Consiglio Nazionale delle Ricerche

Istituto per le Applicazioni del Calcolo "Mauro Picone"

Hyperboliclike encounters in a two-body system: Analytical treatment, recent developments and open problems

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Based on works done in collaboration with:

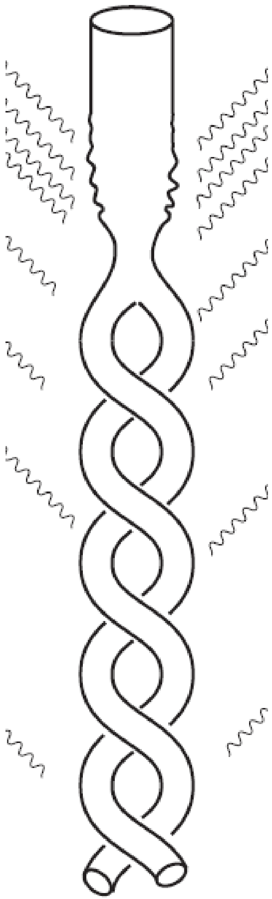
T. Damour, A. Geralico

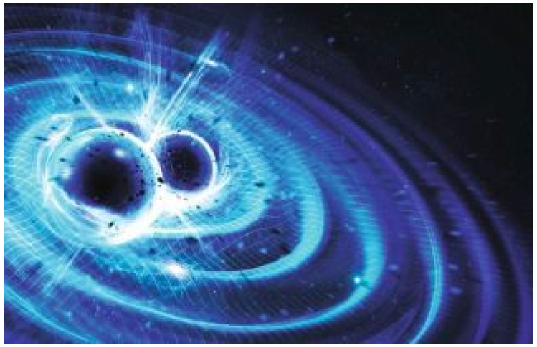
Abstract

The main features of the "Tutti Frutti" method which has allowed reaching the 6PN level of accuracy (modulo a few still unknown parameters) in the relativistic treatment of the two-body system will be briefly introduced and discussed.
Special attention will be devoted to the gauge-invariant information encoded in the scattering angle of hyperboliclike encounters.

Gravitational scattering, inspiral, and radiation, GGI Workshop, May 11, 2021

A fully relativistic treatment of the gravitational two-body problem requires solving the EEs





$$T^{\alpha\beta} = \sum_{a=1,2} m_a \int u_a^\alpha u_a^\beta \, {}^{(4)}\delta(x^\mu - z_a^\mu(\tau)) d\tau$$

$$\boxed{R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}}$$

Energy-momentum tensor of the two bodies

…using some approximation method or NR…

Solve perturbations, reconstruct the metric, find the geodesics, the H, etc

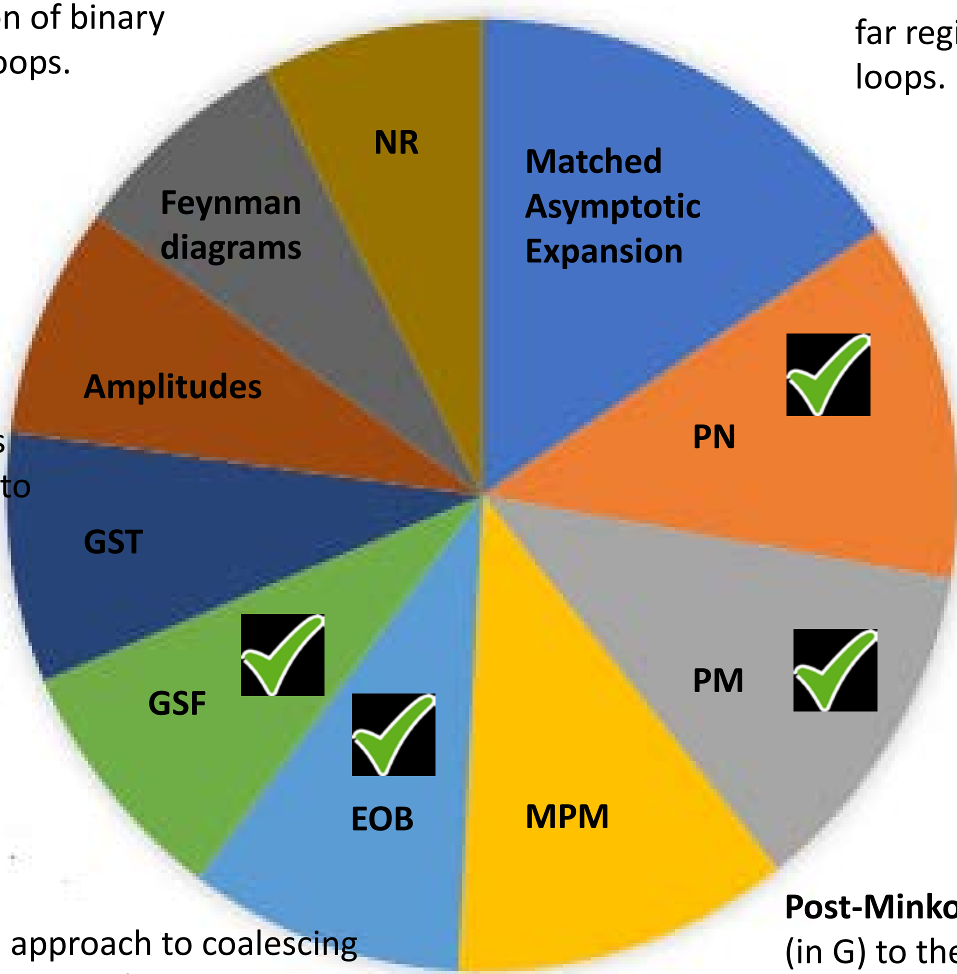
Extract gauge-invariant information

Images freely taken from the web

A list of tools

PN-matched Multipolar post-Minkowskian (MPM) approximation theory of the GW emission of binary systems. Known up to 3 loops.

Matched Asymptotic Expansion approach to the motion of strongly self-gravitating bodies (match between near region and far region)→ UV finiteness up to 5PN=5 loops.



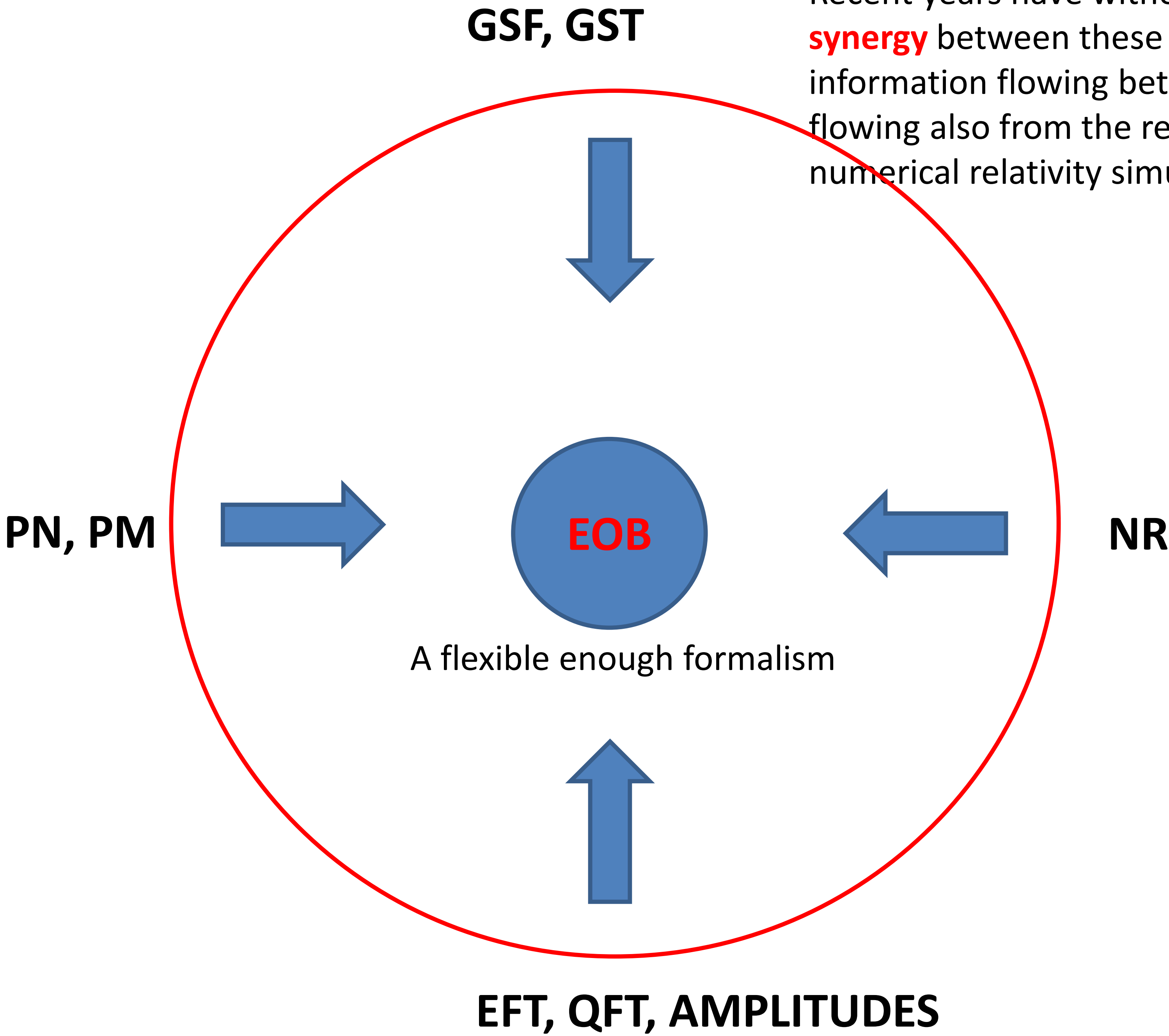
Gravitational Self-Force (GSF, GST). Computations of some GI quantities up to 22 loops.

Post-Newtonian (PN) approximation to the motion of binary systems. Fully known up to 4 PN=4 loops. Potential terms computed @5PN. Recent results @5PN and 6PN.

Effective One-Body (EOB) approach to coalescing binary black holes (and binary NS). Known up to 4PN. Recently extended to 5PN = 5 loops (with 2 unknown) and 6PN =6 loops (2+4 unknown).

Post-Minkowskian (PM) approximation theory (in G) to the motion of binary systems. Metric fully known to 2 PM= $O(G^2)$=1 loop, it led to the first derivation of the 2PN dynamics. Recent results at $O(G^3)$ and $O(G^4)$.

Recent years have witnessed a **useful synergy** between these methods, with information flowing between them (and flowing also from the results of full numerical relativity simulations).



PM approach: recent results

3PM, 2019

Scattering Amplitudes and the Conservative Hamiltonian for Binary Systems at Third Post-Minkowskian Order

Zvi Bern,[1] Clifford Cheung,[2] Radu Roiban,[3] Chia-Hsien Shen,[1] Mikhail P. Solon,[2] and Mao Zeng[4]

$$
\begin{aligned}
H(\boldsymbol{p},\boldsymbol{r}) &= \sqrt{\boldsymbol{p}^2+m_1^2}+\sqrt{\boldsymbol{p}^2+m_2^2}+V(\boldsymbol{p},\boldsymbol{r})\\
V(\boldsymbol{p},\boldsymbol{r}) &= \sum_{i=1}^{\infty} c_i(\boldsymbol{p}^2)\left(\frac{G}{|\boldsymbol{r}|}\right)^i,
\end{aligned}
$$

$$
\begin{aligned}
c_1 &= \frac{\nu^2 m^2}{\gamma^2\xi}(1-2\sigma^2)\\
c_2 &= \frac{\nu^2 m^3}{\gamma^2\xi}\left[\frac{3}{4}(1-5\sigma^2)-\frac{4\nu\sigma(1-2\sigma^2)}{\gamma\xi}-\frac{\nu^2(1-\xi)(1-2\sigma^2)^2}{2\gamma^3\xi^2}\right]\\
c_3 &= \frac{\nu^2 m^4}{\gamma^2\xi}\left[\frac{1}{12}(3-6\nu+206\nu\sigma-54\sigma^2+108\nu\sigma^2+4\nu\sigma^3)\right.\\
&\quad -\frac{4\nu(3+12\sigma^2-4\sigma^4)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\\
&\quad -\frac{3\nu\gamma(1-2\sigma^2)(1-5\sigma^2)}{2(1+\gamma)(1+\sigma)}-\frac{3\nu\sigma(7-20\sigma^2)}{2\gamma\xi}\\
&\quad -\frac{\nu^2(3+8\gamma-3\xi-15\sigma^2-80\gamma\sigma^2+15\xi\sigma^2)(1-2\sigma^2)}{4\gamma^3\xi^2}\\
&\quad \left.+\frac{2\nu^3(3-4\xi)\sigma(1-2\sigma^2)^2}{\gamma^4\xi^3}+\frac{\nu^4(1-2\xi)(1-2\sigma^2)^3}{2\gamma^6\xi^4}\right]
\end{aligned}
$$

4PM, 2021

Scattering Amplitudes and Conservative Binary Dynamics at $\mathcal{O}(G^4)$

Zvi Bern,[1] Julio Parra-Martinez,[2] Radu Roiban,[3] Michael S. Ruf,[4]
Chia-Hsien Shen,[5] Mikhail P. Solon,[1] and Mao Zeng[6]

… about 40y after the 2PM results…

Poincaré-Invariant Gravitational Field and Equations of Motion of Two Pointlike Objects: The Postlinear Approximation of General Relativity

General Relativity and Gravitation, Vol. 13, No. 10, 1981

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JESUS IBANEZ and JESUS MARTIN[1]

Universidad del Pais Vasco, San Sebastian, Spain

Fortschr. Phys. **33** (1985) 8, 417—493

High-Speed Scattering of Charged and Uncharged Particles in General Relativity

Konradin Westpfahl

Class. Quantum Grav. 4 (1987) L185-L188. Printed in the UK

LETTER TO THE EDITOR

Energy–momentum conservation for gravitational two-body scattering in the post-linear approximation

K Westpfahl, R Möhles and H Simonis

PN approach

H: N, 1PN, 2PN

$$\hat{H}_{\leq 4\mathrm{PN}} := (H_{\leq 4\mathrm{PN}} - Mc^2)/\mu$$

$$\hat{H}_{\leq 4\mathrm{PN}}[\mathbf{r},\mathbf{p}] = \hat{H}_{\mathrm{N}}(\mathbf{r},\mathbf{p}) + \hat{H}_{1\mathrm{PN}}(\mathbf{r},\mathbf{p}) + \hat{H}_{2\mathrm{PN}}(\mathbf{r},\mathbf{p}) + \hat{H}_{3\mathrm{PN}}(\mathbf{r},\mathbf{p}) + \hat{H}_{4\mathrm{PN}}[\mathbf{r},\mathbf{p}],$$

Conservative ADM Hamiltonian in the CM system
(including time-sym rr effects into the conservative part of the H)

Standard notation:

$$p = \frac{P_{\mathrm{real}}}{\mu}, \qquad M = m_1 + m_2, \qquad \mu = \frac{m_1 m_2}{M}, \qquad \nu = \frac{\mu}{M}$$

$$\hat{H}_{\mathrm{N}}(\mathbf{r},\mathbf{p}) = \frac{\mathbf{p}^2}{2} - \frac{1}{r}$$

$$c^2 \hat{H}_{1\mathrm{PN}}(\mathbf{r},\mathbf{p}) = \frac{1}{8}(3\nu - 1)(\mathbf{p}^2)^2 - \frac{1}{2}\{(3+\nu)\mathbf{p}^2 + \nu(\mathbf{n}\cdot\mathbf{p})^2\}\frac{1}{r} + \frac{1}{2r^2},$$

$$c^4 \hat{H}_{2\mathrm{PN}}(\mathbf{r},\mathbf{p}) = \frac{1}{16}(1 - 5\nu + 5\nu^2)(\mathbf{p}^2)^3 + \frac{1}{8}\{(5 - 20\nu - 3\nu^2)(\mathbf{p}^2)^2 - 2\nu^2(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - 3\nu^2(\mathbf{n}\cdot\mathbf{p})^4\}\frac{1}{r} + \frac{1}{2}\{(5+8\nu)\mathbf{p}^2 + 3\nu(\mathbf{n}\cdot\mathbf{p})^2\}\frac{1}{r^2} - \frac{1}{4}(1+3\nu)\frac{1}{r^3},$$

H: 3PN

$$
\begin{aligned}
c^6 \hat{H}_{3\mathrm{PN}}(\mathbf{r},\mathbf{p}) = {} & \frac{1}{128}(-5+35\nu-70\nu^2+35\nu^3)(\mathbf{p}^2)^4 + \frac{1}{16}\{(-7+42\nu-53\nu^2-5\nu^3)(\mathbf{p}^2)^3 \\
& + (2-3\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + 3(1-\nu)\nu^2(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - 5\nu^3(\mathbf{n}\cdot\mathbf{p})^6\}\frac{1}{r} \\
& + \left\{\frac{1}{16}(-27+136\nu+109\nu^2)(\mathbf{p}^2)^2 + \frac{1}{16}(17+30\nu)\nu(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \frac{1}{12}(5+43\nu)\nu(\mathbf{n}\cdot\mathbf{p})^4\right\}\frac{1}{r^2} \\
& + \left\{\left(-\frac{25}{8}+\left(\frac{\pi^2}{64}-\frac{335}{48}\right)\nu-\frac{23\nu^2}{8}\right)\mathbf{p}^2 + \left(-\frac{85}{16}-\frac{3\pi^2}{64}-\frac{7\nu}{4}\right)\nu(\mathbf{n}\cdot\mathbf{p})^2\right\}\frac{1}{r^3} \\
& + \left\{\frac{1}{8}+\left(\frac{109}{12}-\frac{21}{32}\pi^2\right)\nu\right\}\frac{1}{r^4}.
\end{aligned}
$$

H: 4PN loc

2014

Damour Jaranowski Schaefer

$$
\begin{aligned}
c^8 \hat{H}^{\rm loc}_{\rm 4PN}(\mathbf{r},\mathbf{p}) \;:=\; c^8 \frac{H^{\rm loc}_{\rm 4PN}(\mathbf{r},\mathbf{p})}{\mu} &= \left(\frac{7}{256} - \frac{63}{256}\nu + \frac{189}{256}\nu^2 - \frac{105}{128}\nu^3 + \frac{63}{256}\nu^4\right)(\mathbf{p}^2)^5 \\
&+ \left\{\frac{45}{128}(\mathbf{p}^2)^4 - \frac{45}{16}(\mathbf{p}^2)^4\nu + \left(\frac{423}{64}(\mathbf{p}^2)^4 - \frac{3}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2\right)\nu^2\right. \\
&+ \left(-\frac{1013}{256}(\mathbf{p}^2)^4 + \frac{23}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 + \frac{69}{128}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{64}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 + \frac{35}{256}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^3 \\
&+ \left.\left(-\frac{35}{128}(\mathbf{p}^2)^4 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^3 - \frac{9}{64}(\mathbf{n}\cdot\mathbf{p})^4(\mathbf{p}^2)^2 - \frac{5}{32}(\mathbf{n}\cdot\mathbf{p})^6\mathbf{p}^2 - \frac{35}{128}(\mathbf{n}\cdot\mathbf{p})^8\right)\nu^4\right\}\frac{1}{r} \\
&+ \left\{\frac{13}{8}(\mathbf{p}^2)^3 + \left(-\frac{791}{64}(\mathbf{p}^2)^3 + \frac{49}{16}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{889}{192}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{369}{160}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu\right. \\
&+ \left(\frac{4857}{256}(\mathbf{p}^2)^3 - \frac{545}{64}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 + \frac{9475}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 - \frac{1151}{128}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^2 \\
&+ \left.\left(\frac{2335}{256}(\mathbf{p}^2)^3 + \frac{1135}{256}(\mathbf{n}\cdot\mathbf{p})^2(\mathbf{p}^2)^2 - \frac{1649}{768}(\mathbf{n}\cdot\mathbf{p})^4\mathbf{p}^2 + \frac{10353}{1280}(\mathbf{n}\cdot\mathbf{p})^6\right)\nu^3\right\}\frac{1}{r^2} \\
&+ \left\{\frac{105}{32}(\mathbf{p}^2)^2 + \left(\left(\frac{2749\pi^2}{8192} - \frac{589189}{19200}\right)(\mathbf{p}^2)^2 + \left(\frac{63347}{1600} - \frac{1059\pi^2}{1024}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2\right.\right. \\
&+ \left.\left(\frac{375\pi^2}{8192} - \frac{23533}{1280}\right)(\mathbf{n}\cdot\mathbf{p})^4\right)\nu + \left(\left(\frac{18491\pi^2}{16384} - \frac{1189789}{28800}\right)(\mathbf{p}^2)^2\right. \\
&+ \left.\left(-\frac{127}{3} - \frac{4035\pi^2}{2048}\right)(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 + \left(\frac{57563}{1920} - \frac{38655\pi^2}{16384}\right)(\mathbf{n}\cdot\mathbf{p})^4\right)\nu^2 \\
&+ \left.\left(-\frac{553}{128}(\mathbf{p}^2)^2 - \frac{225}{64}(\mathbf{n}\cdot\mathbf{p})^2\mathbf{p}^2 - \frac{381}{128}(\mathbf{n}\cdot\mathbf{p})^4\right)\nu^3\right\}\frac{1}{r^3} \\
&+ \left\{\frac{105}{32}\mathbf{p}^2 + \left(\left(\frac{185761}{19200} - \frac{21837\pi^2}{8192}\right)\mathbf{p}^2 + \left(\frac{3401779}{57600} - \frac{28691\pi^2}{24576}\right)(\mathbf{n}\cdot\mathbf{p})^2\right)\nu\right. \\
&+ \left.\left(\left(\frac{672811}{19200} - \frac{158177\pi^2}{49152}\right)\mathbf{p}^2 + \left(\frac{110099\pi^2}{49152} - \frac{21827}{3840}\right)(\mathbf{n}\cdot\mathbf{p})^2\right)\nu^2\right\}\frac{1}{r^4} \\
&+ \left\{-\frac{1}{16} + \left(\frac{6237\pi^2}{1024} - \frac{169199}{2400}\right)\nu + \left(\frac{7403\pi^2}{3072} - \frac{1256}{45}\right)\nu^2\right\}\frac{1}{r^5}.
\end{aligned}
$$

H: 4PN nonloc

(less familiar with NL Hamiltonians…)

$$
\begin{aligned}
H_{\mathrm{nonloc,4PN}} &= -\frac{1}{5}\frac{G^2 M}{c^8} I_{ij}^{(3)}(t)\,\mathrm{Pf}_{2r/c}\int_{-\infty}^{\infty}\frac{dt'}{|t-t'|} I_{ij}^{(3)}(t') \\
&= -\frac{G^2 M}{c^8}\mathrm{Pf}_{2r/c}\int_{-\infty}^{\infty}\frac{dt'}{|t-t'|}\mathcal{F}_{\mathrm{GW}}^{\mathrm{split,N}}(t,t')
\end{aligned}
$$

$$
\mathcal{F}_{\mathrm{GW}}^{\mathrm{split,N}}(t,t') = \frac{1}{5} I_{ij}^{(3)}(t) I_{ij}^{(3)}(t')
$$

The 4PN Hamiltonian has been fully derived and confirmed within three different approaches

ADM, EOB (Damour, Jaranowski, Schaefer)

H (Bernard, Blanchet, Bohe, Faye, Marsat)

EFT (Foffa, Porto, Rothstein, Sturani)

EOB: Buonanno-Damour 1999

Dimensionless inverse radial variable, also carrying a G

$u = GM/(c^2 r)$

$$H = Mc^2\sqrt{1 + 2\nu(\hat{H}_{\rm eff} - 1)}$$

EOB potentials

$$\hat{H}^2_{\rm eff} = A(u;\nu)(1 + p_\phi^2 u^2 + A(u;\nu)\bar{D}(u;\nu)p_r^2 + \hat{Q}(u, p_r;\nu)),$$

$$\hat{Q}(u, p_r;\nu) = p_r^4 q_4(u;\nu) + p_r^6 q_6(u;\nu) + p_r^8 q_8(u;\nu) + p_r^{10} q_{10}(u;\nu) + \ldots$$

DJS gauge: P_r dependence in place of J dependence

P_r vanishes along circular orbits: the DJS gauge is adapted to circular orbits and will imply an expansion in small eccentricity when considering ellipticlike motion

EOB @3PN

Schwarzschild + 1PN

2PN

3PN

$$A(u) = \boxed{1 - 2u} + 2\nu u^3 + \left(\frac{94}{3} - \frac{41\pi^2}{32}\right)\nu u^4$$

$$\bar{D}(u) = \boxed{1} + 6\nu u^2 + (52\nu - 6\nu^2)u^3$$

$$\hat{Q}(\mathbf{r}', \mathbf{p}') = \left(2(4 - 3\nu)\nu u^2\right)(\mathbf{n}' \cdot \mathbf{p}')^4$$

u=1/r

DJS gauge: radial component of the momentum

Canonically equivalent to the one shown above written in ADM or h coordinates

Coefficients: rational numbers plus π^2
polynomiality in u and ν

5 coefficients only instead of 24: a big computational advantage!

Note: PN by definition lives in the weak-field slow-motion region. It is impossible to enter the strong-field region…even with 20 PN orders!

Starting from 4PN the EOB potentials have a nonlocal structure

Terms with logs are genuine 4PN terms

$$A = A^{\rm loc} + A^{\rm nonloc}\,, \qquad \bar{D} = \bar{D}^{\rm loc} + \bar{D}^{\rm nonloc}\,, \qquad \hat{Q} = \hat{Q}^{\rm loc} + \hat{Q}^{\rm nonloc}$$

The EOB potentials complete up to 4PN, EOB coordinates

Transcendental richness…

$$A(u) = 1 - 2u + 2\nu u^3 + \left(\frac{94}{3} - \frac{41\pi^2}{32}\right)\nu u^4 + \left(\left(\frac{2275\pi^2}{512} - \frac{4237}{60} + \frac{128}{5}\gamma_{\rm E} + \frac{256}{5}\ln 2\right)\nu + \left(\frac{41\pi^2}{32} - \frac{221}{6}\right)\nu^2 + \frac{64}{5}\nu \ln u\right)u^5,$$

New kind of terms in u, logs, gamma

$$\bar{D}(u) = 1 + 6\nu u^2 + (52\nu - 6\nu^2)u^3 + \left(\left(-\frac{533}{45} - \frac{23761\pi^2}{1536} + \frac{1184}{15}\gamma_{\rm E} - \frac{6496}{15}\ln 2 + \frac{2916}{5}\ln 3\right)\nu + \left(\frac{123\pi^2}{16} - 260\right)\nu^2 + \frac{592}{15}\nu \ln u\right)u^4,$$

$q_4(u;\nu)$

$$\hat{Q}(\mathbf{r}', \mathbf{p}') = \left(2(4 - 3\nu)\nu u^2 + \left(\left(-\frac{5308}{15} + \frac{496256}{45}\ln 2 - \frac{33048}{5}\ln 3\right)\nu - 83\nu^2 + 10\nu^3\right)u^3\right)(\mathbf{n}' \cdot \mathbf{p}')^4 + \left(\left(-\frac{827}{3} - \frac{2358912}{25}\ln 2 + \frac{1399437}{50}\ln 3 + \frac{390625}{18}\ln 5\right)\nu - \frac{27}{5}\nu^2 + 6\nu^3\right)u^2(\mathbf{n}' \cdot \mathbf{p}')^6 + O[\nu u(\mathbf{n}' \cdot \mathbf{p}')^8]$$

$q_6(u;\nu)$

Higher powers of ν

Schwarzschild is included. To go beyond with PN one should determine other potentials (i.e., both their local and nonlocal parts)!

All the potentials $A(u;\nu)$, $\bar{D}(u;\nu)$, $\hat{Q}(u,p_r;\nu)$ reduce to their Schwarzschild values when $\nu \to 0$: $A(u;0) = 1-2u$, $\bar{D}(u;0) = 1$, $\hat{Q}(u,p_r;0) = 0$, and can be expanded in powers of ν away from the test-mass limit:

$$A(u;\nu) = 1 - 2u + \nu a^{\nu^1}(u) + \nu^2 a^{\nu^2}(u) + \nu^3 a^{\nu^3}(u) + \ldots$$

$$\bar{D}(u;\nu) = 1 + \nu \bar{d}^{\nu^1}(u) + \nu^2 \bar{d}^{\nu^2}(u) + \nu^3 \bar{d}^{\nu^2}(u) + \ldots$$

$$q_4(u;\nu) = \nu q_4^{\nu^1}(u) + \nu^2 q_4^{\nu^2}(u) + \nu^3 q_4^{\nu^3}(u) + \ldots$$

$$q_6(u;\nu) = \nu q_6^{\nu^1}(u) + \nu^2 q_6^{\nu^2}(u) + \nu^3 q_6^{\nu^3}(u) + \ldots$$

$$q_8(u;\nu) = \nu q_8^{\nu^1}(u) + \nu^2 q_8^{\nu^2}(u) + \nu^3 q_8^{\nu^3}(u) + \ldots.$$

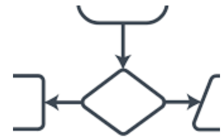
Up now (6PN) we have seen polinomialy in u plus log (u)

A convenient way to order these information: polynomiality in ν

Functions of u are power series with coefficients including log(u) after the 3PN

Beyond the 4PN?

PN, PM, MPM, SF, EOB, EFT, Delaunay averaging formalisms



Capturing information from….

TuttiFrutti

and leading to….

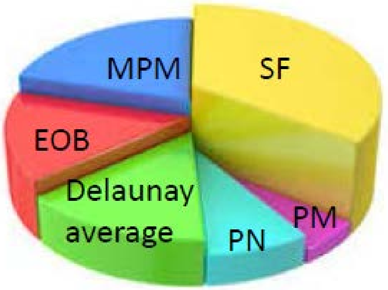
6PN nonlocal part of the two-body Hamiltonian

6PN local part of the two-body Hamiltonian

Nonlocal effects @6PN

1 Evaluate flux split integrals @ 2PN to have the nonlocal Hamiltonian @6PN since nonlocal effects start @4PN

$$H_{\rm nonloc,4PN} = -\frac{G^2 M}{c^8}\mathrm{Pf}_{2r/c}\int_{-\infty}^{\infty}\frac{dt'}{|t-t'|}\mathcal{F}^{\rm split,N}_{\rm GW}(t,t')$$



2 Integrated flux split equals averaged Hamiltonian

Strategy: Computed in h coords and translated in EOB coords

First accomplishment

3 Compare averaged Hamiltonian in h and EOB coordinates

Use of a first GI information

4 Fix the nonlocal part of EOB potentials up to 6PN

Passing from nonloc to loc...

4

Fix the nonlocal part of EOB potentials up to 6PN

Key tool: 1SF+First Law

PHYSICAL REVIEW D **85,** 064039 (2012)

First law of binary black hole mechanics in general relativity and post-Newtonian theory

Alexandre Le Tiec,[1,*] Luc Blanchet,[2,†] and Bernard F. Whiting[3,‡]

$$\nu = \frac{m_1 m_2}{(m_1+m_2)^2}$$ sym mass ratio

Second accomplishment

5

Evaluate z_1 from 1SF, expanded in eccentricity, which incorporates both local and nonlocal effects

Use of a second GI information

6

Evaluate z_1 in EOB using the local part of the Hamiltonian in a parametrized form with unknown coefficients

Third accomplishment

Compare the two and fix the local EOB potentials at $O(\nu)$ since we can use 1SF only

8

Compute the scattering angle in EOB with a parametrized ν dependence of unknown coefficients

Use of a third GI information

9

Use a special «rule» to fix the allowed dependence on ν of the local EOB potentials

10

Determine all missing coefficients except for 2 @5PN and 4 @6PN

GI quantities involved in the TF approach

→ Averaged NL Hamiltonian

→ Redshift

→ Scattering angle

Just the program…details (not many) will follow

Gravitational Self-Force approach/program

This is standard BH perturbation theory a la Regge-Wheeler-Zerilli in the Schwarzschild case or a la Teukolsky in the Kerr spacetime.

The program is standard:

→ Decompose the perturbed metric in spherical harmonics,

→ Fourier-transform in time,

→ Split the 10 radial equations according to their parity, summarize them by a single radial equation (Regge-Wheeler-Zerilli) with Dirac-delta source terms.

→ Solve the RWZ homogeneous (PN solutions and MST solutions, in and up)

→ Solve the RWZ inhomogeneous by the Green function method.

→ Resum in m (ranging from –l to l) and in l (ranging from 0 to infinity).

→ Divergent series in l to be regularized (removal of the singular field).

→ Problem with gauge-dependent low multipoles l=0,1.

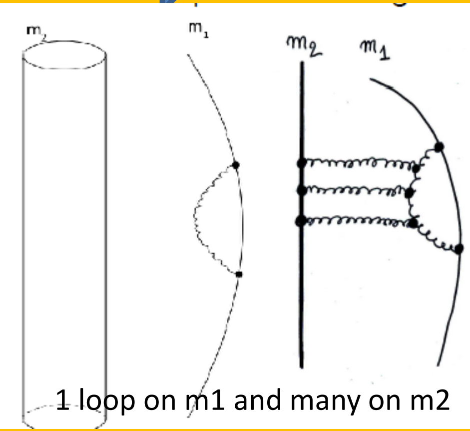
→ Compute a gauge invariant quantity [redshift, periastron advance, gyroscopic precessions..]

→ Convert information in the Hamiltonian formalism.

Expansion in small eccentricity implies series expansion around the circular orbit: e^8
Dirac-delta source terms with eight derivatives of the delta

Maple or Mathematica

Computing z_1 1SF



$$
\begin{aligned}
\delta z_1^{e^8}(u) \;=\;& C_3 u^3 + C_4 u^4 + (C_5^{\rm c} + C_5^{\rm ln} \ln u) u^5 \\
&+ (C_6^{\rm c} + C_6^{\rm ln} \ln u) u^6 + C_{13/2} u^{13/2} \\
&+ (C_7^{\rm c} + C_7^{\rm ln} \ln u) u^7 + C_{15/2} u^{15/2} \\
&+ (C_8^{\rm c} + C_8^{\rm ln} \ln u + C_8^{\rm ln^2} \ln^2 u) u^8 + C_{17/2} u^{17/2} \\
&+ (C_9^{\rm c} + C_9^{\rm ln} \ln u + C_9^{\rm ln^2} \ln^2 u) u^9 \\
&+ (C_{19/2}^{\rm c} + C_{19/2}^{\rm ln} \ln u) u^{19/2} \\
&+ O_{\ln(u)}(u^{10}) \,,
\end{aligned}
$$

$$
z_1 = \langle \frac{\partial H}{\partial m_1} \rangle
$$

External computation, performed by using standard 1SF technology.

C_3	$\frac{15}{64}$
C_4	$\frac{3001}{384} - \frac{287}{4096}\pi^2$
$C_5^{\rm c}$	$\frac{4597}{96} - \frac{162109375}{2304}\ln(5) - \frac{11332791}{1280}\ln(3) + \frac{55}{6}\gamma + \frac{15967961}{90}\ln(2) - \frac{474715}{196608}\pi^2$
$C_5^{\rm ln}$	$\frac{55}{12}$
$C_6^{\rm c}$	$-\frac{9863051}{40320} + \frac{968889010407}{442368}\ln(7) - \frac{64481546637}{114688}\ln(3) - \frac{5977}{240}\gamma - \frac{16605499789}{5040}\ln(2) + \frac{1466047}{196608}\pi^2 + \frac{4761539921875}{3096576}\ln(5)$
$C_6^{\rm ln}$	$-\frac{5977}{480}$

Converting z_1 1SF information in EOB potentials via the first law of the two-body dynamics (N. analytic 1SF encodes both local+nonlocal effects)

$$
\begin{aligned}
q_8^{\nu^1}(u) &= B_1 u + B_2 u^2 + B_{5/2} u^{5/2} \\
&+ B_3 u^3 + B_{7/2} u^{7/2} \\
&+ (B_4^{\rm c} + B_4^{\rm ln} \ln u) u^4 + B_{9/2} u^{9/2} \\
&+ (B_5^{\rm c} + B_5^{\rm ln} \ln u) u^5 + (B_{11/2}^{\rm c} + B_{11/2}^{\rm ln} \ln u) u^{11/2} \\
&+ O_{\ln(u)}(u^6) \,,
\end{aligned}
$$

Subtract the (sum of local and nonlocal) EOB potential obtained from 1SF from their nonlocal parts obtained by using Delaunay averaging to obtain the local part of the EOB potentials.
1SF allows to compute only terms linear in ν: all other coefficients should be included in a parametrized form.

B_1	$-\frac{27734375}{126}\ln(5) + \frac{6591861}{350}\ln(3) + \frac{21668992}{45}\ln(2) - \frac{35772}{175}$
B_2	$\frac{13841287201}{17280}\ln(7) - \frac{393786545409}{156800}\ln(3) - \frac{16175693888}{1575}\ln(2) + \frac{875090984375}{169344}\ln(5) + \frac{5790381}{2450}$
$B_{5/2}$	$+\frac{5994461}{12700800}\pi$

Example of the GSF computation: redshift

$$c_1 = -1, \quad c_2 = -2, \quad c_3 = -5, \quad c_4 = \frac{41}{32}\pi^2 - \frac{121}{3},$$

$$c_5 = -\frac{1157}{15} - \frac{128}{5}\gamma + \frac{677}{512}\pi^2 - \frac{256}{5}\log(2), \quad c_5^{\ln} = -\frac{64}{5},$$

$$c_6 = \frac{1606877}{3150} + \frac{1912}{105}\gamma - \frac{60343}{768}\pi^2 + \frac{7544}{105}\log(2) - \frac{243}{7}\log(3), \quad c_6^{\ln} = \frac{956}{105}, \quad c_{6.5} = -\frac{13696}{525}\pi,$$

$$c_7 = \frac{17083661}{4050} + \frac{102512}{567}\gamma - \frac{1246056911}{1769472}\pi^2 + \frac{2800873}{262144}\pi^4 + \frac{372784}{2835}\log(2) + \frac{1215}{7}\log(3),$$

$$c_7^{\ln} = \frac{51256}{567}, \quad c_{7.5} = \frac{81077}{3675}\pi,$$

$$c_8 = -\frac{2048}{5}\boxed{\zeta(3)} + \frac{12624956532163}{382016250} - \frac{10327445038}{5457375}\gamma + \frac{109568}{525}\gamma^2 - \frac{9041721471697}{2477260800}\pi^2 - \frac{23851025}{16777216}\pi^4 + \frac{438272}{525}\log^2(2)$$
$$- \frac{16983588526}{5457375}\log(2) + \frac{438272}{525}\gamma\log(2) - \frac{2873961}{24640}\log(3) - \frac{1953125}{19008}\log(5),$$

$$c_8^{\ln} = -\frac{5163722519}{5457375} + \frac{109568}{525}\gamma + \frac{219136}{525}\log(2), \quad c_8^{\ln^2} = \frac{27392}{525}, \quad c_{8.5} = \frac{82561159}{467775}\pi,$$

$$c_9 = -\frac{41408}{105}\boxed{\zeta(3)} - \frac{7516581717416867}{34763478750} - \frac{1526970297506}{496621125}\gamma - \frac{108064}{2205}\gamma^2 - \frac{246847155756529}{18496880640}\pi^2 + \frac{22759807747673}{6442450944}\pi^4$$
$$- \frac{2143328}{1575}\log^2(2) + \frac{18954}{49}\log^2(3) - \frac{1363551923554}{496621125}\log(2) - \frac{3574208}{3675}\gamma\log(2) - \frac{2201898578589}{392392000}\log(3)$$
$$+ \frac{37908}{49}\gamma\log(3) + \frac{37908}{49}\log(2)\log(3) + \frac{798828125}{741312}\log(5)$$

This is up to u^9. C Kavanagh et al reached the incredible result u^{22}.

IMPORTANT NOTICE

Computation of the nonlocal (tail-related) parts of the EOB potentials (up to 6PN) is easy!
Beyond the 6PN the nonlocality becomes technically more involved.

$$H^{4+5+6\mathrm{PN}}_{\mathrm{nonloc,h}}(t) \; = \; \frac{G\mathcal{M}}{c^3}\mathrm{Pf}_{2r^h_{12}(t)/c}\int \frac{dt'}{|t-t'|}\mathcal{F}^{\mathrm{split}}_{2\mathrm{PN}}(t,t')\,.$$

$$\begin{aligned} r &= a_r(1-e_r\cos u)\,, \\ \ell &= n(t-t_P) = u - e_t\sin u + f_t \sin V + g_t(V-u)\,, \\ \bar{\phi} &= \frac{\phi-\phi_P}{K} = V + f_\phi \sin 2V + g_\phi \sin 3V\,, \end{aligned} \qquad V(u) = 2\arctan\left[\sqrt{\frac{1+e_\phi}{1-e_\phi}}\tan\frac{u}{2}\right]$$

Integration performed expanding in small eccentricity!

To have H nonloc,h at 6PN one needs the flux split at the 2PN level

2PN Flux split ← 2PN mutipolar moments ← 2PN parametric eqs of ellptic orbits ← Quasi-Keplerian form

t,t

$$\mathcal{F}_{2\mathrm{PN}}(t) = \frac{G}{c^5}\frac{1}{5}I^{(3)}_{ab}(t)I^{(3)}_{ab}(t) + \dots$$ GW flux

$$\mathcal{F}^{\mathrm{split}}_{2\mathrm{PN}}(t,t') = \frac{G}{c^5}\frac{1}{5}I^{(3)}_{ab}(t)I^{(3)}_{ab}(t') + \dots$$

t,t'

Various multipolar contributions as the building block of the flux function: electric quadrupole, electric octupole, magnetic quadrupole, electric hexadecapole, magnetic octupole.

It is the flux split version of the GW flux which enters the nonlocal part of the two-body Hamiltonian!

→ Building blocks of the NL Hamiltonian

$$\langle H^h_{\rm nonloc,h}\rangle \equiv \langle H^{4+5+6{\rm PN}}_{\rm nonloc,h}\rangle = F^h(a^h_r, e^h_t)$$

$$= \langle H^{\rm eob}_{\rm nonloc}\rangle$$

F^h is a GI quantity which can be used to fix completely the NL part of the EOB Hamiltonian

$$\hat{H}^2_{\rm eff} = A(u;\nu)\Big(1 + p_\phi^2 u^2 + A(u;\nu)\bar{D}(u;\nu)p_r^2 + \hat{Q}(u,p_r;\nu)\Big)$$

$$\hat{Q}(u,p_r;\nu) = p_r^4 q_4(u;\nu) + p_r^6 q_6(u;\nu) + p_r^8 q_8(u;\nu) + p_r^{10} q_{10}(u;\nu) + \ldots$$

$$A = A^{\rm loc,h} + A^{\rm nonloc,h}$$
$$\bar{D} = \bar{D}^{\rm loc,h} + \bar{D}^{\rm nonloc,h}$$
$$\hat{Q} = \hat{Q}^{\rm loc,h} + \hat{Q}^{\rm nonloc,h}$$

Fully determined

Still to be determined

TABLE VI: h-route nonlocal EOB coefficients.

$a_7^{\rm nl,c}$	$\left(\frac{206740}{567}\ln(2)+\frac{12664}{105}-\frac{4617}{14}\ln(3)-\frac{5044}{405}\gamma\right)\nu + \left(-\frac{1139672}{945}\ln(2)+\frac{10132}{105}+\frac{10449}{7}\ln(3)+\frac{101272}{315}\gamma\right)\nu^2 + \left(-\frac{112}{5}+32\gamma+\frac{1214624}{945}\ln(2)-\frac{4860}{7}\ln(3)\right)\nu^3\,,$
$a_7^{\rm nl,log}$	$-\frac{2522}{405}\nu+\frac{50636}{315}\nu^2+16\nu^3\,,$
$\bar{d}_6^{\rm nl,c}$	$\left(-\frac{6381680}{189}\ln(2)+\frac{2043541}{2835}+\frac{1765881}{140}\ln(3)-\frac{64096}{45}\gamma+\frac{9765625}{2268}\ln(5)\right)\nu + \left(\frac{28429312}{189}\ln(2)-\frac{3576231}{70}\ln(3)+\frac{167906}{105}+\frac{302752}{105}\gamma-\frac{9765625}{378}\ln(5)\right)\nu^2 + \left(-\frac{9908480}{63}\ln(2)-\frac{744704}{945}+\frac{9765625}{252}\ln(5)+\frac{2944}{3}\gamma+\frac{1275021}{28}\ln(3)\right)\nu^3\,,$
$\bar{d}_6^{\rm nl,log}$	$-\frac{32048}{45}\nu+\frac{151376}{105}\nu^2+\frac{1472}{3}\nu^3$
$q_{45}^{\rm nl,c}$	$\left(\frac{70925884}{63}\ln(2)+\frac{13212013}{5670}-\frac{3873663}{16}\ln(3)-\frac{8787109375}{27216}\ln(5)-\frac{617716}{315}\gamma\right)\nu + \left(\frac{92560887}{280}\ln(3)-\frac{12619052648}{2835}\ln(2)-\frac{1437979}{63}+\frac{632344}{315}\gamma+\frac{7755859375}{4536}\ln(5)\right)\nu^2 + \left(-\frac{177316}{35}+\frac{11263031264}{2835}\ln(2)+\frac{16544}{9}\gamma-\frac{4091796875}{2268}\ln(5)+\frac{2908467}{20}\ln(3)\right)\nu^3$
$q_{45}^{\rm nl,log}$	$-\frac{308858}{315}\nu+\frac{316172}{315}\nu^2+\frac{8272}{9}\nu^3$
$q_{64}^{\rm nl,c}$	$\left(-\frac{211076833264}{14175}\ln(2)-\frac{137711989}{28350}-\frac{9678652821}{5600}\ln(3)+\frac{447248}{1575}\gamma+\frac{153776136875}{23328}\ln(5)+\frac{96889010407}{116640}\ln(7)\right)\nu + \left(\frac{44592947739}{2800}\ln(3)+\frac{2411178384736}{42525}\ln(2)-\frac{126070663}{4725}-\frac{26848}{175}\gamma-\frac{796015515625}{27216}\ln(5)-\frac{96889010407}{19440}\ln(7)\right)\nu^2 + \left(-\frac{40513708}{4725}-\frac{109566260523}{5600}\ln(3)+\frac{1424826953125}{54432}\ln(5)+\frac{96889010407}{12960}\ln(7)+\frac{2368}{5}\gamma-\frac{431564554688}{8505}\ln(2)\right)\nu^3$
$q_{64}^{\rm nl,log}$	$\frac{223624}{1575}\nu-\frac{13424}{175}\nu^2+\frac{1184}{5}\nu^3$
$q_{83}^{\rm nl,c}$	$\left(\frac{5196312336176}{35721}\ln(2)+\frac{175156380279261}{313600}\ln(3)-\frac{63886617280625}{1016064}\ln(5)-\frac{29247366220639}{933120}\ln(7)-\frac{709195549}{132300}\right)\nu + \left(-\frac{177055674739808}{297675}\ln(2)-\frac{43719724468071}{156800}\ln(3)+\frac{366449151015625}{1524096}\ln(5)+\frac{26506549233199}{155520}\ln(7)-\frac{1746293}{70}\right)\nu^2 + \left(\frac{57604236136064}{99225}\ln(2)+\frac{104675833300341}{39200}\ln(3)-\frac{73366198046875}{381024}\ln(5)-\frac{7709596970957}{38880}\ln(7)-\frac{154862}{21}\right)\nu^3$
$q_{83}^{\rm nl,log}$	0

List of the EOB nonlocal coefficients determined by comparison of averages

The nonlocal part of all the EOB potentials (a, $\bar{d}$, q_n) needed @ 6PN is fully known.

How can we determine the local part of the EOB potentials?

→Compute another GI invariant quantity (the redshift) which incorporates both local and nonlocal effects in two different ways:

1) 1SF (SF computations are only linear in ν)
2) Using the full EOB Hamiltonian, connected via the first law of two-body dynamics (Le Tiec-Blanchet-Whiting 2012)

$$z_1 = \langle \frac{\partial H}{\partial m_1} \rangle$$

→Compare the two («subtract» nonlocal from the full local plus nonlocal objects), and identify the local part of the EOB potentials! But only linear in ν!

Determining the local part of the EOB potential @ O(ν^1)

$$a_{4+5+6\mathrm{PN,loc,f}} = \left[\left(\frac{2275}{512}\pi^2 - \frac{4237}{60}\right)\nu + \left(\frac{41}{32}\pi^2 - \frac{221}{6}\right)\nu^2\right]u^5 + \left[\left(-\frac{1026301}{1575} + \frac{246367}{3072}\pi^2\right)\nu + a_{6,f}^{(\nu)}\right]u^6$$
$$+ \left[\left(-\frac{2800873}{262144}\pi^4 + \frac{608698367}{1769472}\pi^2 - \frac{1469618167}{907200}\right)\nu + a_{7,f}^{(\nu)}\right]u^7 ,$$

$$\bar{d}_{4+5+6\mathrm{PN,loc,f}} = \left[\left(\frac{1679}{9} - \frac{23761}{1536}\pi^2\right)\nu + \left(-260 + \frac{123}{16}\pi^2\right)\nu^2\right]u^4 + \left(\frac{331054}{175}\nu - \frac{63707}{512}\nu\pi^2 + \bar{d}_{5,f}^{(\nu)}\right)u^5$$
$$+ \left[\left(\frac{229504763}{98304}\pi^2 + \frac{135909}{262144}\pi^4 - \frac{99741733409}{6350400}\right)\nu + \bar{d}_{6,f}^{(\nu)}\right]u^6 ,$$

$$q_{4,4+5+6\mathrm{PN,loc,f}} = \left(20\nu + q_{43}^{(\nu)}\right)u^3 + \left[\left(-\frac{93031}{1536}\pi^2 + \frac{1580641}{3150}\right)\nu + q_{44,f}^{(\nu)}\right]u^4$$
$$+ \left[\left(\frac{81030481}{65536}\pi^2 - \frac{3492647551}{423360}\right)\nu + q_{45,f}^{(\nu)}\right]u^5 ,$$

$$q_{6,4+5+6\mathrm{PN,loc,f}} = \left(-\frac{9}{5}\nu + q_{62}^{(\nu)}\right)u^2 + \left(\frac{123}{10}\nu + q_{63,f}^{(\nu)}\right)u^3$$
$$+ \left[\left(-\frac{9733841}{327680}\pi^2 - \frac{112218283}{294000}\right)\nu + q_{64,f}^{(\nu)}\right]u^4 ,$$

$$q_{8,5+6\mathrm{PN,loc,f}} = q_{82}(\nu)u^2 + \left(-\frac{7447}{560}\nu + q_{83}^{(\nu)}\right)u^3 ,$$

$$q_{10,6\mathrm{PN,loc,f}} = q_{10,2}(\nu)u^2 .$$

Parametrize the local part of the EOB potential at all orders ν^n (n>1) by using uknown coefficients!

Almost all of them will be determined by using the scattering angle along hyp orbits

And now?

Compute another GI invariant quantity (the scattering angle along hyperboliclike orbits) whose ν structure is known

$$\frac{1}{2}\chi^{\rm loc,f}(\gamma, j;\nu) = \sum_{n\geq 1}\frac{\chi_n^{\rm loc,f}(\gamma;\nu)}{j^n}$$

$$\widetilde{\chi}_n^{\rm loc,f}(\gamma;\nu) \equiv [h(\gamma;\nu)]^{n-1}\,\chi_n^{\rm loc,f}(\gamma;\nu)$$

$$h(\gamma;\nu) \equiv \sqrt{1+2\nu(\gamma-1)} = \frac{H}{Mc^2}\,.$$

$$C_n^{\rm tot}\,:\,\widetilde{\chi}_n^{\rm tot}(\gamma;\nu) = P_{d_n}^{\gamma}(\nu)\;;\;\text{with } d_n \equiv \left[\frac{n-1}{2}\right]$$

[Technical: the introduction of the flexibility function allows both the local and the nonlocal parts to satisfy this rule!]

T. Damour, *Classical and quantum scattering in post-Minkowskian gravity*, Phys. Rev. D **102** (2), 024060 (2020). [arXiv:1912.02139 [gr-qc]].

Using the ν structure of the scattering angle one determines most of the ν structure of the local Hamiltonian

The scattering angle depends on unknown EOB parameters (at either 5PN and 6PN)

$$\frac{1}{2}(\chi(\gamma, j)+\pi) = -\int_0^{u_{\max}} \frac{\partial}{\partial j} p_r(u;\gamma, j)\frac{du}{u^2}$$

$$\begin{aligned}\gamma^2 &= \widehat{\mathcal{E}}^2_{\rm eff} = \widehat{H}^{2\,\rm EG}_{\rm eff,loc,f}(u, p_r, j;\nu)\\ &= H_S^2 + (1-2u)\widehat{Q}^{\rm EG}_{H\,\rm loc,f}(u, H_S;\nu)\end{aligned}$$

$$p_r^2(\gamma, j, u) = \frac{\gamma^2 - (1-2u)\left(1 + j^2u^2 + \widehat{Q}^{\rm EG}_{E\,\rm loc,f}(u,\gamma;\nu)\right)}{(1-2u)^2}$$

$$p_r = p_r^{(0)} + Gp_r^{(1)} + G^2 p_r^{(2)} + \dots,$$

[Here computations are conveniently performed in the energy gauge and not in the DJS gauge]

$$\begin{aligned}\chi_5^{\rm loc,f} = {}& \frac{1}{5p_\infty^5} - \frac{2}{p_\infty^3}\eta^2 + \frac{32-8\nu}{p_\infty}\eta^4\\ &+\left[320 + \left(-\frac{1168}{3} + \frac{41}{8}\pi^2\right)\nu + 24\nu^2\right]p_\infty\eta^6\\ &+\left[640 + \left(\frac{5069}{144}\pi^2 - \frac{227059}{135}\right)\nu\right.\\ &\left.+\left(-\frac{287}{24}\pi^2 + \frac{7342}{9}\right)\nu^2 - 40\nu^3\right]p_\infty^3\eta^8\\ &+\left[\frac{1792}{5} + \left(-\frac{1460479}{525} + \frac{111049}{960}\pi^2\right)\nu\right.\\ &+\left(\frac{41026}{15} - \frac{40817}{640}\pi^2 - \frac{4}{15}\bar{d}_5^{\nu^2}\right)\nu^2\\ &\left.+\left(-\frac{11108}{9} + \frac{451}{24}\pi^2\right)\nu^3 + 56\nu^4\right]p_\infty^5\eta^{10}\\ &+\left[\left(\frac{93031}{2304}\pi^2 - \frac{498343703}{604800}\right)\nu\right.\\ &+\left(\frac{2827607}{1152} - \frac{31633}{768}\pi^2\right)\nu^2\\ &+\left(\frac{205}{16}\pi^2 - \frac{253361}{96}\right)\nu^3 + \frac{212879}{384}\nu^4 + \frac{63}{64}\nu^5\\ &\left.-2q_{3\rm EG}^4 - 4q_{4\rm EG}^3 - \frac{4}{3}q_{5\rm EG}^2\right]p_\infty^7\eta^{12}.\end{aligned}$$

Results

We have computed **3** gauge invariant quantities: averaged GW flux (elliptic), redshift (elliptic), scattering angle (hyperbolic).

We have determined the local 5PN (first) and 6PN (later) EOB Hamiltonian modulo

2 unknown at 5PN and 4 more unknown at 6PN.

Note that thinking of all possible terms entering the 6PN real Hamiltonian, we have determined 147 terms of the 151 total number!

@5PN instead the real Hamiltonian contains 97 coefficients of which we have determined 95!

The missing terms are second order in ν in the EOB potentials: 1SF cannot help!
One needs 2SF which is the next challenge in SF

PHYSICAL REVIEW LETTERS **124,** 021101 (2020)

Second-Order Self-Force Calculation of Gravitational Binding Energy in Compact Binaries

Adam Pound,[1] Barry Wardell,[2] Niels Warburton,[2] and Jeremy Miller[1]

What remains….

$$A(u;\nu) = 1 - 2u + \nu a^{\nu^1}(u) + \nu^2 a^{\nu^2}(u) + \nu^3 a^{\nu^3}(u) + \dots$$

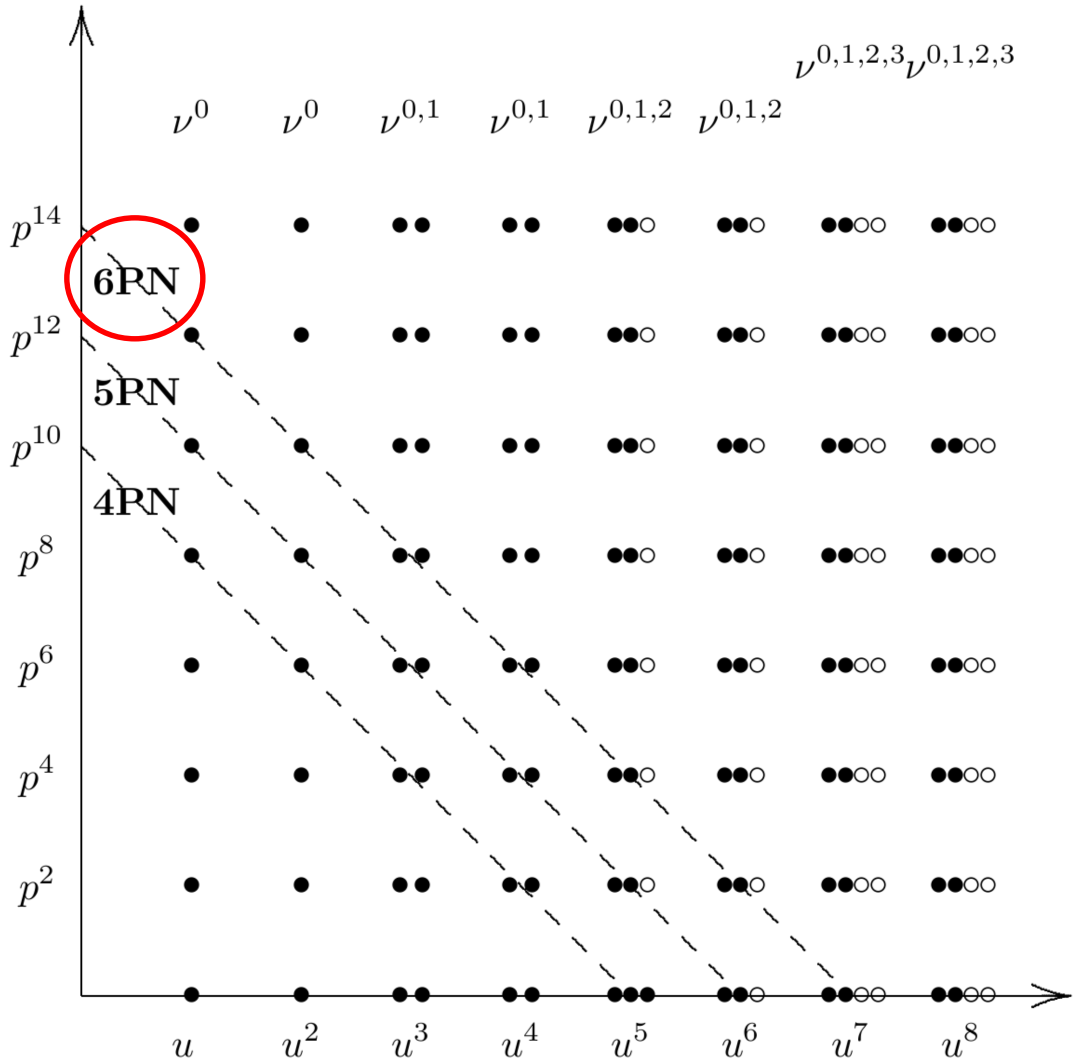
$$\bar{D}(u;\nu) = 1 + \nu \bar{d}^{\nu^1}(u) + \nu^2 \bar{d}^{\nu^2}(u) + \nu^3 \bar{d}^{\nu^2}(u) + \dots$$

Reminder of EOB potentials

List of the f-route EOB potentials in p_r-gauge.

$a_5^{\mathrm{loc,f}}$	$\left(-\frac{4237}{60} + \frac{2275}{512}\pi^2\right)\nu + \left(\frac{41}{32}\pi^2 - \frac{221}{6}\right)\nu^2$
$a_6^{\mathrm{loc,f}}$	$\left(-\frac{1026301}{1575} + \frac{246367}{3072}\pi^2\right)\nu + a_6^{\nu^2}\nu^2 + 4\nu^3$
$a_7^{\mathrm{loc,f}}$	$\left(-\frac{2800873}{262144}\pi^4 + \frac{608698367}{1769472}\pi^2 - \frac{1469618167}{907200}\right)\nu + a_7^{\nu^2}\nu^2 + a_7^{\nu^3}\nu^3$
$\bar{d}_4^{\mathrm{loc,f}}$	$\left(\frac{1679}{9} - \frac{23761}{1536}\pi^2\right)\nu + \left(\frac{123}{16}\pi^2 - 260\right)\nu^2$
$\bar{d}_5^{\mathrm{loc,f}}$	$\left(\frac{331054}{175} - \frac{63707}{512}\pi^2\right)\nu + \bar{d}_5^{\nu^2}\nu^2 + \left(-\frac{205}{16}\pi^2 + \frac{1069}{3}\right)\nu^3$
$\bar{d}_6^{\mathrm{loc,f}}$	$\left(\frac{229504763}{98304}\pi^2 + \frac{135909}{262144}\pi^4 - \frac{99741733409}{6350400}\right)\nu + \bar{d}_6^{\nu^2}\nu^2 + \left(\frac{45089}{72} - \frac{44489}{1536}\pi^2 - \bar{d}_5^{\nu^2} - 15a_6^{\nu^2}\right)\nu^3 - 48\nu^4$
$q_{43}^{\mathrm{loc,f}}$	$20\nu - 83\nu^2 + 10\nu^3$
$q_{44}^{\mathrm{loc,f}}$	$\left(\frac{1580641}{3150} - \frac{93031}{1536}\pi^2\right)\nu + \left(-\frac{2075}{3} + \frac{31633}{512}\pi^2\right)\nu^2 + \left(640 - \frac{615}{32}\pi^2\right)\nu^3$
$q_{45}^{\mathrm{loc,f}}$	$\left(\frac{81030481}{65536}\pi^2 - \frac{3492647551}{423360}\right)\nu + q_{45}^{\nu^2}\nu^2 + \left(-\frac{14}{3}\bar{d}_5^{\nu^2} + \frac{36677}{1152}\pi^2 - \frac{474899}{216}\right)\nu^3 + \left(\frac{1435}{32}\pi^2 - \frac{7375}{6}\right)\nu^4$
$q_{62}^{\mathrm{loc,f}}$	$-\frac{9}{5}\nu - \frac{27}{5}\nu^2 + 6\nu^3$
$q_{63}^{\mathrm{loc,f}}$	$\frac{123}{10}\nu - \frac{69}{5}\nu^2 + 116\nu^3 - 14\nu^4$
$q_{64}^{\mathrm{loc,f}}$	$\left(-\frac{9733841}{327680}\pi^2 - \frac{112218283}{294000}\right)\nu + \left(\frac{156397}{1280}\pi^2 - \frac{21996581}{21000}\right)\nu^2 + \left(\frac{6977}{6} - \frac{29665}{256}\pi^2\right)\nu^3 + \left(\frac{287}{8}\pi^2 - \frac{3640}{3}\right)\nu^4$
$q_{82}^{\mathrm{loc,f}}$	$\frac{6}{7}\nu + \frac{18}{7}\nu^2 + \frac{24}{7}\nu^3 - 6\nu^4$
$q_{83}^{\mathrm{loc,f}}$	$-\frac{7447}{560}\nu - \frac{963}{56}\nu^2 - \frac{117}{10}\nu^3 - 147\nu^4 + 18\nu^5$
$q_{10,2}^{\mathrm{loc,f}}$	$-\frac{11}{21}\nu - \frac{11}{7}\nu^2 - \frac{20}{7}\nu^3 - \frac{5}{3}\nu^4 + 6\nu^5$

What is needed to complete the 2 body dynamics at a fixed PM level



1PM 2PM 3PM 4PM 5PM 6PM...

Each vertical column of dots describes the PN expansion (keyed by powers of p^2) of an energy-dependent function parametrizing the scattering angle.

The various columns at a given PM level correspond to increasing powers of ν.

D. Bini, T. Damour and A.Geralico,
``Sixth post-Newtonian local-in-time dynamics of binary systems,"
Phys. Rev. D 102, no.2, 024061 (2020)
[arXiv:2004.05407 [gr-qc]].

Schematic representation of the irreducible information contained, at each PM level (keyed by a power of u = GM/r), in the local dynamics.

The next challenge(s)…and open problems

→ Improve the present knowledge of the nonlocal part of χ ($\mathrm{N^nLO}$).

This implies the evaluation of the nl part of the Hamiltonian averaged along hyperbolic orbits (that is of the flux-split integrals expanded in large eccentricity) and can be done both in the time domain and in the frequency domain (with difficulties in both cases…)

→ Compute rr contributions to χ (presently @ 2PN)

This implies the use of the rr χ expression for time sym scattering obtained in 2012

PHYSICAL REVIEW D **86,** 124012 (2012) BD and T. Damour

Gravitational radiation reaction along general orbits in the effective one-body formalism

($N^n LO$) Time domain computations

One needs to compute flux split integrals (with a singularity line). Expand in large eccentricity and work in the time domain.

$$\int_{[-1,1]\times[-1,1]} dT\,dT' \frac{\mathcal{G}(T,T')}{|T-T'|}\,,$$

The arising integrals correspond to very large expressions (numerical computation and then recovery of the analytical results by using the PSLQ algorithm; analytical computations in terms of harmonic polylogs).

We reached the NNNLO level, $O(G^7)$ of accuracy (in the expansion in large eccentricity).

At NNLO we found large integrands , difficult to evaluate even numerically (fruitful collaboration with HEP people: **S. Laporta** and **P. Mastrolia**).

Q_{20}	524.7672921802125843427359557031017584761419995573690119377287112384988398300977120939070371581 9606083170623899520567705206794678374496647513473011101045588318417017082934721207112410611 3165 8613485679

PSLQ

Q_{20}	$\frac{25883}{1800}+\frac{22333}{140}\mathrm{K}-\frac{625463}{3360}\pi-\frac{361911}{560}\pi\ln 2+\frac{99837}{160}\pi\zeta(3)$

Noticeably, in the intermediate results the Catalan constant and ζ(3) enter (enriching the transcendental structure of the result). However, the first cancels out in the scattering angle while BOTH cancel out in the periastron advance, leading to a rational coefficient only (at NNLO).

The NNLO accomplishment

Example of integral entering our computation (in its reduced, final form)

$$J(x) \equiv \int_0^1 dT\,(1-x^2) \times$$

$$\times \frac{16\,\mathrm{arctanh}^3(T) - 3\,\mathrm{Li}_3\left[\left(\frac{(1-T)(1-x)}{(1+T)(1+x)}\right)^2\right]}{2i(T+x)(T+1/x)}$$

Expression in terms of HPLs w=4

sought for result

$$J(i) = -\frac{1}{2}\pi^2\,\mathrm{K} + \frac{9}{2}\pi\zeta(3)$$

Three-log function...
Polylogs in general

At NNLO?

Fitting is easy only once you know the structure of the expected result!

One of these integrands

«About 2-300 times this block»

```
(-86222/315*(-1+T)*
   (-((Tp^12+567504/43111*Tp^10+181497/43111*Tp^8-6429152/
         105476/43111*Tp*(Tp^12-1627314/26369*Tp^10-5379177
         524393/43111*(Tp^12-4160988/524393*Tp^10-24738249/
         6509256/43111*Tp*(-3308411/271219*Tp^8+1662010/271
         386007/43111*(1+Tp)*(-1+Tp)*(Tp^12+6987756/128669*
         21516708/43111*Tp*(-6616822/1793059*Tp^10-6616822/
```

Collaboration with S. Laporta and P. Mastrolia

```
Q40num = 2324.\
29455187550792447967610323846731159860960126009152871136693341059681 55\
96516949527851354984100960313536312955605534132663609242010754814059 64\
23359396092951014959192592441452972186125214023366531144692768025865 85\
58189419149709539123866389339865336003125403243701821797176239443816 29\
79148877927509427603253001916964253799935581589339405677197494555122 95\
065402497571665902497480782514780235722738 3231
```

```
Q40PSLQ = -(21644756692/7640325) + (13986515191 Pi)/
  19051200 - (5616 Pi^2)/25 + (122517 Pi^4)/
  2240 - (11235344 Zeta[3])/51975
```

Frequency domain computations

Expressions involving BesselK Functions, with the integration variable appearing either in the argument or in the order.

$$
\begin{aligned}
\mathcal{K}_{\rm N}(u) \;=\; & -\frac{32}{5}\frac{\nu^2}{\bar{a}_r^2}\frac{p^2}{u^4}e^{-i\pi p}\left\{u^2(p^2+u^2+1)(p^2+u^2)K_{p+1}^2(u)\right.\\
& -2u\left[\left(p-\frac{3}{2}\right)u^2+p(p-1)^2\right](p^2+u^2)K_p(u)K_{p+1}(u)\\
& \left.+2\left[\frac{1}{2}u^6+\left(2p^2-\frac{3}{2}p+\frac{1}{6}\right)u^4+\left(\frac{5}{2}p^4-\frac{7}{2}p^3+p^2\right)u^2+p^4(p-1)^2\right]K_p^2(u)\right\}
\end{aligned}
$$

Expansion in large eccentricity involves derivatives with respect to the order [unknown beyond the third order]

$$
K_p(u) \;=\; K_0(u)+\frac{1}{2}p^2\left.\frac{\partial^2 K_\nu(u)}{\partial\nu^2}\right|_{\nu=0}
$$

Various tricks

Use of various integral transform, like the Mellin transform

$$
\begin{aligned}
\int_0^\infty du\,\mathcal{I}_{\rm 1PN}^{\rm NLO}(u) \;=\; & -\frac{96}{5\pi}\int_{-\infty}^{\infty}dv\arctan\left(\tanh\frac{v}{2}\right)\cosh v\left[\sinh v(g_{K_0\cos}(5;v)+2g_{K_1\cos}(6;v))\right.\\
& \left.+(\cosh^2 v-2)(g_{K_0\sin}(6;v)+g_{K_1\sin}(5;v))\right]\\
=\; & \int dv\arctan\left(\tanh\left(\frac{v}{2}\right)\right)\frac{\sinh v}{\cosh^4 v}\left(-\frac{4032}{5}+\frac{2448}{\cosh^2 v}\right)\\
=\; & \frac{2048}{25},
\end{aligned}
$$

The status of the art

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Higher derivatives of the Bessel functions with respect to the order

Yu. A. Brychkov

$$\left.\frac{\partial^2 K_\nu(z)}{\partial \nu^2}\right|_{\nu=n} = \left.\frac{\partial^2 K_\nu(z)}{\partial \nu^2}\right|_{\nu=-n} = \frac{1}{2}[\ln z - \ln(-z)]$$
$$\times \left[\ln z + \ln\left(-\frac{z}{4}\right) + 2\mathrm{C}\right] K_n(z) + \frac{n!}{2}\left(\frac{z}{2}\right)^{-n} \sum_{k=0}^{n-1} \frac{(z/2)^k}{k!(n-k)}$$
$$\times \{\psi(n-k) - [\ln z - \ln(-z)] + \mathrm{C}\} K_k(z) + \frac{z}{4}\sum_{k=0}^{n}\binom{n}{k}$$
$$\times \frac{(\frac{3}{2})_k}{[(k+1)!(k+1)]^2}(-2z)^k[\ln z - \ln(-z)][zK_{n-k}(z) - 2(n-k)K_{n-k-1}(z)]$$
$$\times\ {}_3F_4\left(\begin{matrix} k+1, k+1, k+\frac{3}{2}; z^2 \\ k+2, k+2, k+2, k+2 \end{matrix}\right)$$
$$-(-1)^n \frac{\pi^{3/2}}{2}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{z}{2}\right)^{-k} I_{n-k}(z) G_{4,6}^{4,1}\left(-z^2 \,\middle|\, \begin{matrix} 0, \frac{1}{2}, \frac{1}{2}, 1 \\ 0, 0, 0, 0, \frac{1}{2}, k \end{matrix}\right)$$
$$-\frac{\sqrt{\pi}}{2}\sum_{k=0}^{n}\binom{n}{k}\left(\frac{z}{2}\right)^{-k}\{(-1)^k K_{n-k}(z) + (-1)^n[\ln z - \ln(-z)] I_{n-k}(z)\}$$
$$\times G_{3,5}^{3,1}\left(-z^2 \,\middle|\, \begin{matrix} \frac{1}{2}, -\frac{1}{2}, 1 \\ 0, 0, 0, -\frac{1}{2}, k \end{matrix}\right). \tag{7.3}$$

MejerG functions

Radiation-Reaction?

B-D 2012

$$\delta^{(\mathrm{RR})}\chi = \frac{1}{2}\left(\frac{\partial \chi^{(\mathrm{conserv})}(\tilde{E},j)}{\partial \tilde{E}}\delta^{(\mathrm{RR})}E + \frac{\partial \chi^{(\mathrm{conserv})}(\tilde{E},j)}{\partial j}\delta^{(\mathrm{RR})}j\right)$$

Known at 2PN
and recently used @ 3PM by T. Damour to solve the paradoxical situation concerning the HE limit of χ.

Obtained in the case of first order rr effects, and in absence of CM total momentum recoil

Passing from the conservative case to the case of presence of rr one uses the parametric eqs of the orbit and a variation of the constants method. Actually the constants which vary are the energy, the angular momentum and the angular dispacement of the apsidal line. The latter, however, does not contribute due to symmetry reasons up to the level in which the above mentioned limitations apply.

The quantity $c_\lambda(t)$ corresponds in our above simplified treatment to the direction of the vector $\mathbf{A}(t)$. We found above that the direction of $\mathbf{A}(t)$ did not include a secular change under the influence of $\mathcal{F}$, because of symmetry reasons linked, finally, to the time-odd character of $\mathcal{F}$. This fact has a correspondent in $c_\lambda(t)$. Indeed, Ref. [61] found that there were no secular changes in $c_\lambda(t)$ [and $c_\ell(t)$] precisely because $dc_\lambda(t)/dt$ is an odd function of ϕ, around the periastron, and remarked that this was linked to the time-odd character of $\mathcal{F}$.

We are currently generalizing this result!
Work in progress @ G^3, G^4 and beyond but @ a fixed PN accuracy

Blumlein, Maier, Marquard, Schaefer et al 2020

Complete agreement with the recent work by J. Blumlein, A. Maier, P. Marquard, G. Schafer, arXiv: 2010.13672 @5PN.

See Eqs. 63-64 there, with the determination of the π^2 part of the 5PN missing coefficients in TF.

$$\boxed{\begin{aligned} \bar{d}_5 &= r_{d_5} + \frac{306545}{512}\pi^2 \\ a_6 &= r_{a_6} + \frac{25911}{256}\pi^2 \end{aligned}}$$

Typo in our 6PN nonloc paper arXiv:2007.11239 [gr-qc]

Eq. (8.27) first line $-\frac{155}{12}\ln 2 \quad \to \quad -\frac{155}{112}\ln 2$

Conclusions and plans for future works

- Can TF method be applied to reach 7PN? → In principle yes…but one finds 6 more undetermined parameters..2+4+6
- Push forward the knowledge of the nonlocal in time dynamics and obtain more information about the nonlocal part of the scattering angle…
- 5PN h-coordinate-based potential Hamiltonian [See the conservative radiation Foffa-Sturani pioneering works, as well as the recent works by J. Blumlein, et al.]
- Scattering angle from 1SF? [L. Barack et al. @Capra 2020, S. Hopper, 2018]
- Full scattering angle @ 4PM from amplitudes? [Potential graviton contributions recently obtained by Z. Bern et al., 2021]
  New and promising avenues from amplitudues [G. Veneziano et al., R. Porto et al., J. Parra Martinez et al., F. Vernizzi et al., etc.]
- Extension of results to spinning bodies or in general to bodies with a given multipolar structure [See e.g. A. Buonanno, J. Steinhoff, J. Vines, etc.]

Thanks for your kind attention!