Integrals for post-Minkowskian classical dynamics

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Motivation: amplitudes approach to two-body dynamics in general relativity.

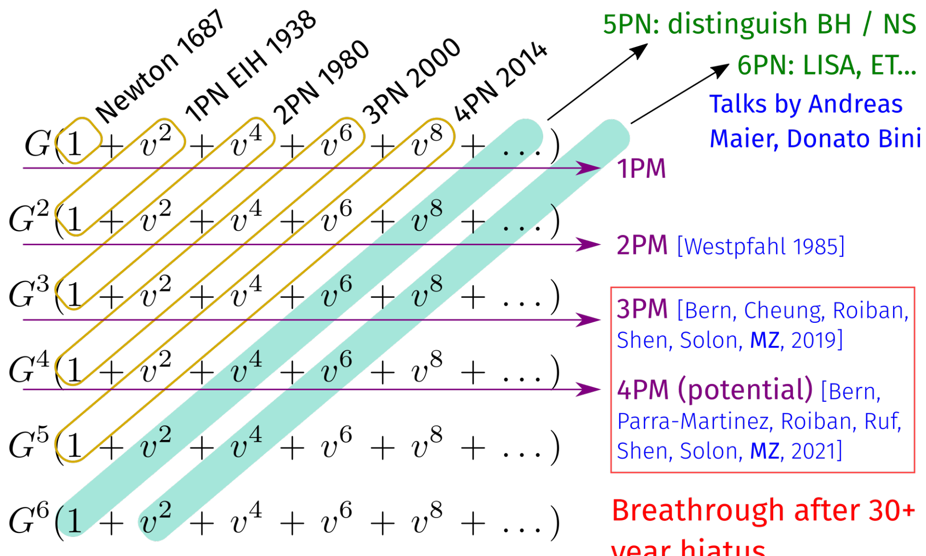
Eikonal method, NR EFT, KMOC formalism, analytic continuation...

Post-Newtonian expansion: small-velocity limit. Integration methods mature.
(simple numbers, pi, zeta values etc.)
- 4PN / 5-loop known, rapid progress at even higher orders.

Post-Minkowskian expansion: exact velocity dependence. Nontrival functions: polylogarithms, elliptic integrals etc. **Integration a key bottleneck.**

Integrand construction: talk by Radu Roibian *Also: talks by Carlo Heissenberg, Ludovic Planté*

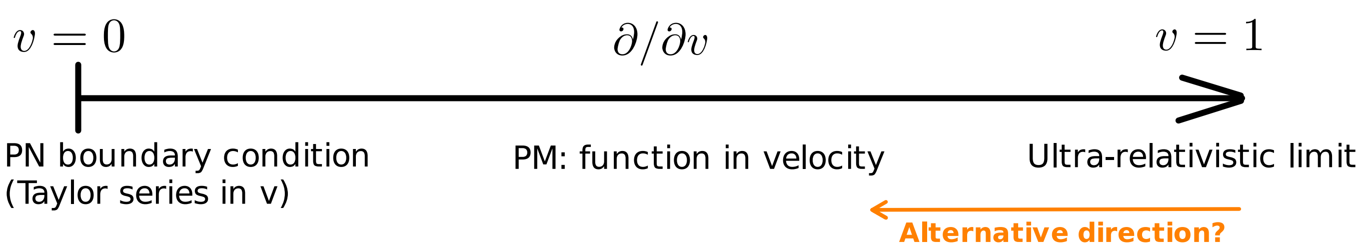
NEW RESULTS FOR CONSERVATIVE DYNAMICS



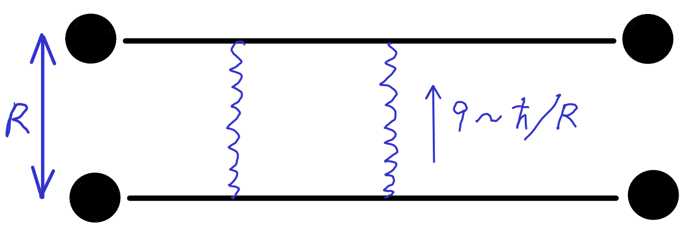
Breathrough after 30+ year hiatus.

[adapted from Mikhail Solon's slide]

Connection between PN and PM integration methods: differential equations



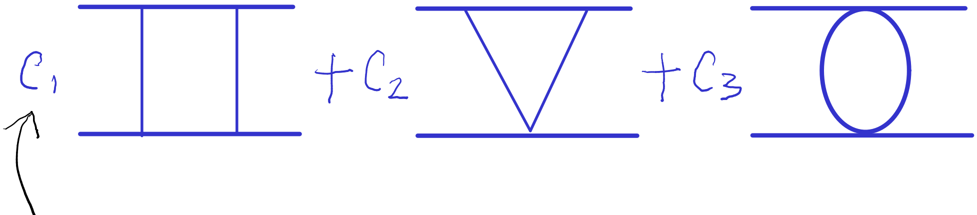
Tools: (1) expansion by regions. *Beneke, Smirnov*
- exchanged graviton momentum $\sim \hbar/R \ll m_1, m_2$



Physically, iterated exchange builds up classical momentum exchange. (c.f. eikonal approximation, and e.o.m. in NR EFT)

(2) Integration-by-parts reduction. Complicated integrals reduced to simpler ones like scalar integrals. *Chetyrkin; Laporta...*

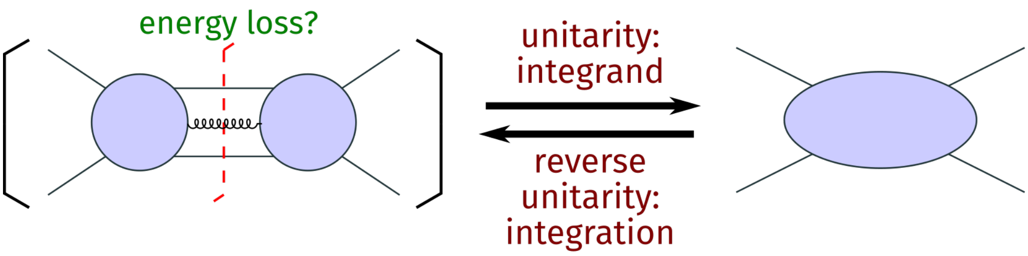
With numerators from e.g. Feynman rules



Rational functions of s, t, m_1, m_2

(3) Differential equations, based on IBP. *Kotikov; Bern, Dixon, Kosower; Gehrmann, Remmidi; Henn...*

(4) Reverse unitarity: cutting rules on steroids. Re-use loop integral techniques for phase space integrals. *Anastasiou, Melnikov...*



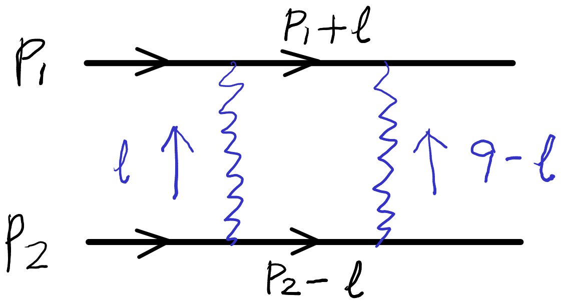
$$2\pi i\,\delta(k^2 - m^2) = \frac{1}{(k^2 - m^2 - i\epsilon)} - \frac{1}{(k^2 - m^2 + i\epsilon)}$$

relativistic mass-shell condition for phase-space

propagator for virtual particles

- Allows us to reuse methods for loop integrals, e.g. IBP, differential equations.

Asymptotic expansion of Feynman integrals

$\lim\limits_{|q|\ll|p_i|} \int d^d \ell$ []

No brainer: Taylor expansion in |q| / |p|.

But how do you treat l ? It may be comparable with |q|, or |p|, or in between.

[Beneke, Smirnov, '98]

Method of regions: the full integral as a series in small |q| to all orders, is a sum over two contributions.

(1) soft region $|q|, |l| \ll |p|$. Contains non-analytic behavior, e.g. $1/q^2$, $\log(-q^2)$.

Taylor expansion in small $|q|/|p|, |l|/|p|$, then integrate over ALL l.

e.g. $1/[(\ell+p_1)^2-m_1^2] = 1/[2p_1\cdot\ell+\ell^2] = 1/(2p_1\cdot\ell)+\cdots$

$|q| \ll |l| \sim |p|$.

(will fine-tune the expansion strategy later)

(2) hard region

Gives Taylor series in q^2. Contact interaction in position space.

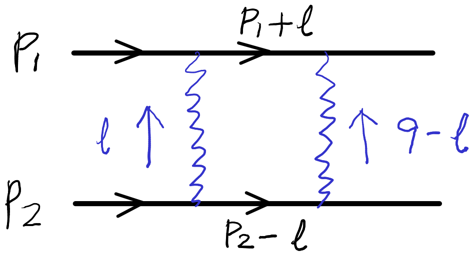
Taylor expansion in small $|q|/|p|$, then integrate over ALL l.

e.g. $1/[(\ell+p_1)^2-m_1^2]$ is unexpanded,

while $1/(\underset{\text{small}}{q}-\ell)^2 = 1/\ell^2+\cdots$

Missing OVERLAP contributions; vanishes as scaleless integrals in dim. reg., in Beneke & Smirnov's expansion prescription.

Later: for velocity expansion, will use alternative principal value / symmetrization prescription



Aside:

Rigorous justification for asymptotic expansions is intricate. For example, consider the soft region,

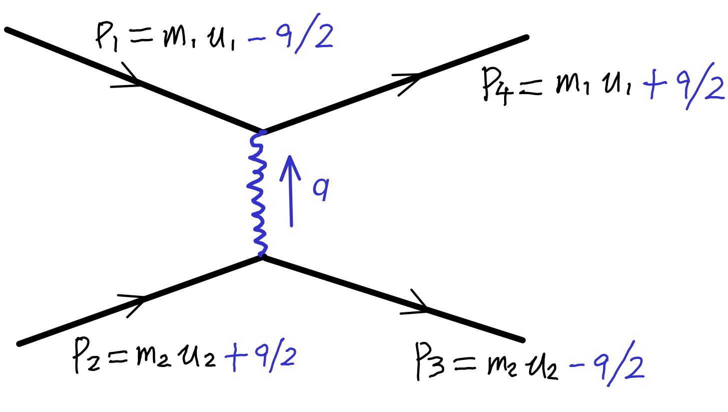
$$1/[(l+p_1)^2-m_1^2] = 1/[2p_1\cdot l+l^2] = 1/(2p_1\cdot l)+\cdots$$

But $|l| \ll |p_1|$ does NOT nessesarily imply $l^2 \ll 2p_1\cdot l$! For example, the latter may become small if p_1 is purely timelike while l is purely spacelike.

This may be a tiny part of integration volume, but the denominator diverges here...
Massive-massless scattering: special region avoided by contour deformation
[Akhoury, Saotome, Sterman, 1308.5204v3]
Fully massive generalization?

Symmetric parametrization for soft region

[Glauber; Polkinghorne; Neill & Rothstein]



$$u_1\cdot q = u_2\cdot q = 0, u_1\cdot u_1 = u_2\cdot u_2 = 1,$$

$$u_1\cdot u_2 = y, \quad q^2 = -t$$

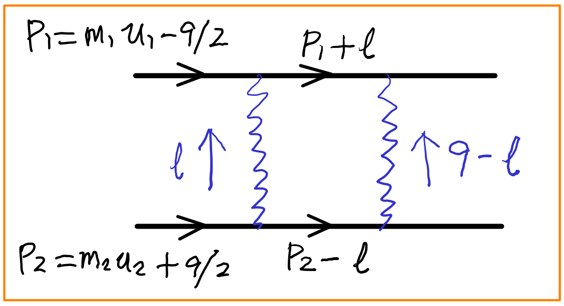
dependence fixed by mass dimension

The only nontrivial parameter which the master integrals depend on.

Used to be s/t, m_1^2/t, m_2^2/t.

Function of 3 variables ⟶ Function of 1 variable. Enormous reduction in complexity.

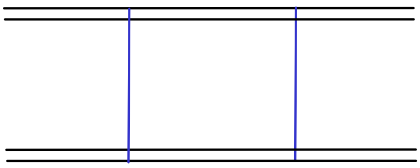
One-loop integrals in soft expansion



$$\frac{1}{2p_1\cdot l + l^2} = \frac{1}{2m_1 u_1\cdot l + (l^2 - q\cdot l)} = \frac{1}{m_1}\frac{1}{2u_1\cdot l} - \frac{1}{m_1^2}\frac{l^2 - q\cdot l}{(2u_1\cdot l)^2}$$

$$\frac{1}{l^2}\frac{1}{(q-l)^2}\frac{1}{(p_1+l)^2 - m_1^2}\frac{1}{(p_2-l)^2 - m_2^2}$$

$$\longrightarrow \frac{1}{l^2}\frac{1}{(q-l)^2}\frac{1}{(2u_1\cdot l + i0)}\frac{1}{(-2u_2\cdot l + i0)} + \dots$$

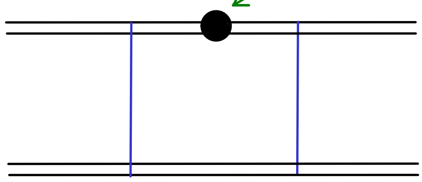


a master integral

Double line = linear propagator

Higher orders in the expansion: will have e.g.

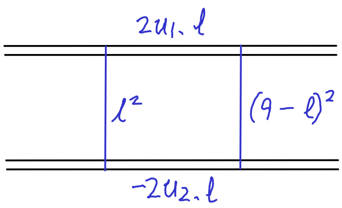
squared linear propagator



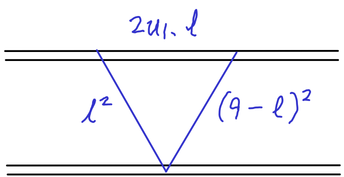
• *Numerators*

Recall that the more complicated integrals evarporate after IBP reduction.

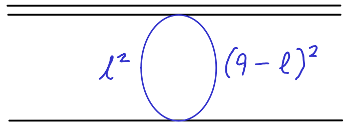
All masters at one loop

$I_{\text{box}} =$ 

linearized box

$I_{\text{tri}} =$ 

linearized triangle

$I_{\text{bub}} =$ 

bubble

The entire one-loop 4-scalar amplitude is a linear combination of these integrals.

Evaluating soft integrals: (1) velocity expansion

Further series expansion around small-velocity limit. *[Parra-Martinez, Ruf, MZ, '20].*

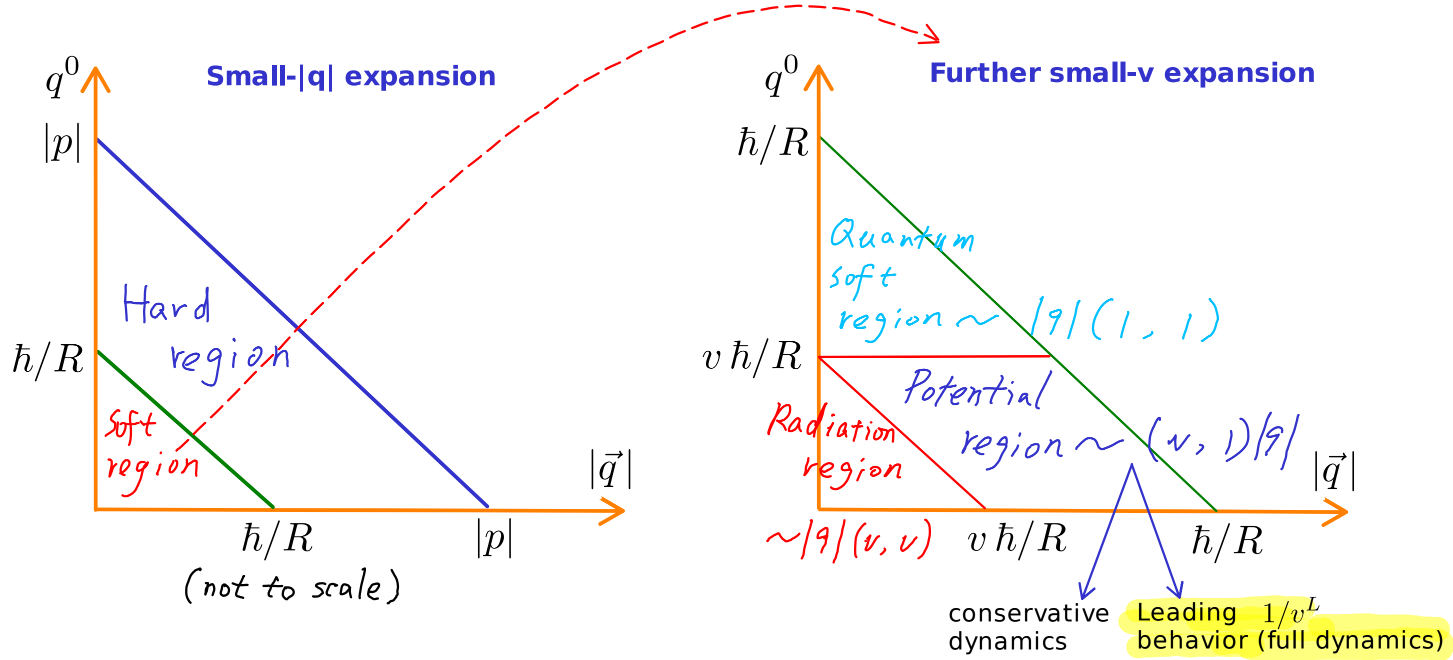
Initially done in opposite order of expansions, in *[Cheung, Bern, Roiban, Solon, Shen, MZ, '19].*

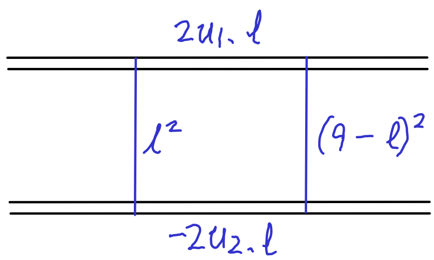
Most well established for conservative dynamics.

Again sum over expansions in several regions. At one and two loops, conservative dynamics comes from only the potential region, in a suitable definition of potential region.

Intuition: exchange of gravitons dominated by spatial momenta, of order $|\vec{\ell}| \sim |q| \sim \hbar/R$.

Energy component suppressed $\ell^0 \sim v|\vec{\ell}| \sim v|q|$.



$I_{\text{box}} =$ 

$u_1 \approx p_1/m_1 \sim (1, \vec{v})$

$(p_1+\ell)^2 - m_1^2 \approx 2m_1 u_1 \cdot \ell$

is put on-shell when $\ell^0 \approx \vec{\ell} \cdot \vec{v}$.

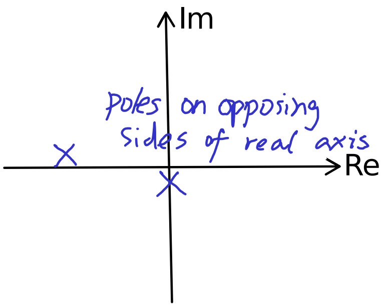
Linearized box integral in potential region: $(l^0, \vec{l}) \sim (qv, q), \quad q = (0, \vec{q})$. Taylor expansion:

$$\frac{1}{l^2} = \frac{1}{-|\vec{l}|^2} + \ldots, \qquad \frac{1}{(q-l)^2} = \frac{1}{-|\vec{q}-\vec{l}|^2} + \ldots,$$

$2u_1 \cdot l, \quad -2u_2 \cdot l$ remain unexpanded

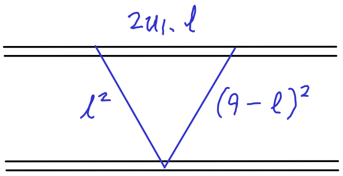
Specializing to frame $u_1 = (1, 0, 0, 0), \quad u_2 = (\sqrt{1+v^2}, 0, 0, v)$,

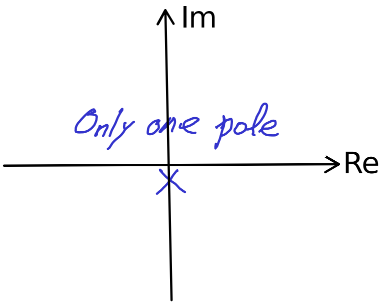
We have $$\int d^3\vec{l} \int dl^0 \, \frac{1}{-|\vec{l}|^2} \, \frac{1}{-|\vec{q}-\vec{l}|^2} \, \underbrace{\frac{1}{2l^0 + i0} \, \frac{1}{-2\sqrt{1+v^2}\, l^0 + v l_z + i0}}_{\text{well-defined contour integral in } l^0}$$



How about the triangle integal? It's a bit more complicated.

$$I_{\text{tri}} =$$



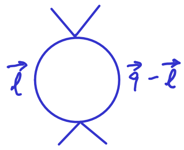


$$\int d^3\vec{l} \int dl^0 \, \frac{1}{-|\vec{l}\,|^2} \, \frac{1}{-|\vec{q}-\vec{l}\,|^2} \, \frac{1}{2l^0+i0}$$

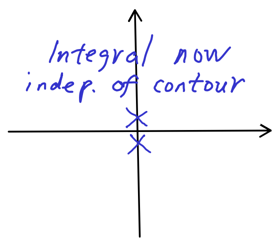
The integral has no dependence on v (though IBP reduction coefficients do), making it strange to talk about the potential region.

Nevertheless, we use a symmetrization prescription, averaging over $l^0 \leftrightarrow -l^0$.

$$\int d^3\vec{l} \int dl^0 \, \frac{1}{-|\vec{l}\,|^2} \, \frac{1}{-|\vec{q}-\vec{l}\,|^2} \, \frac{1}{2l^0+i0} = \int d^3\vec{l} \underbrace{\frac{1}{-|\vec{l}\,|^2} \, \frac{1}{-|\vec{q}-\vec{l}\,|^2}} \underbrace{\int dl^0 \, \frac{1}{2}\left(\frac{1}{2l^0+i0} + \frac{1}{-2l^0+i0}\right)}_{\int -\frac{1}{2}(2\pi i)\,\delta(2l^0)\, dl^0}$$



well known 3D bubble integral

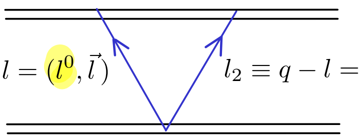


If using Beneke & Smrinov's prescription, this is a scaleless integral set to zero; the triangle integral would be fully captured by quantum soft region.

Our symmetrization prescription coincides with CONSERVATIVE dynamics at 1 and 2 loops.

One loop:

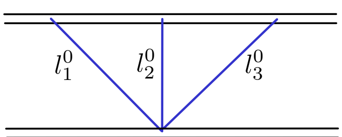
$2u_1 \cdot l = 2l^0$ with $u_1 = (1, \vec{0}\,)$



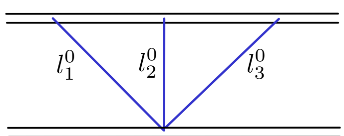
$l_1 \equiv l = (l^0, \vec{l}\,)$ $\quad l_2 \equiv q - l = (-l^0, \vec{q}-\vec{l}\,), \quad q^0 = 0$

Symmetrize over l_1^0 and l_2^0, eliminates contour ambiguity.

Two loops:



$\sim \frac{1}{6}(-2\pi i)^3$

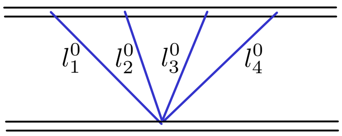


$\sim -\frac{1}{3}(-2\pi i)^3$

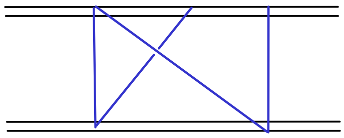
Symmetrize over l_1^0, l_2^0, and l_3^0, eliminates contour ambiguity.

Three loops

Symmetrize over $l^0_{1,2,3,4}$? Does NOT fully eliminate contour ambiguity!



$\sim \frac{1}{24}(-2\pi i)^3$



$\sim \frac{1}{24}(-2\pi i)^3$ or $\frac{1}{8}(-2\pi i)^3$ depending on contour choice

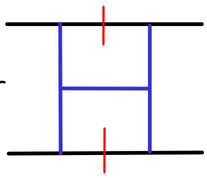
Energy integrals become well defined after summing over diagrams; can still assign symmetry factors to individual diagrams after velocity expansion & IBP.

Evaluating soft integrals: (2) differential equations

In simple case, can promote velocity series to exact functions.

Generally method: differential equations. Well established in loop integration literature. Recently imported into post-Minkowskian gravity.

First version: *[Cheung, Bern, Roiban, Ruf, Solon, MZ, '19]*

$\frac{\partial}{\partial v}$  $= \ldots$

LHS & RHS as functions of s, t, m_1, m_2, then take limit $t \to 0$.

Only applied to $H + \bar{H}$. IBP reduction time consuming.

Simplified version: soft expansion first, obtain DEs with only nontrivial dependece on v.

Parra-Martinez, Ruf, MZ, '20. Application in worldline PM EFT: Porto, Kalin, '20.
Solutions with soft boundary conditions: Di Vecchia, Heissenberg, Russo, Veneziano, '21,
Herrmann, Parra-Martinez, Ruf, MZ, '21.
Further development in soft DEs: Bjerrum-Bohr, Damgaard, Plante, Vanhove, '21

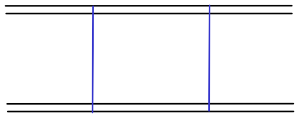
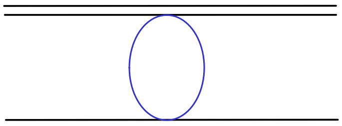
General structure of DEs: start with master integrals, grouped in a column vector $\vec{I}$.

Derivatives $\partial \vec{I} / \partial v$ reduced to original set of masters, with rational (in Mandelstams) coefficients.

$$\frac{\partial \vec{I}}{\partial v} = M \cdot \vec{I}.$$

singularity sturcture determines function space.

Example:

$$\frac{\partial}{\partial v}\left(v \right.$$  $$\left.\right) = \frac{\partial \log(\sqrt{1+v^2}-v)}{\partial v} \times$$ 

Vanishes in potential region defined in terms of matter propagator residues

$= -\frac{1}{\varepsilon} i\pi$, constant in potential region.

Explains nontrivial magic cancellation at every higher order in v,
in direct expansion.

If working in soft region without truncation to potential region, RHS is a v-indep. constant,

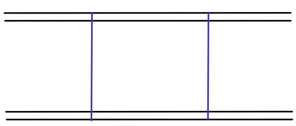
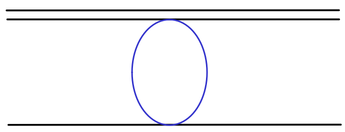
$$= -\frac{1}{\varepsilon}\left[i\pi + \log\left(\sqrt{1+v^2} - v\right)\right]$$

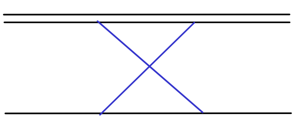
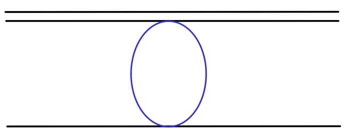
leading order in v expansion
purely from potential region

$$-v + \frac{v^3}{6} - \frac{3v^5}{40} \ldots$$

Box + crossed box = const. in both potential region and full soft region. Can we see this at the level of differential equaitons?

See also [Bjerrum-Bohr, Damgaard, Plante, Vanhove, '21].
Combining in Feynman parametrization: Cristofoli, Damgaard, Di Vecchia, Heissenberg, '20]

$$\frac{\partial}{\partial v}\left(v \right.$$  $$\left.\right) = \frac{\partial \log(\sqrt{1+v^2}-v)}{\partial v} \times$$ 

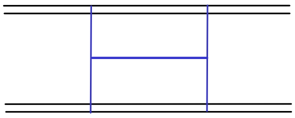
$$\frac{\partial}{\partial v}\left(v \right.$$  $$\left.\right) = (-1)\frac{\partial \log(\sqrt{1+v^2}-v)}{\partial v} \times$$ 

$$\Longrightarrow \frac{\partial}{\partial v}\left(\underbrace{v \cdot I_{box} + v \cdot I_{xbox}}_{} \right) = 0$$

constant, due to localization on
matter poles.

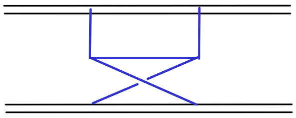
Two-loop case

H

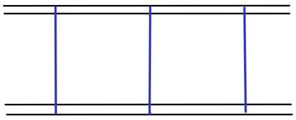


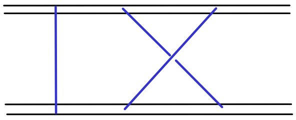
H integral: known even before expansion, as a function of s, t, m_1, m_2. [Bianchi, Leoni, '16; Kreer, Weinzierl, '21]

$\bar{H}$



$H + \bar{H}$ in potential region:
DEs for unexpanded integrals, then take small-t limit in *[Cheung, Bern, Roiban, Shen, Solon, MZ, '19]*





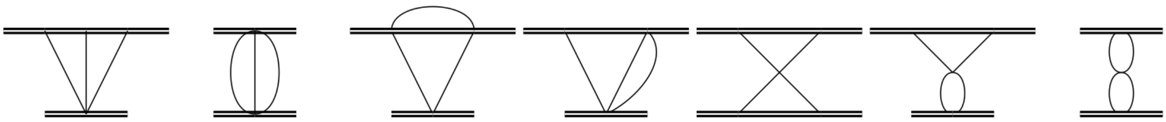
All master integrals contained in these diagrams and contact sub-diagrams

Calcuated from simplified DEs (along w/ all other 2-loop diagrams) in soft expansion:
[Parra-Martinez, Ruf, MZ, '20]

Soft integrals without further expansion into potential region:
[Di Vecchia, Heissenberg, Russo, Veneziano, '21; Herrmann, Parra-Martinez, Ruf, MZ, '21]

Boundary conditions for soft region

(1) single-scale integrals, i.e. no velocity dependence.



(a) (b) (c) (d) (e) (f) (g)

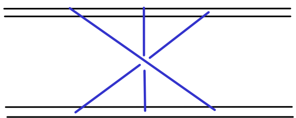
symmetrization trick reduces to lower spacetime dimension

Iterated one-loop integrals

one-loop × one-loop

(2) Regularity conditions:

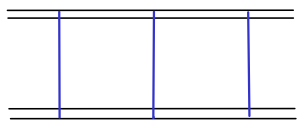
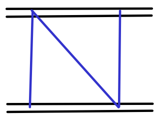
Planar integrals have no singularities in the Euclidean region.
In practice, this means u-channel planar integrals are non-singular at any $v < c$.



And when such integrals are multiplied by $v = \sqrt{y^2 - 1}$,

they have to vanish at y=1 (or y=-1 in terms of s-channel).

(3) Leading small-v behavior from potential region

$$\frac{\partial}{\partial v}\left[v^2 \right.$$  $$\left. \right] = \text{simpler integrals}$$  etc.

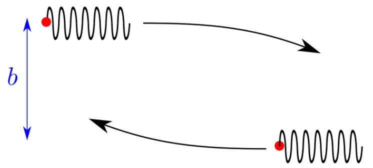
leading term from potential region

$$\frac{1}{\epsilon^2}\frac{\pi^2}{2} - \frac{1}{\epsilon}\frac{\pi^3}{12} + \mathcal{O}(\epsilon^0)\ldots$$

solving DE gives higher orders in v

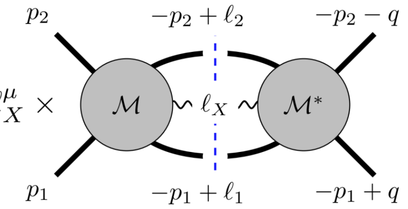
Phase space integrals for e.g. KMOC formalism

[Kosower, Maybee, O'Connell, '18]



$$|\mathrm{in}\rangle \sim \int_{\boldsymbol{q}_T} e^{i\boldsymbol{b}\cdot\boldsymbol{q}_T} |\boldsymbol{p}+\boldsymbol{q}_T\rangle$$

$$\sum_X \int \mathrm{d}\Phi_{2+X}\, \ell_X^\mu \times$$



$|\mathrm{in}\rangle$ $|\mathrm{out}\rangle$ $\langle\mathrm{out}|$ $\langle\mathrm{in}|$

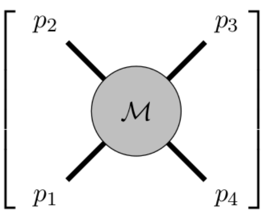
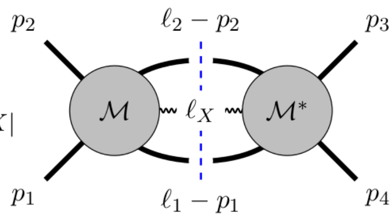
Momentum transfer << momentum spread of wavepacket
Just picks up relative phase $\sim e^{i\boldsymbol{b}.\boldsymbol{q}}$ $\Longrightarrow$ Fourier transform over $\boldsymbol{q}$.

Unitarity relates phase space integrals to virtual / loop integrals.

Unitarity of S-matrix:

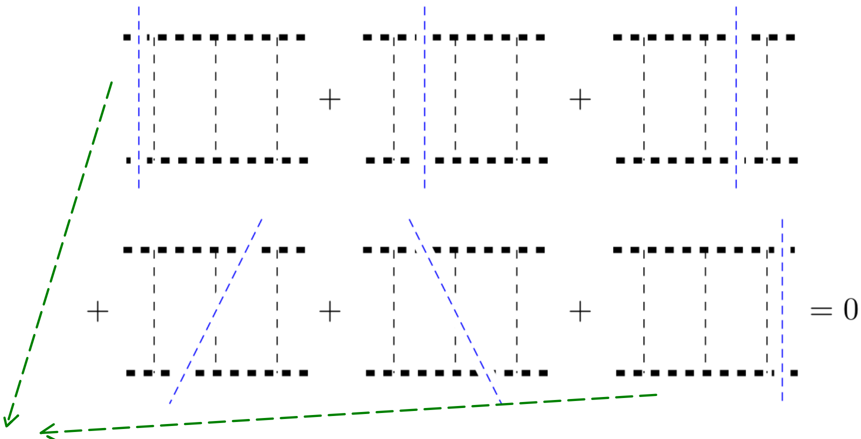
$$S = 1 + iT$$

$$SS^\dagger = 1 \implies 2\,\mathrm{Im}\,T = -i(T - T^\dagger) = TT^\dagger$$

$$2\,\mathrm{Im}\left[\right.$$  $$\left.\right] = \sum_X \int \mathrm{d}\widetilde{\Phi}_{2+|X|}$$ 

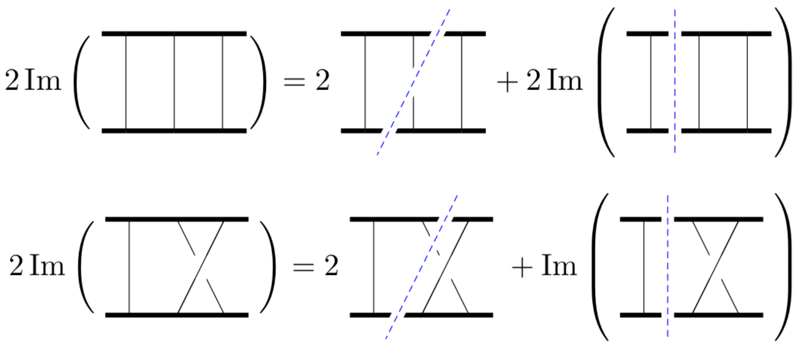
RHS is generally a sum over all s-channel Cutkosky cuts.

For example, in phi3 theory with a heavy scalar and a light scalar,

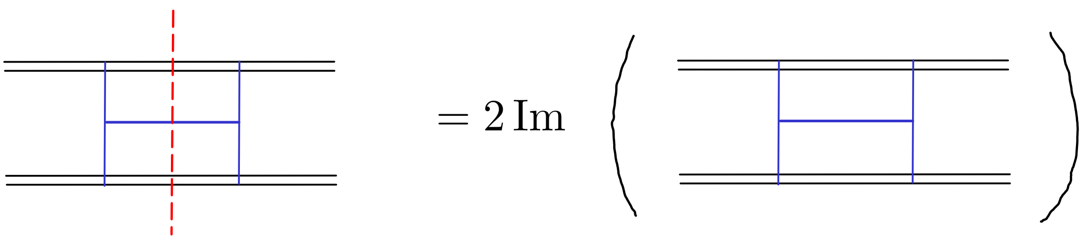


LHS. Remaining 4 cuts are for RHS.

After stripping off factors of i from vertices & propagators, relations for scalar integrals:

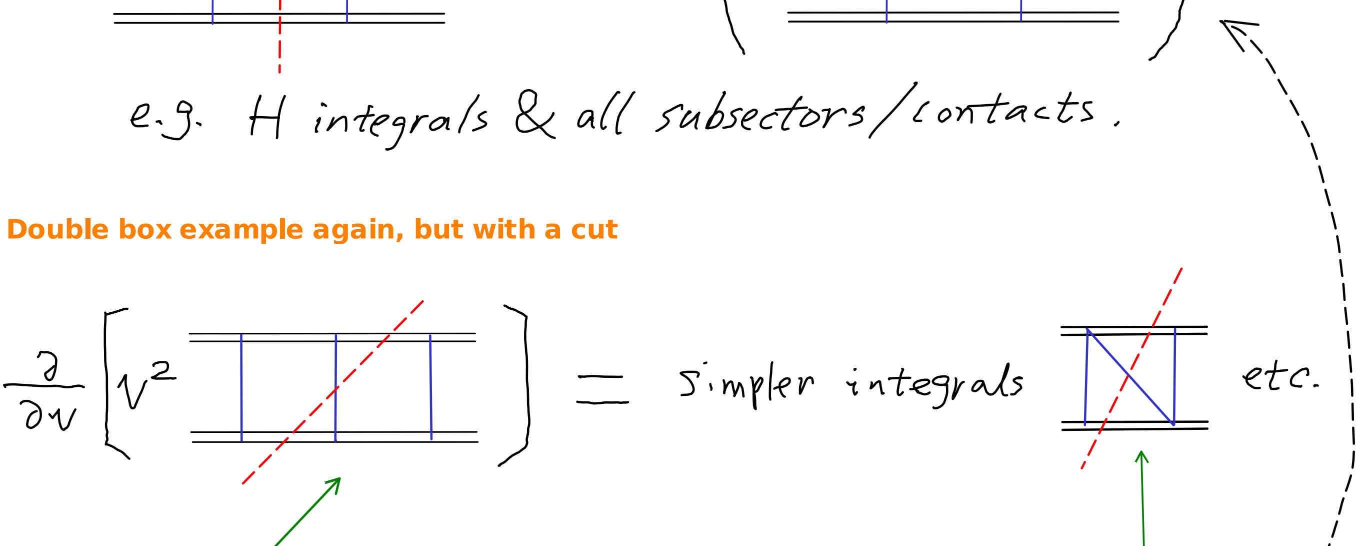


When a diagram has only one Cutskosky cut, instantly read off the phase space integral



e.g. H integrals & all subsectors/contacts.

Double box example again, but with a cut

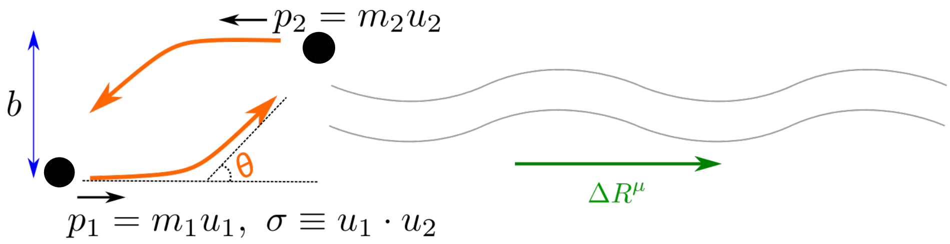


emitted graviton in radiation region.
Phase space volume vanishes near threshold.
Power counting predicts zero static limit.

Known!

Result for radaiated energy at 3rd-post-Minkowskian order

talk by Enrico Herrmann & Michael Ruf



$$\Delta R^\mu = \frac{G^3 m_1^2 m_2^2}{|b|^3} \frac{u_1^\mu + u_2^\mu}{\sigma + 1} \mathcal{E}(\sigma) + \mathcal{O}(G^4)\,.$$

$$\mathcal{E}(\sigma) = f_1 + f_2 \log\left(\frac{\sigma+1}{2}\right) + f_3 \frac{\sigma \,\mathrm{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}}\,,$$

$$f_1 = \frac{210\sigma^6 - 552\sigma^5 + 339\sigma^4 - 912\sigma^3 + 3148\sigma^2 - 3336\sigma + 1151}{48\left(\sigma^2-1\right)^{3/2}}\,,$$

$$f_2 = -\frac{35\sigma^4 + 60\sigma^3 - 150\sigma^2 + 76\sigma - 5}{8\sqrt{\sigma^2-1}}\,, \quad f_3 = \frac{\left(2\sigma^2-3\right)\left(35\sigma^4 - 30\sigma^2 + 11\right)}{8(\sigma^2-1)^{3/2}}$$