The gravitational potential at fifth post-Newtonian order

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In collaboration with

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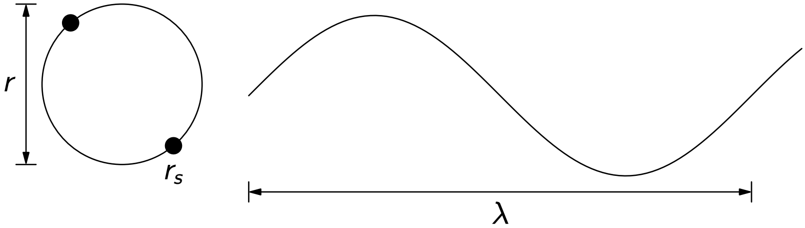
Compact binary systems

Post-Newtonian expansion

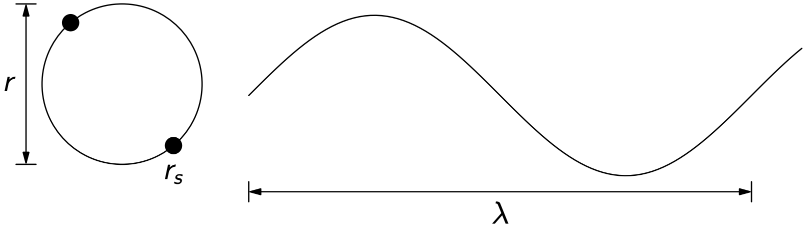


- Masses comparable: $m \equiv m_1 \sim m_2$
  Generalisation to different masses straightforward
- Nonrelativistic system: $v \ll 1$
- Virial theorem: $mv^2 \sim \frac{Gm^2}{r}$

Post-Newtonian (PN) expansion:
Combined expansion in $v \sim \sqrt{Gm/r} \ll 1$

Post-Newtonian expansion

Scales



- $\omega_{\mathrm{GW}} = \frac{2v}{r} \Rightarrow \boxed{\lambda \sim \frac{r}{v}}$
- $r_s = 2Gm \Rightarrow \boxed{r_s \sim r v^2}$

Post-Newtonian expansion

Conservative dynamics

Various Approaches:

- Traditional: up to fourth order (4PN)

  [Damour, Jaranowski, Schäfer 2014-2015]

  [Bernard, Blanchet, Bohé, Faye, Marchant, Marsat 2015-2017]

- "Tutti Frutti": partial 5PN & 6PN

  [Bini, Damour, Geralico 2019-2020]

- Effective Field Theory [Goldberger, Rothstein 2004-2006]

  4PN [Foffa, Porto, Rothstein, Sturani 2019; Blümlein, Maier, Marquard 2020]

  (partial) 5PN & 6PN $\rightarrow$ this talk

Effective Field Theory

Idea: construct simpler, but equivalent theory

Starting from a full theory (general relativity)

1. Identify relevant scales and
   expand action in small scaling parameter

$$v \sim \sqrt{Gm/r} \ll 1$$

General relativity

General relativity action:

$$S_{\text{GR}}[g^{\mu\nu}] = S_{\text{EH}} + S_{\text{GF}} + S_{\text{matter}}$$

With $\eta^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, $g = \det(g^{\mu\nu})$:

- Einstein-Hilbert action:

$$S_{\text{EH}} = \frac{1}{16G\pi} \int d^d x \sqrt{-g} R$$

- *Harmonic* gauge $\partial_\mu \sqrt{-g} g^{\mu\nu} = 0$:

$$S_{\text{GF}} = -\frac{1}{32G\pi} \int d^d x \sqrt{-g} \Gamma_\mu \Gamma^\mu \,, \qquad \Gamma^\mu = g^{\alpha\beta} \Gamma^\mu{}_{\alpha\beta}$$

- Assume point-like matter, no spin:

$$S_{\text{matter}} = -\sum_{a=1}^{2} m_a \int d\tau_a$$

General relativity

Post-Newtonian expansion

Expand S_{GR} in $v \sim \sqrt{Gm/r} \ll 1$, e.g.

$$-m_a \int d\tau_a = -m_a \int dt \sqrt{-g_{\mu\nu} \frac{\partial x_a^\mu}{\partial t} \frac{\partial x_a^\nu}{\partial t}} = -m_a \int dt \sqrt{-g_{00}} + \mathcal{O}(v_a)$$

Coupling to spatial components of metric suppressed

Temporal Kaluza-Klein decomposition [Kol, Smolkin 2010]

$$g^{\mu\nu} = e^{2\phi} \begin{pmatrix} -1 & A_j \\ A_i & e^{-c_d \phi}(\delta_{ij} + \sigma_{ij}) - A_i A_j \end{pmatrix}, \qquad c_d = 2\frac{d-2}{d-3}$$

Flat spacetime for $\sqrt{Gm/r} \to 0$:

Expand in $\phi, A_i, \sigma_{ij} \sim \sqrt{Gm/r} \sim v$

General relativity

Expanded action

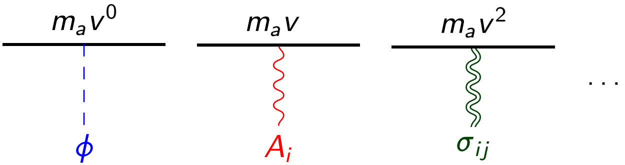
$$S_{\text{GR}}[\phi, A_i, \sigma_{ij}] =$$

$$\sum_{a=1}^{2} \int dt \left(m_a + \frac{1}{2} m_a v_a^2 + \mathcal{O}(v^4) \right)$$

$$+ \sum_{a=1}^{2} m_a \int dt \left(-\phi + v_{ai} A_i + v_{ai} v_{aj} \sigma_{ij} - \frac{1}{2} \phi^2 + \dots \right)$$

$$+ \int \frac{d^d x}{32\pi G} \left[-c_d (\partial_\mu \phi)^2 + (\partial_\mu A_i)^2 + \frac{1}{4} (\partial_\mu \sigma_{ii})^2 - \frac{1}{2} (\partial_\mu \sigma_{ij})^2 + \dots \right]$$

Consider as quantum field theory:



Effective Field Theory

Starting from a full theory (general relativity)

1. Identify relevant scales and expand action in small scaling parameter
2. Decompose fields into modes, characterised by scaling of momentum (components)

[Beneke, Smirnov 1999]

$$\phi = \phi_{\text{pot}} + \phi_{\text{rad}}, \quad A = A_{\text{pot}} + A_{\text{rad}}, \quad \sigma = \sigma_{\text{pot}} + \sigma_{\text{rad}}$$

- Potential: $k_0 \sim \frac{v}{r}, \vec{k} \sim \frac{1}{r}$
- Radiation/ultrasoft: $k^\mu \sim \frac{v}{r}$
- hard, soft modes: negligible quantum corrections

Radiation fields have wavelength $\lambda \sim \frac{r}{v} \gg r$
$\Rightarrow$ multipole expansion in terms with both modes,
e.g. $\phi_{\text{rad}}(t, \vec{x}) = \phi_{\text{rad}}(t, \vec{0}) + x_i \partial_i \phi_{\text{rad}}(t, \vec{0}) + \ldots$

Effective Field Theory

Starting from a full theory (general relativity)

1. Identify relevant scales and
   expand action in small scaling parameter
2. Decompose fields into modes,
   characterised by scaling of momentum (components)
3. Write down effective field theory action:
   Nonrelativistic general relativity [Goldberger, Rothstein 2004-2006]
   - Absorb potential modes into near-zone potential
   - Keep radiation/ultrasoft modes

   Ansatz:

$$S_{\text{NRGR}} = S_{\text{matter}} + S_{\text{mixed}} + S_{\text{radiation}}$$

   - $S_{\text{matter}} = \int dt \, (M + T - V_{\text{NZ}})$
   - S_{mixed}: coupling of radiation modes to matter
   - $S_{\text{radiation}}$: pure radiation modes

Effective Field Theory

Starting from a full theory (general relativity)

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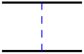
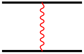
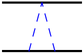
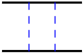
   expand action in small scaling parameter
2. Decompose fields into modes,
   characterised by scaling of momentum (components)
3. Write down effective field theory action
4. Determine parameters from equivalence between effective and full theory up to higher PN orders: matching

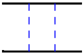
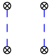
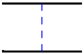
Nonrelativistic effective theory

Potential matching

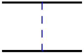
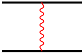
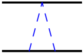
$$V_{\mathrm{NZ}} = i\frac{d}{dt}\log\left[\int \mathcal{D}\phi_{\mathrm{pot}}\mathcal{D}A_{\mathrm{pot}}\mathcal{D}\sigma_{\mathrm{pot}}\; e^{iS_{\mathrm{GR}} - i\int dt(M+T)}\Big|_{\phi_{\mathrm{rad}}=A_{\mathrm{rad}}=\sigma_{\mathrm{rad}}=0}\right]$$

$= i\frac{d}{dt}\log\Big[\, 1 +$  $+$  $+ \frac{1}{2}\times$  $+ \frac{1}{2}\times$  $+ \cdots \Big]$

Matter lines are no propagators:

 $=$  $= \Big($  $\Big)^2$

No contributions from factorising diagrams: cf. [Fischler 1977]

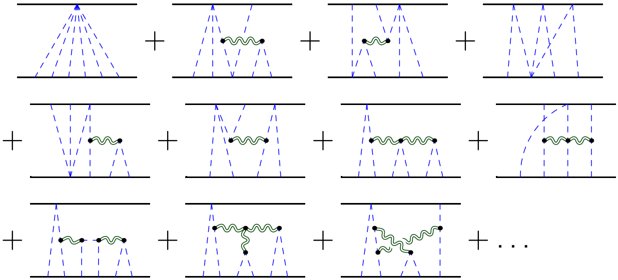
$V_{\mathrm{NZ}} = i\frac{d}{dt}\Big[$  $+$  $+ \frac{1}{2}\times$  $+ \dots \Big]$

Potential matching

5PN calculation

1. Generate diagrams with up to 5 loops with `QGRAF` [Nogueira 1991]
2. Discard unwanted diagrams, e.g. graviton loops

$$-i \int dt\, V_{5\text{PN}} =$$



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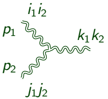

$$= \frac{i}{32 m_{\mathrm{Pl}}} \left(\tilde{V}^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} + \tilde{V}^{i_2 i_1, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} \right)$$

$$\tilde{V}^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} = V^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} + V^{i_1 i_2, j_2 j_1, k_1 k_2}_{\sigma\sigma\sigma} + V^{i_1 i_2, j_1 j_2, k_2 k_1}_{\sigma\sigma\sigma} + V^{i_1 i_2, j_2 j_1, k_2 k_1}_{\sigma\sigma\sigma}$$

$$\begin{aligned}
V^{i_1 i_2, j_1 j_2, k_1 k_2}_{\sigma\sigma\sigma} \overset{v=0}{\equiv} & (\vec{p}_1^{\,2} + \vec{p}_1 \cdot \vec{p}_2 + \vec{p}_2^{\,2}) \Big(- \delta^{j_1 j_2} (2 \delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2}) \\
& \quad + 2 \big[\delta^{i_1 j_1} (4 \delta^{i_2 k_1} \delta^{j_2 k_2} - \delta^{i_2 j_2} \delta^{k_1 k_2}) - \delta^{i_1 i_2} \delta^{j_1 k_1} \delta^{j_2 k_2} \big] \Big) \\
& + 2 \Big\{ 4 (p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2}) \delta^{i_1 j_1} \delta^{j_2 k_1} \\
& \quad + 2 \big[(p_1^{i_1} + p_2^{i_1}) p_2^{i_2} \delta^{j_1 k_1} \delta^{j_2 k_2} - p_1^{k_1} p_2^{k_2} \delta^{i_1 j_1} \delta^{i_2 j_2} \big] \\
& \quad + \delta^{j_1 j_2} \big[p_1^{k_1} p_2^{k_2} \delta^{i_1 i_2} + 2 (p_1^{k_2} p_2^{i_2} - p_1^{i_2} p_2^{k_2}) \delta^{i_1 k_1} - (p_1^{i_1} + p_2^{i_1}) p_2^{i_2} \delta^{k_1 k_2} \big] \\
& \quad + p_2^{j_2} \Big(4 p_1^{i_2} \delta^{i_1 k_1} \delta^{j_1 k_2} + p_1^{j_1} (2 \delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2}) \\
& \qquad + 2 \big[\delta^{i_1 j_1} (p_1^{i_2} \delta^{k_1 k_2} - 2 p_1^{k_2} \delta^{i_2 k_1}) - p_1^{k_2} \delta^{i_1 i_2} \delta^{j_1 k_1} \big] \Big) \\
& \quad + p_1^{j_2} \Big(p_1^{j_1} (2 \delta^{i_1 k_1} \delta^{i_2 k_2} - \delta^{i_1 i_2} \delta^{k_1 k_2}) - 4 p_2^{i_2} \delta^{i_1 k_1} \delta^{j_1 k_2} \\
& \qquad + 2 \big[p_2^{k_2} \delta^{i_1 i_2} \delta^{j_1 k_1} + \delta^{i_1 j_1} (2 p_2^{k_2} \delta^{i_2 k_1} - p_2^{i_2} \delta^{k_1 k_2}) \big] \Big) \Big\}
\end{aligned}$$

Potential matching

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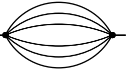
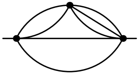
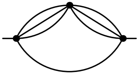
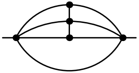
4. Reduce massless propagators to master integrals using Laporta's algorithm [Chetyrkin, Tkachov 1981, Laporta 2000] implemented in `crusher`

$V_{\mathrm{NZ}}^{\mathrm{5PN}} \overset{v=0}{=} c_0$  $+\, c_1$  $+\, c_2$ 

$+\, c_3$  $+\, \mathcal{O}(\epsilon)$

c_j: Laurent series in $\epsilon = \frac{3-d}{2}$,
polynomials in m_1, m_2, r^{-1}, G^{-1}

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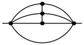
5. Insert known (factorising) master integrals

[Lee, Mingulov 2015; Damour, Jaranowski 2017]



$$= 6\pi^{7/2}\left[\frac{2}{\epsilon} - 4 - 4\ln(2) + \mathcal{O}(\epsilon^1)\right]$$

Potential matching

5PN calculation

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   [Lee, Mingulov 2015; Damour, Jaranowski 2017]

$$V_{5\mathrm{PN}}^{\mathrm{NZ}} \stackrel{v=0}{=} \frac{G^6}{r^6} m_1 m_2 \left[\frac{5}{16}(m_1^5 + m_2^5) + \frac{91}{6} m_1 m_2 (m_1^3 + m_2^3) + \frac{653}{6} m_1^2 m_2^2 (m_1 + m_2) \right]$$

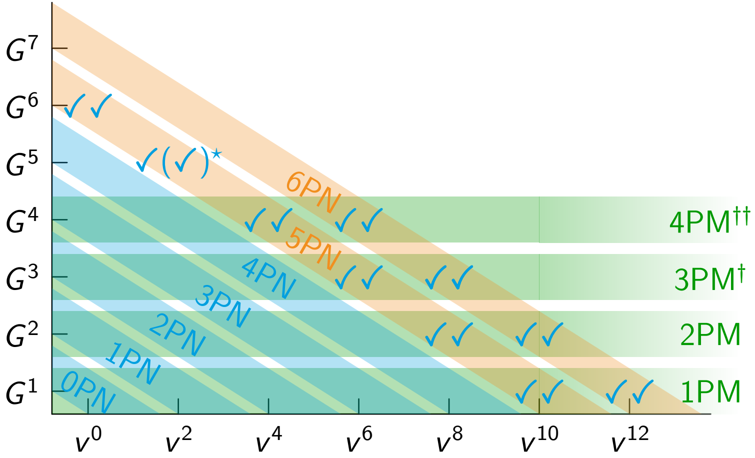
5PN $v = 0$: [Foffa, Mastrolia, Sturani, Sturm, Torres Bobadilla 2019; Blümlein, Maier, Marquard 2019]

New:

- complete 5PN [Blümlein, Maier, Marquard, Schäfer, 2020]
- 6PN up to 3 loops: $v^{\geq 6}$ [Blümlein, Maier, Marquard, Schäfer, 2020-2021]

Near-zone potential

Cross checks



† [Bern, Cheung, Roiban, Shen, Solon 2019; Cheung Solon 2020; Kälin, Liu, Porto 2020]

†† [Bern, Martinez, Roiban, Ruf, Shen 2021]

⋆ [Foffa, Sturani, Torres Bobadilla 2021]

Classical action

V_{NZ} is not physical:

- Gauge dependent
- Infrared divergence at $\geq$4PN

$\Rightarrow$ combine with contribution from radiation/ultrasoft modes

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Construct classical post-Newtonian action *without* any field:

$$S_{PN} = \int dt \; L[\vec{x}_a, \vec{v}_a] = \int dt \; (M + T - V)$$

Absorb radiation modes radiation/ultrasoft modes into far-zone potential V_{FZ} (“tail”)

$$V = V_{NZ} + V_{FZ}$$

Far-zone potential

Matching equation for far-zone potential:

$$V_{\mathrm{FZ}} = i\frac{d}{dt}\log\left[\int \mathcal{D}\phi_{\mathrm{rad}}\mathcal{D}A_{\mathrm{rad}}\mathcal{D}\sigma_{\mathrm{rad}}\; e^{i(S_{\mathrm{mixed}}+S_{\mathrm{radiation}})}\right]$$

Matter-radiation interaction in NRGR:

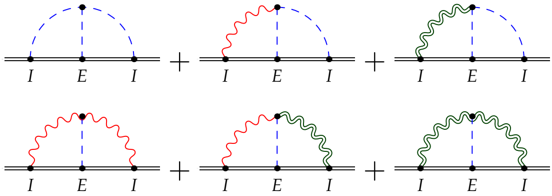
$$S_{\mathrm{mixed}} = \frac{1}{2}\int d^dx\, T^{\mu\nu}\delta g_{\mu\nu} + \mathcal{O}(\delta g_{\mu\nu}^2), \qquad \delta g_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$$

$\delta g_{\mu\nu}$ multipole expanded
$\Rightarrow$ ϕ, A_i, σ_{ij} coupling to multipole moments $E, P_i, L_i, I_{ij}, \ldots$
after integration by parts e.g. [Ross 2012]

Far-zone potential

Matching at 4PN, conservative part:

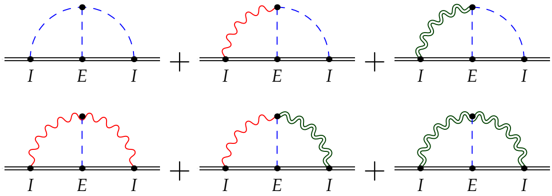
$$-i\int dt\, V_{\rm FZ}^{\rm 4PN} =$$



Far-zone potential

Matching at 4PN, conservative part:

$$-i\int dt\, V_{\mathrm{FZ}}^{\mathrm{4PN}} =$$



At 5PN: [Foffa, Sturani 2019–2021]

- Additional multipole moments J_{ij}, O_{ijk}
- More 2-loop diagrams
- 1PN corrections to E, I_{ij} in d dimensions

  [Marchand, Henry, Larrouturou, Marsat, Faye, Blanchet 2020]

Classical Hamiltonian

- Combine potentials $V = V_{\mathrm{NZ}} + V_{\mathrm{FZ}}$:

$$L_{\mathrm{PN}}[\vec{x}_a, \vec{v}_a, \vec{a}_a, \vec{x}_a^{(3)}, \ldots] = M + T - V$$

- Eliminate higher time derivatives:
  - Total derivatives $L_{\mathrm{PN}} \to L_{\mathrm{PN}} + \frac{d}{dt} F$
  - Multiple zeroes $L_{\mathrm{PN}} \to L_{\mathrm{PN}} + F Z_1 Z_2 \cdots$
    with $\delta(F Z_1 Z_2 \cdots) = 0$ at current PN order
  - Coordinate shifts $\vec{x}_a \to \vec{x}_a + \Delta\vec{x}_a$ with $\Delta\vec{x}_a = \mathcal{O}(1\mathrm{PN})$
- Transform to classical Hamiltonian $H = \vec{p}_a \vec{v}_a - L$
  in centre-of-mass frame
- Canonical transformations with generator g for comparisons
  and simpler expressions

$$H \to e^{D_g} H\,, \qquad D_g f = \{f, g\}$$

Classical Hamiltonian

$$\mu = \frac{m_1 m_2}{M}\,, \quad p = \frac{\vec{p}_1}{\mu} = -\frac{\vec{p}_2}{\mu}\,, \quad r = \frac{|x_1 - x_2|}{GM}\,, \quad n = \frac{\vec{r}}{r}$$

$$\begin{aligned}
\frac{H_{5PN}^{\text{pole free}} - M}{\mu} = &- \frac{21p^{12}}{1024} + \frac{5}{16r^6} - \frac{125p^2}{16r^5} - \frac{499p^4}{64r^4} - \frac{161p^6}{32r^3} - \frac{445p^8}{256r^2} - \frac{77p^{10}}{256r} \\
&+ \frac{\mu}{M}\Bigg[\frac{231p^{12}}{1024} - \frac{279775133}{529200r^6} - \frac{1450584679p^2}{2116800r^5} + \frac{2010713771p^4}{1411200r^4} + \frac{11206267p^6}{141120r^3} + \frac{937p^8}{32r^2} \\
&+ \frac{805p^{10}}{256r} + \ln\left(\frac{r}{r_0}\right)\left(\frac{64}{105r^6} - \frac{18944p^2}{105r^5} + \frac{1796p^4}{105r^4} + \frac{19136(p.n)^2}{105r^5} - \frac{10664p^2(p.n)^2}{105r^4}\right. \\
&\left. + \frac{2748(p.n)^4}{35r^4}\right) + \pi^2\left(\frac{70399}{1152r^6} + \frac{65291p^2}{1152r^5} - \frac{1328147p^4}{12288r^4} - \frac{7719p^6}{4096r^3} + \frac{6649(p.n)^2}{576r^5}\right. \\
&+ \frac{5042575p^2(p.n)^2}{6144r^4} + \frac{58887p^4(p.n)^2}{4096r^3} - \frac{3293913(p.n)^4}{4096r^4} - \frac{89625p^2(p.n)^4}{4096r^3} \\
&\left. + \frac{42105(p.n)^6}{4096r^3}\right) - \frac{34541593(p.n)^2}{2116800r^5} - \frac{2395722563p^2(p.n)^2}{282240r^4} - \frac{62196341p^4(p.n)^2}{78400r^3} \\
&- \frac{589p^6(p.n)^2}{16r^2} - \frac{35p^8(p.n)^2}{256r} + \frac{631107353(p.n)^4}{78400r^4} + \frac{31226291p^2(p.n)^4}{23520r^3} \\
&+ \frac{8951p^4(p.n)^4}{384r^2} - \frac{563921(p.n)^6}{960r^3} - \frac{5117p^2(p.n)^6}{320r^2} + \frac{159(p.n)^8}{28r^2}\Bigg] + \frac{\mu^2}{M^2}\Bigg[-\frac{231p^{12}}{256} \\
&+ \frac{72454}{225r^6} + \frac{1353196483p^2}{529200r^5} - \frac{787300061p^4}{264600r^4} + \frac{3605263p^6}{29400r^3} - \frac{11535p^8}{128r^2} - \frac{2865p^{10}}{256r} \\
&- \ln\left(\frac{r}{r_0}\right)\left(\frac{256}{105r^6} + \frac{3392p^2}{105r^5} - \frac{432p^4}{35r^4} - \frac{2992(p.n)^2}{105r^5} - \frac{6824p^2(p.n)^2}{105r^4} + \frac{496(p.n)^4}{7r^4}\right)
\end{aligned}$$

Classical Hamiltonian

$$+\pi^2\left(\frac{5453}{768r^6}-\frac{121315p^2}{768r^5}+\frac{2076041p^4}{12288r^4}+\frac{29987p^6}{4096r^3}+\frac{200359(p.n)^2}{768r^5}-\frac{172311p^4(p.n)^2}{4096r^3}\right.$$

$$\left.-\frac{5962205p^2(p.n)^2}{6144r^4}+\frac{2617363(p.n)^4}{4096r^4}+\frac{127125p^2(p.n)^4}{4096r^3}+\frac{14175(p.n)^6}{4096r^3}\right)$$

$$-\frac{857318207(p.n)^2}{264600r^5}+\frac{34200172759p^2(p.n)^2}{2116800r^4}-\frac{5034763p^4(p.n)^2}{9800r^3}+\frac{4969p^6(p.n)^2}{64r^2}$$

$$+\frac{275p^8(p.n)^2}{256r}-\frac{4989943687(p.n)^4}{352800r^4}+\frac{2674877p^2(p.n)^4}{7840r^3}+\frac{925p^4(p.n)^4}{24r^2}+\frac{15p^6(p.n)^4}{128r}$$

$$\left.-\frac{25649(p.n)^6}{3360r^3}-\frac{8331p^2(p.n)^6}{160r^2}+\frac{751(p.n)^8}{28r^2}\right]+\frac{\mu^3}{M^3}\left[\frac{1617p^{12}}{1024}-\frac{238966727p^2}{151200r^5}\right.$$

$$+\frac{127702733p^4}{84672r^4}+\frac{108551131p^6}{4233600r^3}+\frac{16283p^8}{256r^2}+\frac{3995p^{10}}{256r}+\pi^2\left(-\frac{2339p^2}{192r^5}+\frac{98447p^4}{3072r^4}\right.$$

$$-\frac{20259p^6}{1024r^3}-\frac{16111(p.n)^2}{192r^5}+\frac{131231p^2(p.n)^2}{1536r^4}+\frac{106947p^4(p.n)^2}{1024r^3}-\frac{361499(p.n)^4}{1024r^4}$$

$$\left.-\frac{30075p^2(p.n)^4}{1024r^3}-\frac{65625(p.n)^6}{1024r^3}\right)+\frac{758233181(p.n)^2}{151200r^5}-\frac{10374288811p^2(p.n)^2}{705600r^4}$$

$$-\frac{2207947669p^4(p.n)^2}{1411200r^3}+\frac{177p^6(p.n)^2}{256r^2}-\frac{221p^8(p.n)^2}{64r}+\frac{12810612439(p.n)^4}{705600r^4}$$

$$+\frac{355111837p^2(p.n)^4}{94080r^3}-\frac{125225p^4(p.n)^4}{768r^2}-\frac{3p^6(p.n)^4}{128r}-\frac{13905527(p.n)^6}{4480r^3}$$

$$\left.+\frac{136977p^2(p.n)^6}{1280r^2}-\frac{15p^4(p.n)^6}{128r}-\frac{289839(p.n)^8}{4480r^2}-\frac{35p^2(p.n)^8}{256r}\right]$$

Classical Hamiltonian

$$
\begin{aligned}
&+\frac{\mu^4}{M^4}\Bigg[-\frac{1155p^{12}}{1024}-\frac{593p^6}{32r^3}+\frac{6649p^8}{256r^2}-\frac{1615p^{10}}{256r}+\frac{549p^4(p.n)^2}{32r^3}-\frac{62143p^6(p.n)^2}{256r^2}+\frac{867p^8(p.n)^2}{256r}\\
&-\frac{5749p^2(p.n)^4}{96r^3}+\frac{652381p^4(p.n)^4}{768r^2}-\frac{3p^6(p.n)^4}{64r}-\frac{17623(p.n)^6}{240r^3}-\frac{1178329p^2(p.n)^6}{1280r^2}\\
&-\frac{45p^4(p.n)^6}{128r}+\frac{1443091(p.n)^8}{4480r^2}+\frac{105p^2(p.n)^8}{128r}\Bigg]+\frac{\mu^5}{M^5}\Bigg[\frac{231p^{12}}{1024}-\frac{63p^{10}}{256r}-\frac{35p^8(p.n)^2}{256r}\\
&-\frac{15p^6(p.n)^4}{128r}-\frac{15p^4(p.n)^6}{128r}-\frac{35p^2(p.n)^8}{256r}-\frac{63(p.n)^{10}}{256r}\Bigg]+\frac{17(p.n)^2}{4r^5}+\frac{29p^2(p.n)^2}{8r^4}\\
&+\frac{21p^4(p.n)^2}{16r^3}+\frac{5p^6(p.n)^2}{32r^2}-\frac{(p.n)^4}{8r^4}+V^{5PN}_{\text{FZ,fin, non-log}}
\end{aligned}
$$

$\Rightarrow$ observables: energy, periastron advance

Classical Hamiltonian

Cross checks:

- ✓ $\frac{\mu^0}{M^0}$ agrees with Schwarzschild limit
- ✓ $\frac{\mu^1}{M^1}$, $\frac{\mu^2}{M^2}$: poles cancel between near zone and far zone
- ✓ $\frac{\mu^3}{M^3}$, $\frac{\mu^4}{M^4}$, $\frac{\mu^5}{M^5}$ agree with [Bini, Damour, Geralico 2020]
- $\frac{\mu^1}{M^1}$ agrees with [Bini, Damour, Geralico 2020]
  after finite renormalisation $J_{ij} \to \left(1 - \frac{1}{3}\epsilon\right) J_{ij}$
- No comparable result for $\frac{\mu^2}{M^2}$

Conclusion

- Powerful methods from particle physics for post-Newtonian expansion:
  - Effective field theories
  - Techniques for multiloop calculations
- Latest results:
  - Complete 5PN (near-zone) potential
  - Partial 6PN potential
- Open questions in combination with far zone/tail