Loop Integrals in the Soft Region & the Gravity Eikonal

GGI WORKSHOP
"Gravitational Scattering Inspiral & Radiation"

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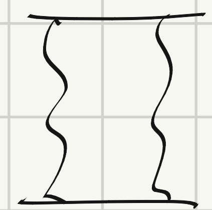
(Mostly) based on

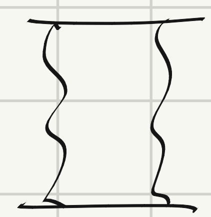
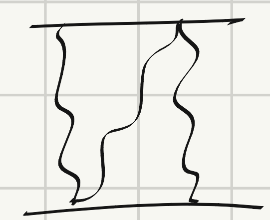
2008.12743
2101.05772
2104.03256

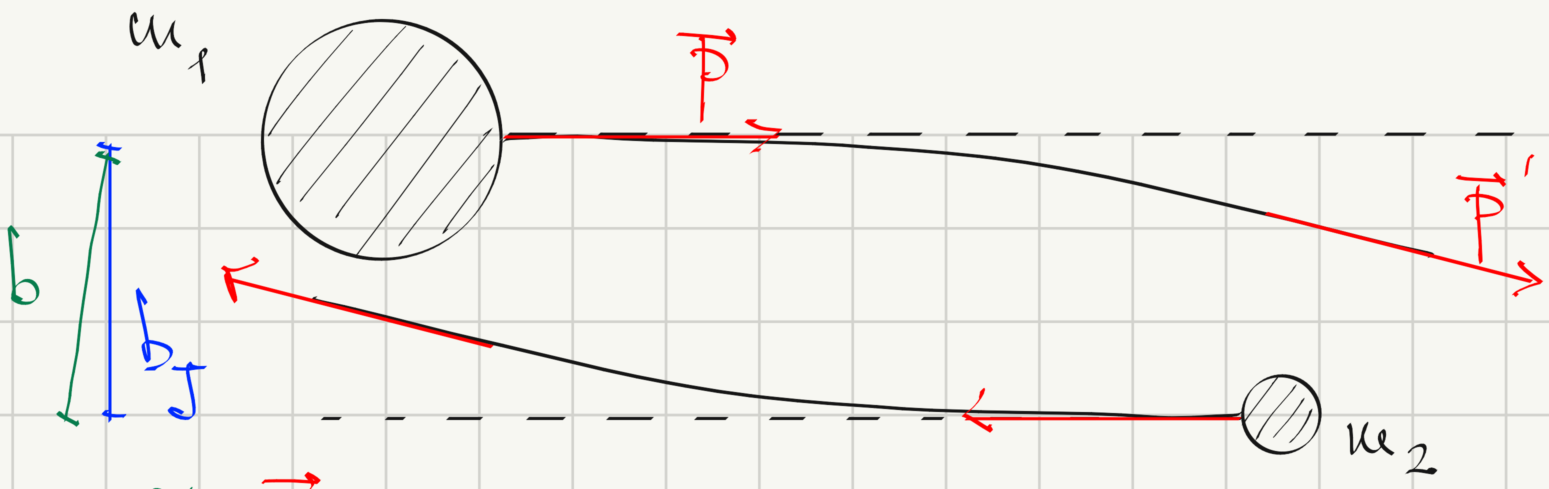
with P. Di Vecchia, R. Russo and G. Veneziano

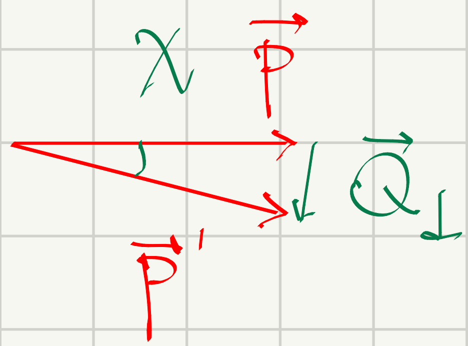
Outline:

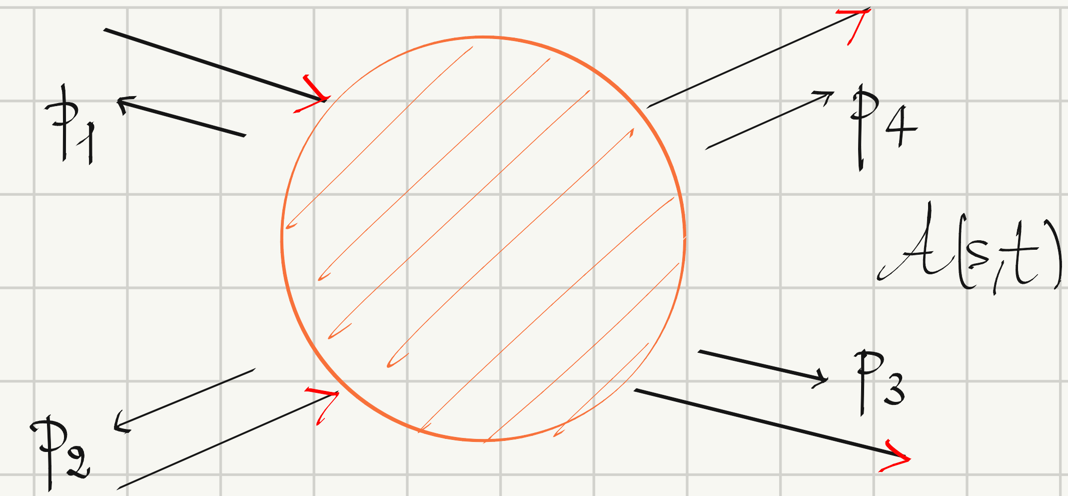
- **Reminder of the Eikonal Approach**
  - $\mathcal{A}(s,t) \longrightarrow \tilde{\mathcal{A}}(s,b) \longrightarrow \chi$
  - Soft vs Hard region: 

- **Some Recent Results**

- **Boundary Conditions:**  & 







$$Q = 2p \sin\frac{\chi}{2} \qquad b_J = b\cos\frac{\chi}{2}$$

$$s = E^2 = -(p_1+p_2)^2 \qquad q = p_1 + p_4$$

$$\sigma = -\frac{p_1 p_2}{m_1 m_2} = \frac{1}{\sqrt{1-v^2}} \qquad t = -q^2$$

$$m_{1,2} \lesssim m_* \lesssim E$$

$G m_*^2 \gg \hbar$ classical $\qquad \frac{G m_*^2}{\hbar} \gg 1$

$b \gg G m_*$ PM $\qquad \frac{G m_*}{b} \ll 1$

$$\mathcal{A} = \mathcal{A}_0 + \mathcal{A}_1 + \mathcal{A}_2 + \cdots$$
$$\quad G \qquad G^2 \qquad G^3$$

$$\frac{\hbar}{m_*} \ll G m_* (b^{2\epsilon}) \ll b \qquad D = 4 - 2\epsilon$$

$$q \ll m_*$$

$$\tilde{\mathcal{A}}(s,b) = \frac{1}{4E|\vec{p}|}\int e^{ibq_\perp}\mathcal{A}(s,-q_\perp^2)\frac{d^{2-2\epsilon}q_\perp}{(2\pi)^{2-2\epsilon}} \qquad q_\perp p_1 = 0 = p_2 q_\perp$$

EIKONAL: $1 + i\tilde{\mathcal{A}} = e^{2i\delta}(1 + 2i\Delta) = \left[1 + i\tilde{\mathcal{A}}_0 + i\tilde{\mathcal{A}}_1 + i\tilde{\mathcal{A}}_2 + \cdots\right]$

$$\delta = \delta_0 + \delta_1 + \delta_2 + \cdots$$

$$\sim \frac{G m_*^2}{\hbar}\left(1 + \frac{G m_*}{b} + \left(\frac{G m_*}{b}\right)^2 + \cdots\right)$$

$$\Delta = \Delta_1 + \Delta_2 + \cdots$$

$$\sim \left(\frac{\hbar}{G m_*}\right)^{0,1,2,\cdots} \left[\left(\frac{G m_*}{b}\right)^{\#}\right]$$

$$i\tilde{A}_0 = 2i\delta_0 \qquad \text{1PM}$$

$$i\tilde{A}_1 = \frac{(2i\delta_0)^2}{2!} + 2i\delta_1 + 2i\Delta_1 \qquad \text{2PM}$$

$$i\tilde{A}_2 = \underbrace{\frac{(2i\delta_0)^3}{3!}}_{i\tilde{A}_{n-1} \sim \frac{(2i\delta_0)^n}{n!}} + 2i\delta_0\, \underset{\text{ok}}{2i\delta_1} + \left[2i\delta_2 + 2i\delta_0\, 2i\Delta_1\right] + 2i\Delta_2 \qquad \text{3PM}$$

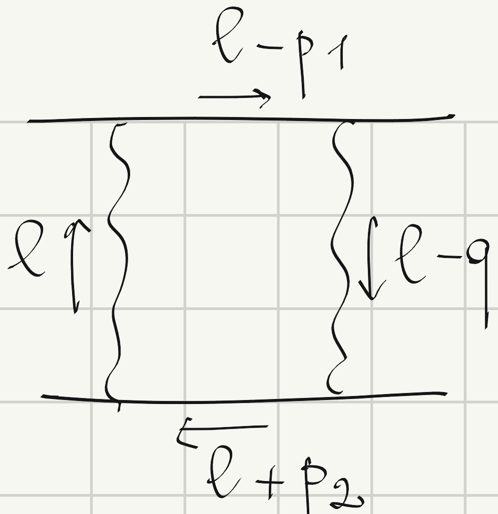
$$\delta_2 = \mathrm{Re}\,\delta_2 + i\,\underset{\mathcal{O}\left(\frac{1}{\epsilon}\right)}{\mathrm{Im}\,\delta_2}$$

but we can deal with it

$\left\{ \begin{array}{l} \text{ZFL of spectrum} \\ \text{RR effects} \end{array} \right.$

$$\int e^{-ibQ_\perp + 2i\delta}\, d^{2-2\epsilon} b \longrightarrow Q_\perp = \frac{\partial}{\partial b}\,\mathrm{Re}\,2\delta$$

* Small-q in $A(s,t)$
* Neglect $(q^2)^{0,1,2,\cdots}$



$$= \int \frac{1}{(\ell^2 - 2p_1\ell - i0)(\ell^2 + 2p_2\ell - i0)(\ell^2 - i0)((\ell-q)^2 - i0)} = I$$

$q \ll m_* \quad (\sim p_1, p_2 \ldots)$ $\qquad p_1 \simeq -m_1 u_1 \qquad u_1^2 = -1 = u_2^2$

$\qquad p_2 \simeq -m_2 u_2 \qquad \sigma = -u_1 u_2$

* HARD: $\ell \sim O(m_*)$

$$I^h \simeq \int \frac{1}{(\ell^2 - 2p_1\ell)(\ell^2 + 2p_2\ell)(\ell^2)^2}$$

q-indep

$(q^2)^{0,1,\ldots}$

$\Rightarrow$ throw away

* SOFT: $\ell \sim O(q)$ $\quad I^s \simeq \displaystyle\int \frac{1}{m_1 m_2 \underset{3}{(2u_1\ell - i0)}\underset{4}{(-2u_2\ell - i0)}\underset{1}{(\ell^2 - i0)}\underset{2}{((\ell-q)^2 - i0)}}$

$$= \frac{1}{m_1 m_2} \int \frac{dt_1\, dt_2}{t_{12}^{1-\epsilon}}\, e^{-\frac{t_1 t_2}{t_{12}} q^2} \left[\int dx_3\, dx_4\, e^{-\left(x_3^2 + 2(-\sigma - i0)\, x_3 x_4 + x_4^2\right)} \right]$$

$$= \frac{1}{m_1 m_2} \frac{\Gamma(1+\epsilon)}{(q^2)^{1+\epsilon}} \frac{\Gamma(-\epsilon)^2}{\Gamma(-2\epsilon)} \left[\frac{i\pi - \operatorname{arccosh}\sigma}{2\sqrt{\sigma^2 - 1}} \right] \simeq \frac{i\pi}{2\tau} \quad \text{as } \tau \equiv \sqrt{\sigma^2 - 1} \to 0$$

Some Recent Results: massive eikonal

1901.04424 + 1908.01493 — Potential (CONSERVATIVE) χ_{3PM} ($\mathrm{Re}\,\delta_2$) GR

2005.04236 — Potential $\mathrm{Re}\,\delta_2$ $\mathcal{N}=8$
+ ODE for soft integrals

2008.12743 — Full $\mathrm{Re}\,\delta_2$ $\mathcal{N}=8$

2101.05772 (2105.04594) — $(\mathrm{Im}\,\delta_2)\,|\,O(\frac{1}{\epsilon})$ $\mathcal{N}=8$ & GR

2101.07254 — Potential I_r ($\mathrm{Re}\,\delta_3$) GR

2104.03256 (2104.0395) — Full δ_2 $\mathcal{N}=8$, Full $\mathrm{Im}\,\delta_2$ GR

2104.04510 — Full δ_2 $\mathcal{N}=8$ (more efficient)

2105.05218 — Full δ_2 GR (more efficient)

Full δ_2 GR

ODE: $\frac{d}{d\sigma} \vec{I}(\sigma) = M(\sigma;\epsilon)\, \vec{I}(\sigma)$

actually much simpler in suitable variables
($\to$ canonical form)

BC: $\vec{I}(\sigma) \underset{\sigma\to 1}{\sim}$? Back to the box

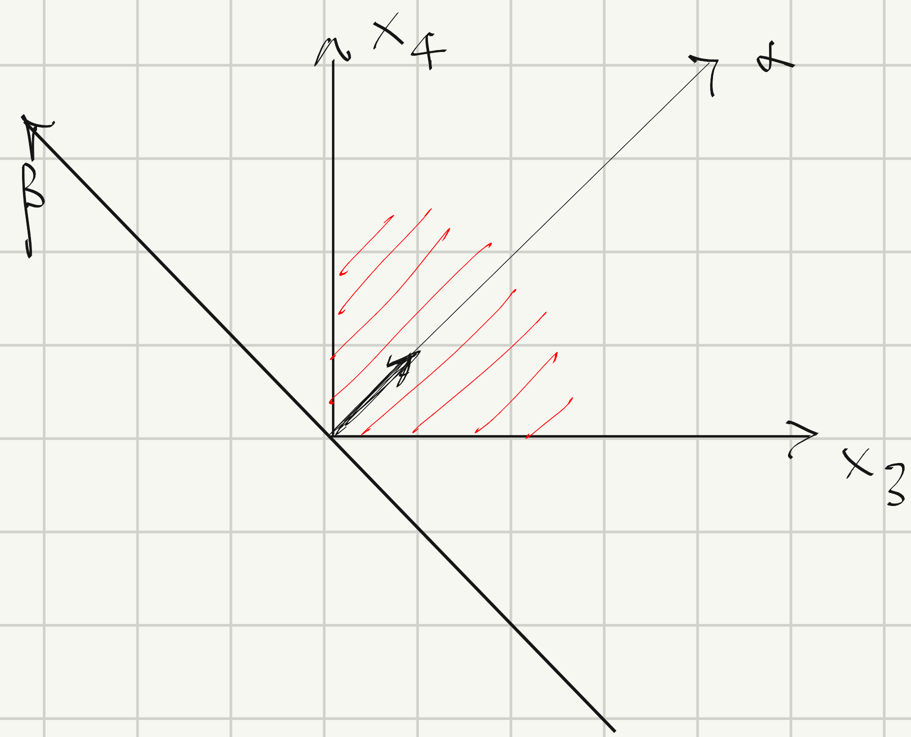
$$\int dx_3 dx_4\, e^{-\left(x_3^2 + 2(-\sigma - i0)\, x_3 x_4 + x_4^2\right)} \xrightarrow{\sigma \simeq 1} (x_3 - x_4)^2 \qquad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$

zero mode

$$\begin{cases} x_4 + x_3 = \alpha \\ x_4 - x_3 = \beta \end{cases}$$

$\sigma \simeq 1 + \frac{\tau^2}{2}$

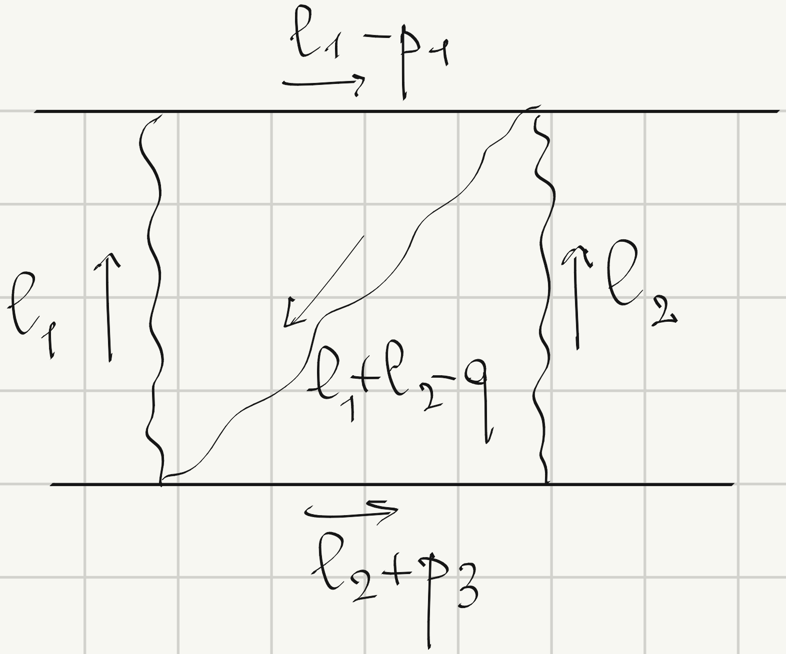
$\tau = \sqrt{\sigma^2 - 1}$



$$\frac{1}{2} \int_{-\infty}^{+\infty} d\beta \int_{|\beta| \simeq 0}^{\infty} d\alpha\, e^{-\left(\beta^2 + \frac{1}{4}\alpha^2(-\tau^2 - i0)\right)}$$

$$\begin{cases} \beta \sim \mathcal{O}(\tau^0) \\ \alpha \sim \mathcal{O}(\tau^{-1}) \end{cases}$$

$$= \frac{1}{2}\sqrt{\pi}\, \frac{1}{2} \sqrt{\frac{4\pi}{-\tau^2 - i0}} = \frac{i\pi}{2\tau}$$



$$\overset{\text{SOFT}}{\sim} \int_{\ell_1}\int_{\ell_2} \frac{1}{m_1 m_2 (2u_1\ell_1)(2u_2\ell_2)\,\ell_1^2\,\ell_2^2\,(\ell_1+\ell_2-q)^2}$$

$p_1 \simeq -m_1 u_1$

$p_2 \simeq -m_2 u_2$

5 4 1 2 3

$$= \frac{1}{m_1 m_2}\int \frac{dt_1\,dt_2\,dt_3}{T^{1-\epsilon}}\, e^{-\frac{t_1 t_2 t_3}{T} q^2} \left[\int dx_4\,dx_5\, e^{-\left(t_{23}x_5^2 + 2t_3(-\sigma - i0)x_4 x_5 + t_{13}x_4^2\right)}\right]$$

$T = t_1 t_2 + t_2 t_3 + t_3 t_1 \qquad t_{12} = t_1 + t_2$ etc. $\rightarrow$ Discriminant $= T$

* ORDINARY:

$\tau \to 0 \quad t_{1,2,3} \sim \mathcal{O}(\tau^0)$

$$[\ldots]^\sigma = \int dx_4\,dx_5\, e^{-\left(t_{23}x_5^2 - 2t_3 x_4 x_5 + t_{13}x_4^2\right)}$$ = REAL & FINITE

3-particle cut $\Rightarrow$ Imaginary part? MISSING!

* SINGULAR:

$\tau \to 0 \quad t_{1,2} \sim \mathcal{O}(\tau^0)$

$t_3 \sim \mathcal{O}(\tau^{-2})$

$$[\ldots]^S = \int dx_4\,dx_5\, e^{-t_3(x_4-x_5)^2 + \ldots}$$

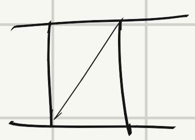
$$\begin{pmatrix} t_3 & -t_3 \\ -t_3 & t_3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$$ zero mode

$$\int dx_4\, dx_5\; e^{-\left(t_{23}x_5^2+2t_3(-\tau-i0)x_4x_5+t_{13}x_4^2\right)} \qquad \begin{aligned}\alpha &= x_5+x_4\\ \beta &= x_5-x_4\end{aligned}$$

$$=\frac{1}{2}\int_{-\infty}^{+\infty} d\beta \int_{|\beta|\simeq 0}^{+\infty} d\alpha\; e^{-\left[t_3\beta^2+\frac{\alpha^2}{4}\left(t_{12}+t_3(-\tau^2-i0)\right)+\cancel{(t_2-t_1)\frac{\alpha\beta}{2}}+\dots\right]}$$

$$\beta\sim O(\tau) \qquad \alpha\sim O(\tau^0)$$

$$=\frac{1}{2}\sqrt{\frac{\pi}{t_3}}\,\frac{1}{2}\sqrt{\frac{4\pi}{t_{12}+t_3(-\tau^2-i0)}}=\frac{i\pi}{2\sqrt{t_3}\sqrt{t_3\tau^2+(-t_{12}+i0)}}$$



$$^{S}\simeq\frac{1}{m_1m_2}\int\frac{dt_1\,dt_2\,dt_3}{(t_3t_{12})^{1-\epsilon}}\,e^{-\frac{t_1t_2}{t_{12}}q^2}\,\frac{i\pi}{2\sqrt{t_3}\sqrt{t_3\tau^2+(-t_{12}+i0)}}\qquad \text{factorized}$$

$$=\frac{1}{2m_1m_2}\,\frac{\Gamma(2\epsilon)}{(q^2)^{2\epsilon}}\,\frac{\Gamma(1-2\epsilon)^2}{\Gamma(2-4\epsilon)}\left(-i\pi e^{i\pi\epsilon}\,\tau^{1-2\epsilon}\right)\frac{\Gamma(1-\epsilon)\Gamma\left(-\frac{1}{2}+\epsilon\right)}{\Gamma\left(\frac{1}{2}\right)}$$

Conclusions & Outlook

- Full $\delta_2 \to \chi_{3PM}$ in $\mathcal{N}=8$ and in GR
- Systematic Evaluation of SOFT BC @ 2-loop order

$(m=0)$ $\mathcal{N}=8$

- (Of course) 3 loops? Issues with exponentiation/ choice of $-i0$ prescription
- Inclusion of real radiation in eikonal framework

Thank You!