Motivation: amplitudes approach to two-body dynamics in general relativity.

**Eikonal method, NR EFT, KMOC formalism, analytic continuation...**

**Post-Newtonian expansion:** small-velocity limit. Integration methods mature.
(simple numbers, pi, zeta values etc.)
- 4PN / 5-loop known, rapid progress at even higher orders.

**Post-Minkowskian expansion:** exact velocity dependence. Nontrival functions:
polylogarithms, elliptic integrals etc. **Integration a key bottleneck.**

**NEW RESULTS FOR CONSERVATIVE DYNAMICS**

Connection between PN and PM integration methods: differential equations

\[
v = 0 \quad \frac{\partial}{\partial v} \quad v = 1
\]

**PN boundary condition**
(Taylor series in v)

**PM: function in velocity**

Ultra-relativistic limit

**Tools:** (1) expansion by regions.
- exchanged graviton momentum \( \sim \hbar / R \ll m_1, m_2 \)

Physically, iterated exchange builds up classical momentum exchange.
(c.f. eikonal approximation, and e.o.m. in NR EFT)
(2) **Integration-by-parts reduction.** Complicated integrals reduced to simpler ones like scalar integrals.  

\[
\begin{align*}
\mathcal{C}_1 & + \mathcal{C}_2 \quad \mathcal{C}_3 \\
\text{Rational functions of } s, t, m_1, m_2
\end{align*}
\]

*With numerators from e.g. Feynman rules*

(3) **Differential equations, based on IBP.**

(4) **Reverse unitarity:** cutting rules on steroids. Re-use loop integral techniques for phase space integrals.  

\[
2\pi i \delta(k^2 - m^2) = \frac{1}{(k^2 - m^2 - i\epsilon)} - \frac{1}{(k^2 - m^2 + i\epsilon)}
\]

- Allows us to reuse methods for loop integrals, e.g. IBP, differential equations.
Asymptotic expansion of Feynman integrals

\[
\lim_{|q| \ll |p_i|} \int d^d l \left[ \begin{array}{c}
\rightarrow \quad \rightarrow \quad \downarrow \quad \downarrow \quad \leftarrow \quad \leftarrow \quad \uparrow \quad \uparrow \quad \downarrow \quad \downarrow
\end{array} \right]
\]

No brainer: Taylor expansion in \(|q| / |p|\).
But how do you treat \(l\)? It may be comparable with \(|q|\), or \(|p|\), or in between.

\[|q| \sim |p|\]

\[\frac{1}{|q| / |p|}, \frac{|l|}{|p|}, |q|, |l| \sim |p|\]

\[\text{Method of regions: the full integral is a sum over two contributions.}\]

(1) soft region \(|q|, |l| \ll |p|\). \hspace{1cm} \text{Contains non-analytic behavior, e.g. } 1/|q^2|, \log(-q^2).

Taylor expansion in small \(|q|/|p|, |l|/|p|\), then integrate over ALL \(l\).

\[\frac{1}{(l+p_i)^2-m^2} = \frac{1}{2p_i \cdot l + l^2} = \frac{1}{2p_i \cdot l} + \ldots\]

\[|q| \ll |l| \sim |p|.\]

(2) hard region

Gives Taylor series in \(q^2\). Contact interaction in position space.

Taylor expansion in small \(|q|/|p|\), then integrate over ALL \(l\).

\[\text{e.g. } \frac{1}{(l+p_i)^2-m^2} \text{ is unexpanded,}\]

\[\text{while } \frac{1}{(q-l)^2} = \frac{1}{l^2} + \ldots\]

Missing OVERLAP contributions; vanishes as scaleless integrals in dim. reg., in Beneke & Smirnov's expansion prescription.

\[\text{Later: for velocity expansion, will use alternative principal value / symmetrization prescription}\]
Rigorous justification for asymptotic expansions is intricate. For example, consider the soft region,

\[ \frac{1}{(l + p_1)^2 - m_1^2} = \frac{1}{2p_1 \cdot l + l^2} = \frac{1}{2p_1 \cdot l} + \ldots \]

But \(|l| \ll |p_1|\) does NOT necessarily imply \(l^2 \ll 2p_1 \cdot l\)!


**Aside:**

This may be a tiny part of integration volume, but the denominator diverges here...

Massive-massless scattering: special region avoided by contour deformation

[Akhoury, Saotome, Sterman, 1308.5204v3]

Fully massive generalization?

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**Symmetric parametrization for soft region**

[Glauber; Polkinghorne; Neill & Rothstein]

\(P_1 = m_1 u_1 - q/2\)

\(P_4 = m_1 u_1 + q/2\)

\(q\)

\(P_2 = m_2 u_2 + q/2\)

\(P_3 = m_3 u_2 - q/2\)

\(u_1 \cdot q = u_2 \cdot q = 0, u_1 \cdot u_1 = u_2 \cdot u_2 = 1,\)

\(u_1 \cdot u_2 = y, \quad q^2 = -t\)

The only nontrivial parameter which the master integrals depend on.

Used to be \(s/t, m_1/t, m_2/t\).

Function of 3 variables → Function of 1 variable. Enormous reduction in complexity.
One-loop integrals in soft expansion

Higher orders in the expansion: will have e.g.

\[
\frac{1}{l^2} \frac{1}{(q-l)^2} \frac{1}{(p_1 + l)^2 - m_1^2} \frac{1}{(p_2 - l)^2 - m_2^2} \]

\[
\frac{1}{l^2} \frac{1}{(q-l)^2} \frac{1}{(2u_1 \cdot l + i0)} \frac{1}{(-2u_2 \cdot l + i0)} + \ldots
\]

Recall that the more complicated integrals evarporate after IBP reduction.

All masters at one loop

\[ I_{\text{box}} = \frac{2u_1 \cdot l}{l^2 (q - l)^2} \]

\[ I_{\text{tri}} = \frac{2u_1 \cdot l}{l^2 (q - l)^2} \]

\[ I_{\text{bub}} = \frac{\lambda^2}{(q - l)^2} \]

The entire one-loop 4-scalar amplitude is a linear combination of these integrals.

Evaluating soft integrals: (1) velocity expansion

Further series expansion around small-velocity limit. [Parra-Martinez, Ruf, MZ, ’20].

Initially done in opposite order of expansions, in [Cheung, Bern, Roiban, Solon, Shen, MZ, ’19].

Most well established for conservative dynamics.
Again sum over expansions in several regions. At one and two loops, conservative dynamics comes from only the potential region, in a suitable definition of potential region.

Intuition: exchange of gravitons dominated by spatial momenta, of order $|\ell| \sim |q| \sim h/R$.

Energy component suppressed $l^0 \sim v|\ell| \sim v|q|$. $ullet$ $\leftarrow \frac{R}{\bullet}$

Linearized box integral in potential region: $(l^0, \ell) \sim (qv, q)$, $q = (0, \bar{q})$. Taylor expansion:

\[
\begin{align*}
\frac{1}{l^2} &= \frac{1}{-|\ell|^2} + \cdots, \\
\frac{1}{(q-l)^2} &= \frac{1}{-|q-\ell|^2} + \cdots,
\end{align*}
\]

$2u_1 \cdot l, \ -2u_2 \cdot l$ remain unexpanded

Specializing to frame $u_1 = (1, 0, 0, 0), \quad u_2 = (\sqrt{1+v^2}, 0, 0, v)$,

We have $\int d^3\ell \int d^0l \frac{1}{-|\ell|^2} \frac{1}{-|q-\ell|^2} \frac{1}{2l^0 + i0} \frac{1}{-2\sqrt{1+v^2}l^0 + vl_z + i0}$

Well-defined contour integral in $l^0$
How about the triangle integral? It’s a bit more complicated.

\[
I_{\text{tri}} = \pm \frac{2u_1 \cdot l}{(q-l)^2}
\]

\[
\int d^3l \int dl^0 \frac{1}{-|l|^2} - \frac{1}{-|q-l|^2} - \frac{1}{2l^0 + i0}
\]

The integral has no dependence on \( v \) (though IBP reduction coefficients do), making it strange to talk about the potential region.

Nevertheless, we use a symmetrization prescription, averaging over \( l^0 \leftrightarrow -l^0 \).

\[
\int d^3l \int dl^0 \frac{1}{-|l|^2} - \frac{1}{-|q-l|^2} - \frac{1}{2l^0 + i0} = \int d^3\bar{l} \int d\bar{l}^0 \frac{1}{-|\bar{l}|^2} - \frac{1}{-|q-\bar{l}|^2} \int dl^0 \frac{1}{2} \left( \frac{1}{2l^0 + i0} + \frac{1}{-2l^0 + i0} \right) \int \frac{1}{2} (2\pi i) \delta(2\ell^0) d\ell^0
\]

If using Beneke & Smirnov’s prescription, this is a scaleless integral set to zero; the triangle integral would be fully captured by quantum soft region.

Our symmetrization prescription coincides with CONSERVATIVE dynamics at 1 and 2 loops.

One loop:

2u_1 \cdot l = 2^0 with u_1 = (1, \bar{0})

\( l_1 \equiv l = (l^0, \bar{l}) \)
\( l_2 \equiv q - l = (-l^0, \bar{q} - \bar{l}), \quad q^0 = 0 \)

Symmetrize over \( l_1^0 \) and \( l_2^0 \), eliminates contour ambiguity.

Two loops:

\( l_1^0, l_2^0, l_3^0 \sim \frac{1}{6} (-2\pi i)^3 \)

Symmetrize over \( l_1^0, l_2^0, \) and \( l_3^0 \), eliminates contour ambiguity.
Three loops

Symmetrize over $l_1, l_2, l_3, l_4$? Does NOT fully eliminate contour ambiguity!

$\sim \frac{1}{24}(-2\pi)^3$ or $\frac{1}{8}(-2\pi)^3$ depending on contour choice

Energy integrals become well defined after summing over diagrams; can still assign symmetry factors to individual diagrams after velocity expansion & IBP.

Evaluating soft integrals: (2) differential equations

In simple case, can promote velocity series to exact functions.


First version: [Cheung, Bern, Roiban, Ruf, Solon, MZ, '19]

\[
\frac{\partial}{\partial v} = \ldots \\
\text{LHS & RHS as functions of } s, t, m_1, m_2, \text{ then take limit } t \to 0. \\
\text{Only applied to } H + \bar{H}. \text{ IBP reduction time consuming.}
\]

Simplified version: soft expansion first, obtain DEs with only nontrivial dependence on $v$.

Solutions with soft boundary conditions: Di Vecchia, Heissenberg, Russo, Veneziano, '21,
Herrmann, Parra-Martinez, Ruf, MZ, '21.
Further development in soft DEs: Bjerrum-Bohr, Damgaard, Plante, Vanhove, '21

General structure of DEs: start with master integrals, grouped in a column vector $\vec{I}$.

Derivatives $\partial \vec{I} / \partial v$ reduced to original set of masters, with rational (in Mandelstams) coefficients.

\[
\frac{\partial \vec{I}}{\partial v} = M \cdot \vec{I}
\]

Singularity structure determines function space.
Example:

\[ \frac{\partial}{\partial v}(v) = \frac{\partial \log(\sqrt{1 + v^2} - v)}{\partial v} \times \]

\[ = -\frac{i}{\epsilon} \pi \text{, constant in potential region.} \]

Explains nontrivial magic cancellation at every higher order in \( v \), in direct expansion.

If working in soft region without truncation to potential region, RHS is a \( v \)-indep. constant,

\[ = -\frac{i}{\epsilon} \left( i \pi + \log(\sqrt{1 + v^2} - v) \right) \]

leading order in \( v \) expansion purely from potential region

\[-v + \frac{v^3}{6} - \frac{3v^5}{40} \ldots\]

Box + crossed box = const. in both potential region and full soft region. Can we see this at the level of differential equaitons?

See also \[Bjerrum-Bohr, Damgaard, Plante, Vanhove, '21\].
Combining in Feynman parametrization: Cristofoli, Damgaard, Di Vecchia, Heissenberg, '20]
**Two-loop case**

H integral: known even before expansion, as a function of $s, t, m_1, m_2$. 
[Bianchi, Leoni, '16; Kreer, Weinzierl, '21]

$H + \bar{H}$ in potential region:

DEs for unexpanded integrals, then take small-t limit in
[Cheung, Bern, Roiban, Shen, Solon, MZ, '19]

All master integrals contained in these diagrams and contact sub-diagrams

Calculated from simplified DEs (along w/ all other 2-loop diagrams) in soft expansion:
[Parra-Martinez, Ruf, MZ, '20]

Soft integrals without further expansion into potential region:
[Di Vecchia, Heissenberg, Russo, Veneziano, '21; Herrmann, Parra-Martinez, Ruf, MZ, '21]

**Boundary conditions for soft region**

(1) single-scale integrals, i.e. no velocity dependence.

(a) symmetrization trick reduces to lower spacetime dimension
(b) Iterated one-loop integrals
(c) one-loop \times one-loop

(2) Regularity conditions:

Planar integrals have no singularities in the Euclidean region.
In practice, this means u-channel planar integrals are non-singular at any $v < c$.

And when such integrals are multiplied by $v = \sqrt{y^2 - 1}$, they have to vanish at $y=1$ (or $y=-1$ in terms of s-channel).
(3) Leading small-\(v\) behavior from potential region

\[
\frac{2}{\partial v} \left[ v^2 \right] = \text{Simpler integrals etc.}
\]

leading term from potential region
\[
\frac{1}{\epsilon^2} \frac{\pi^2}{2} - \frac{1}{\epsilon} \frac{\pi^3}{12} + \mathcal{O}(\epsilon^0) \ldots
\]
solving DE gives higher orders in \(v\)

Phase space integrals for e.g. KMOC formalism

Unitarity relates phase space integrals to virtual / loop integrals.

Unitarity of S-matrix:

\[
S = 1 + iT
\]

\[
SS^\dagger = 1 \implies 2 \text{Im} T = -i(T - T^\dagger) = TT^\dagger
\]

\[
2 \text{Im} \left[ \begin{array}{c}
p_2 \\ p_3 \\ p_1 \\ p_4
\end{array} \right] = \sum_X \int d\Phi_X \left[ M^\dagger \times \ell_X \sim M^* \right]
\]

RHS is generally a sum over all s-channel Cutkosky cuts.
After stripping off factors of $i$ from vertices & propagators, relations for scalar integrals:

$$2 \text{Im} \left( \begin{array}{c} \text{box} \\ \text{box} \end{array} \right) = 2 \left( \begin{array}{c} \text{box} \\ \text{box} \end{array} \right) + 2 \text{Im} \left( \begin{array}{c} \text{box} \\ \text{box} \end{array} \right)$$

$$2 \text{Im} \left( \begin{array}{c} \text{box} \\ \text{box} \end{array} \right) = 2 \left( \begin{array}{c} \text{box} \\ \text{box} \end{array} \right) + \text{Im} \left( \begin{array}{c} \text{box} \\ \text{box} \end{array} \right)$$

When a diagram has only one Cutkosky cut, instantly read off the phase space integral

$$\begin{array}{c} \text{box} \\ \text{box} \end{array} = 2 \text{Im} \left( \begin{array}{c} \text{box} \\ \text{box} \end{array} \right)$$

\text{e.g.} $H$ integrals & all subsectors/contacts.

\text{Double box example again, but with a cut}

$$\frac{\partial}{\partial \nu} \left[ \begin{array}{c} \text{box} \\ \text{box} \end{array} \right] = \text{simpler integrals}$$

\text{emitted graviton in radiation region.}
\text{Phase space volume vanishes near threshold.}
\text{Power counting predicts zero static limit.}
Result for radiated energy at 3rd-post-Minkowskian order

talk by Enrico Herrmann & Michael Ruf

\[ \Delta R^\mu = \frac{G^3 m_1^2 m_2^2}{|b|^3} \frac{u_1^\mu + u_2^\mu}{\sigma + 1} \mathcal{E}(\sigma) + \mathcal{O}(G^4). \]

\[ \mathcal{E}(\sigma) = f_1 + f_2 \log \left( \frac{\sigma + 1}{2} \right) + f_3 \frac{\sigma \arcsinh \sqrt{\frac{\sigma - 1}{2}}}{\sqrt{\sigma^2 - 1}}, \]

\[ f_1 = \frac{210\sigma^6 - 552\sigma^5 + 339\sigma^4 - 912\sigma^3 + 3148\sigma^2 - 3336\sigma + 1151}{48(\sigma^2 - 1)^{3/2}}, \]

\[ f_2 = -\frac{35\sigma^4 + 60\sigma^3 - 150\sigma^2 + 76\sigma - 5}{8\sqrt{\sigma^2 - 1}}, \quad f_3 = \frac{(2\sigma^2 - 3)(35\sigma^4 - 30\sigma^2 + 11)}{8(\sigma^2 - 1)^{3/2}}. \]