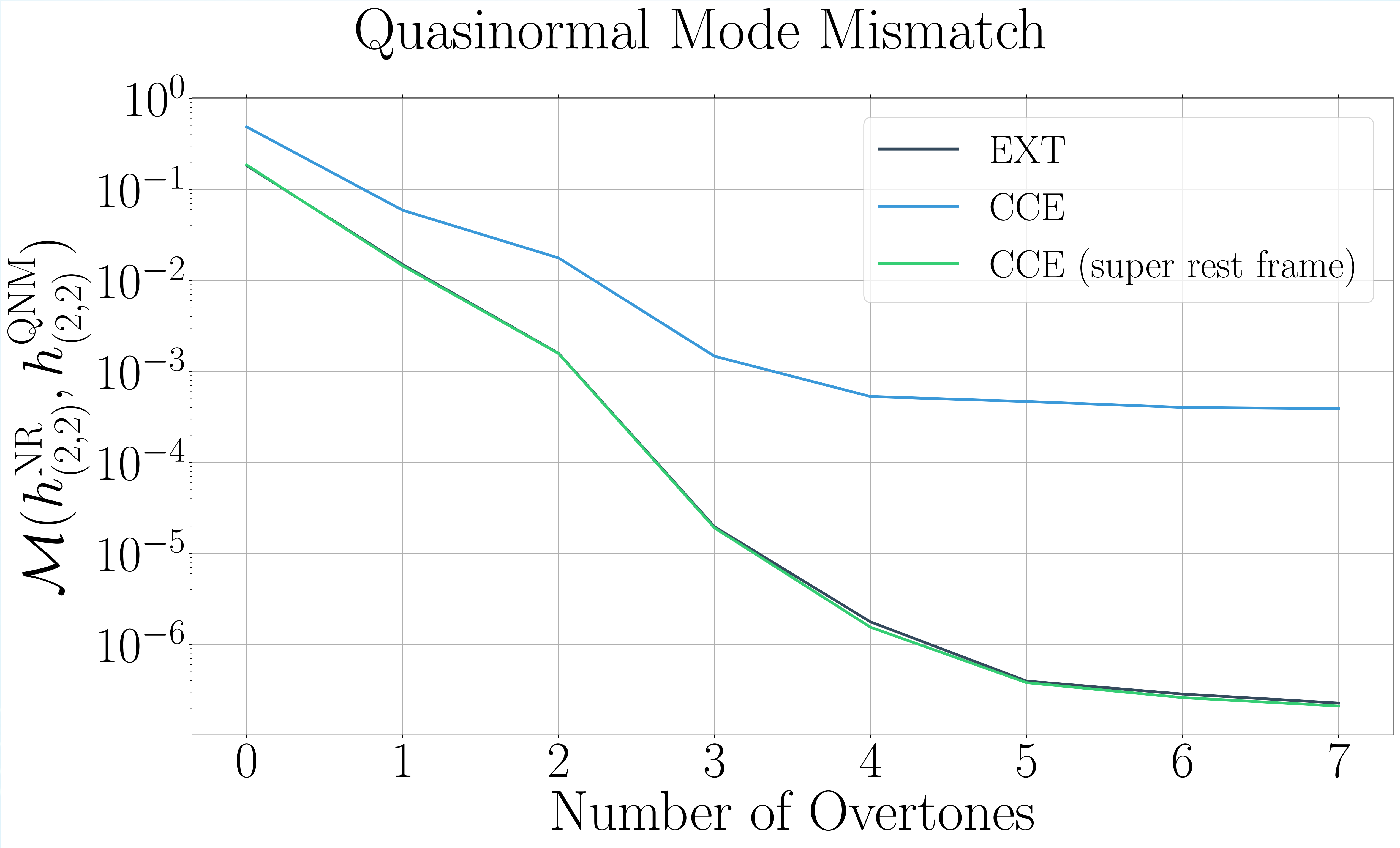


Combining NR and PN Waveforms

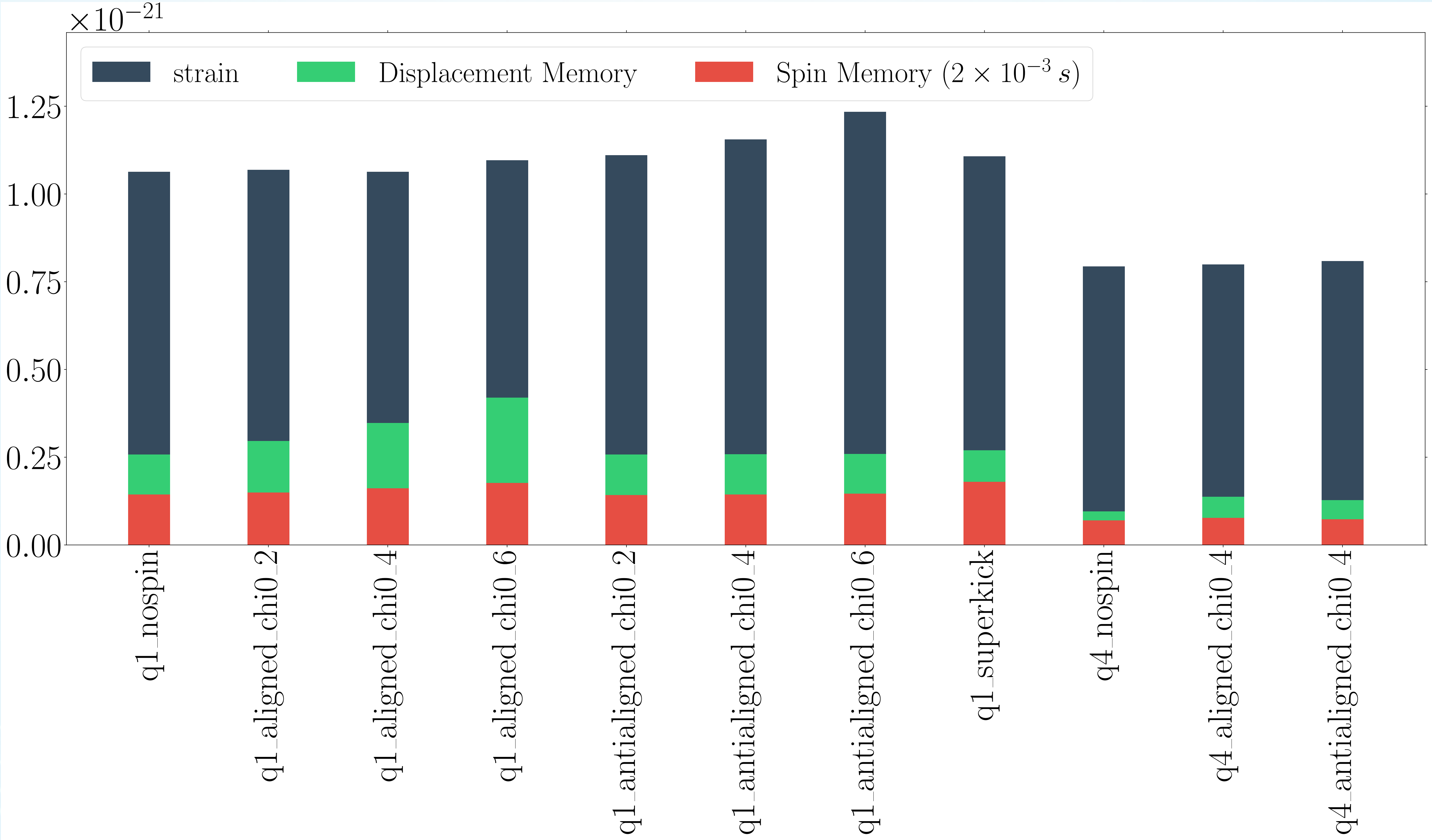


$$h^{\text{hybrid}} = h^{\text{PN}} + f\left(\frac{u-u_1}{u_2-u_1}\right)\left(h^{\text{NR}} - h^{\text{PN}}\right), \quad \text{where} \quad f(x) = \begin{cases} 0 & x \leq 0, \\ \left(1 + \exp\left[\frac{1}{x-1} + \frac{1}{x}\right]\right)^{-1} & 0 < x < 1, \\ 1 & x \geq 1. \end{cases}$$

Benefits of mapping $\mathscr{I}^+$ to the super rest frame



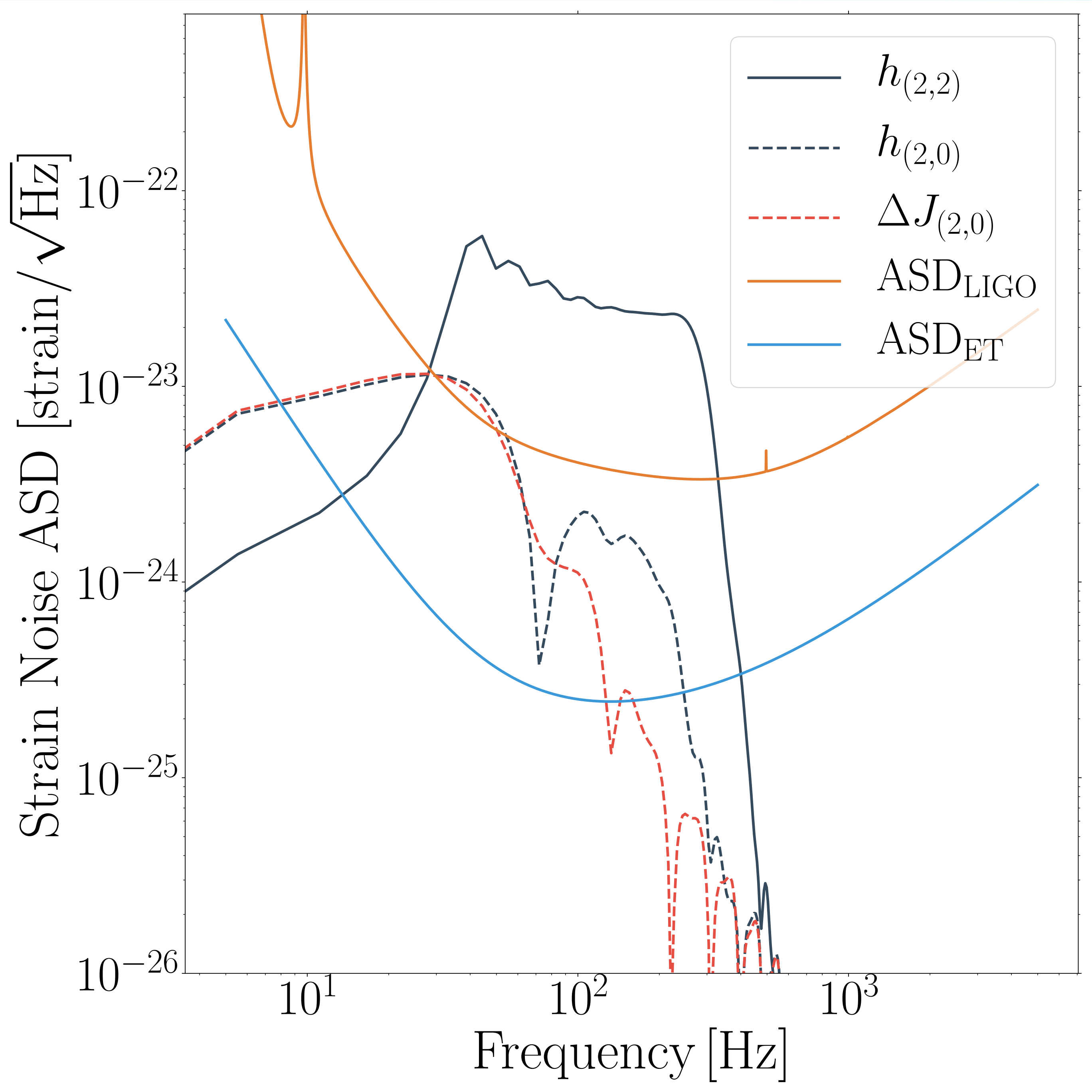
Example Memory Values



Detectability of Gravitational Memory

It won't be easy!

- Can we even detect memory with an interferometer?
- LIGO SNRs for an idealized GW150914 event:
  - Displacement: ~ 2
  - sub-leading: ~ 0.05
- Will rely on "stacking" events
  - Need to break a memory sign degeneracy
  - Need $\sim \mathcal{O}(2000)$ events



Summary

- Can now produce waveforms with memory
- Waveforms without memory can be corrected
- Waveform accuracy can be tested using the BMS balance laws
- Fixing the BMS frame (via the super rest frame) is critical for modeling and analysis
- Detectability estimates improve with NR waveforms

Phys. Rev. D 102, 104007 (2020), (arXiv: 2007.11562), K. Mitman, *et al.*

Phys. Rev. D 103, 024031 (2021), (arXiv: 2011.01309), K. Mitman, *et al.*

"Fixing the BMS Frame of Numerical Relativity Waveforms" (on arXiv later this week), K. Mitman, *et al.*