Heavy Scattering Amplitudes

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GGI meeting on Gravitational scattering, inspiral and radiation

2021

UCLouvain

Based on

- **On-shell Heavy Particle Effective Theory**

  RA, Kays Haddad and Andreas Helset

  JHEP 05 (2020) 051 [hep-th/2001.09164]

- **Tidal effects for spinning particles**

  RA, Kays Haddad and Andreas Helset

  JHEP 03 (2021) 097 [hep-th/2012.05256]

Outline

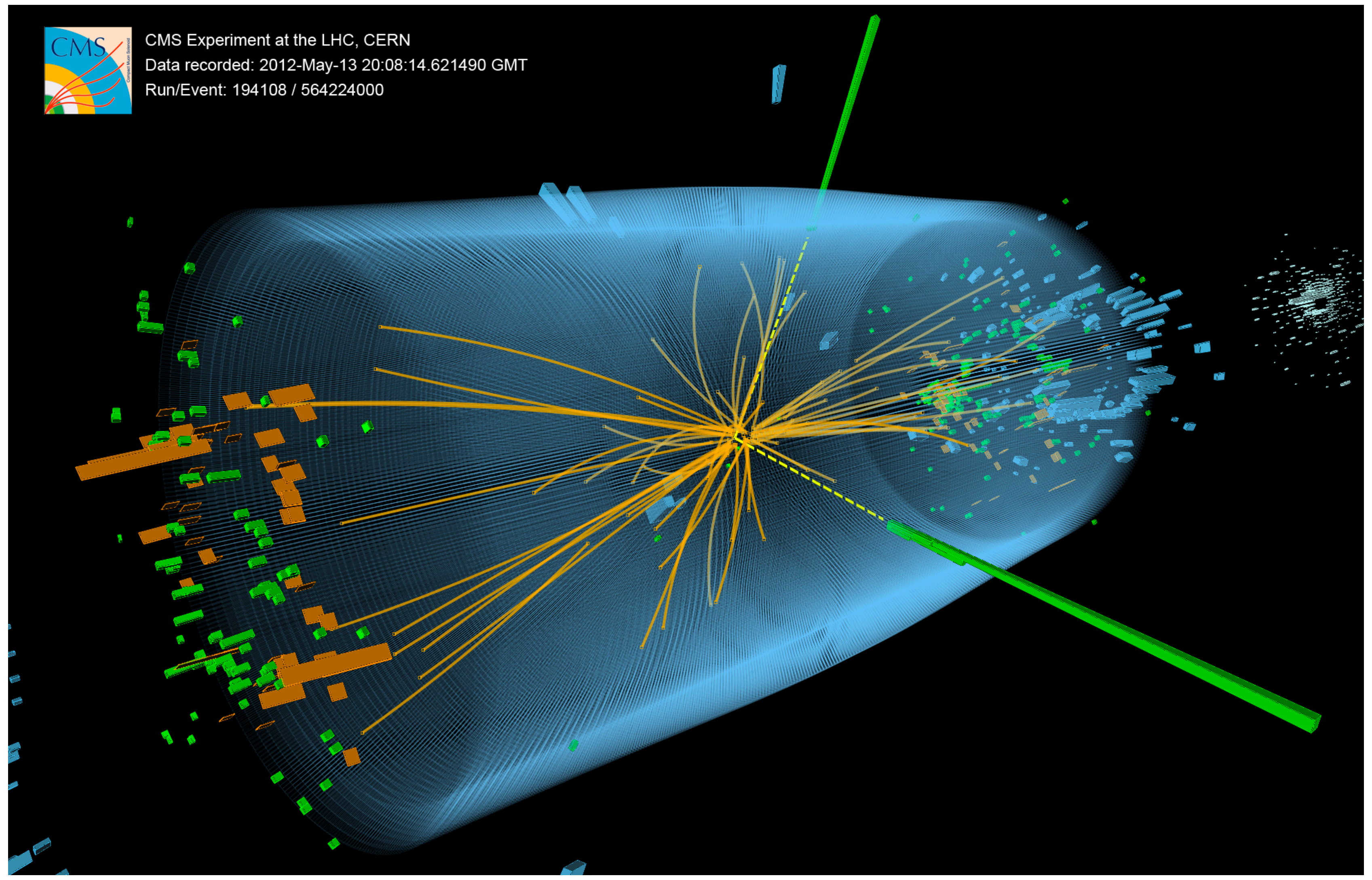
Part 1

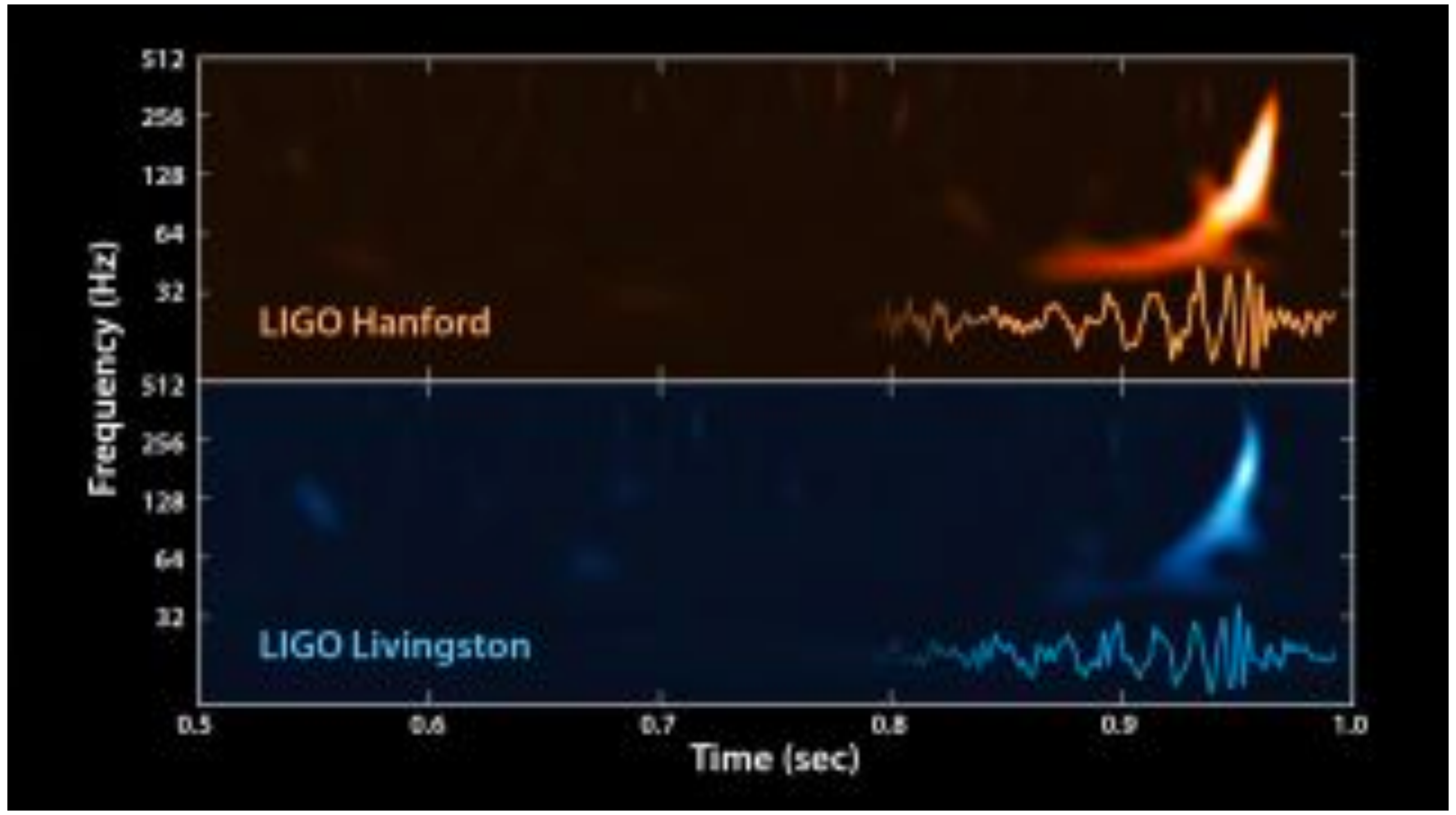
- HQET and HBET from [Daamgard, Haddad, Helset]
- Massive spinor-helicity variables
- On-shell HPET variables and three-point amplitudes
- Kerr black holes as heavy particles
- Classical limits and spin-universality at tree-level

Part 2

- Hilbert series and tidal action construction
- Scattering amplitudes for tidally deformed particles
- Classical observables and eikonal phase

Use particle physics methods to calculate classical gravity





Here, we use Heavy-Quark Effective Theory

Heavy Quark Effective Theory

Limit of QCD describing heavy quark interacting with light degrees of freedom

heavy quark momenta

$$p^\mu = mv^\mu + k^\mu$$

light d.o.f. carry momenta smaller than heavy quark mass

$$|k^\mu| \sim \mathcal{O}(\Lambda_{\mathrm{QCD}}) \text{ where } \Lambda_{\mathrm{QCD}} \ll m_Q.$$

where

$$v^2 = 1$$

on-shell condition

$$v \cdot k = -\frac{k^2}{2m}$$

the residual momenta scales with hbar

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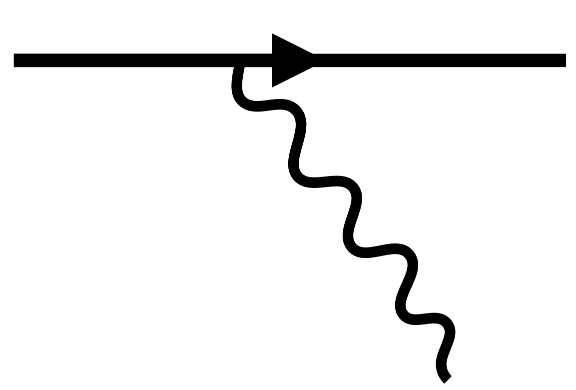
For the quark-fields $$\psi(x) \to e^{-im_Q v\cdot x}\frac{1+v}{2}Q_v(x)$$

QCD Lagrangian $$\bar{\psi}(i\not{D} - m_Q)\psi \to i\bar{Q}_v v\cdot D\frac{1+\not{v}}{2}Q_v$$

Heavy Black Hole Effective Theory

[Damgaard, Haddad, Helset, 19']



Makes explicit the hbar dependence in loops

Propagators: spin-0 and spin-1/2

$$D_v^{s=0}(k) = \frac{i}{\hbar v \cdot k}$$

$$D_v^{s=\frac{1}{2}}(k) = \frac{i}{\hbar v \cdot k}\frac{1+\not{v}}{2}.$$

Explicit $\hbar$ power-counting

$$\begin{aligned} i\mathcal{M}^{(2)}_{\text{bubble}} &\sim \frac{G^2}{\hbar^2}\hbar^4 \int d^4\bar{l}\,\frac{1}{\hbar^2\bar{l}^2}\frac{1}{\hbar^2(\bar{l}+\bar{q})^2} + \mathcal{O}(\hbar^{-1}) \\ &= \mathcal{O}(\hbar^{-2}). \end{aligned}$$

$$\begin{aligned} i\mathcal{M}^{(2)}_{\text{triangle}} &\sim \frac{G^2}{\hbar^2}\hbar^4 \int d^4\bar{l}\,\frac{1}{\hbar^2\bar{l}^2}\frac{1}{\hbar^2(\bar{l}+\bar{q})^2}\frac{1}{\hbar v\cdot(\bar{l}+\bar{k})} + \mathcal{O}(\hbar^{-2}) \\ &= \mathcal{O}(\hbar^{-3}). \end{aligned}$$

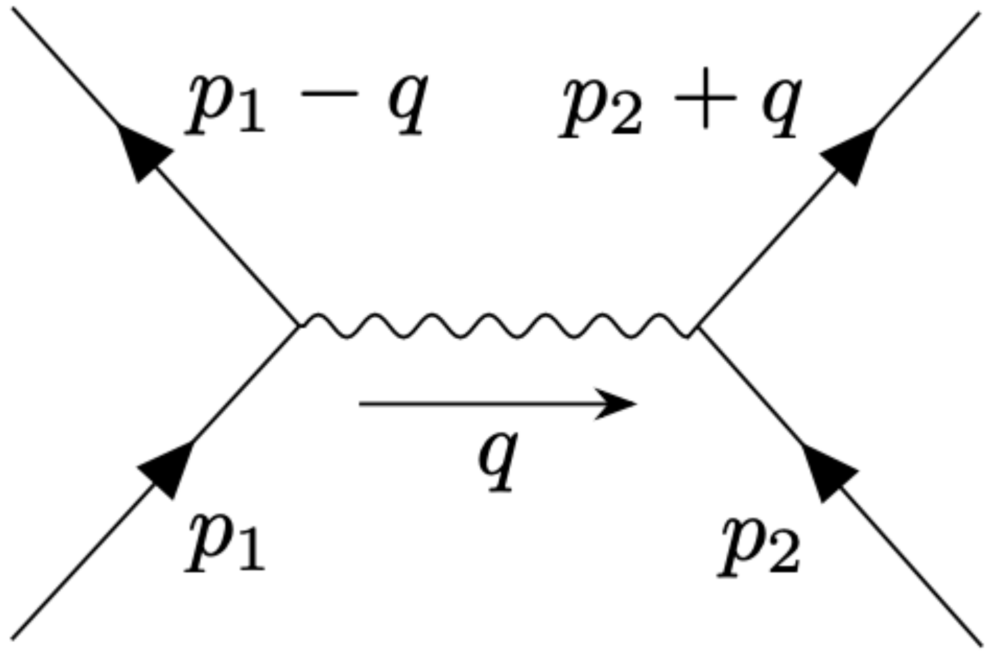
$$\begin{aligned} i\mathcal{M}^{(2)}_{\text{(crossed-)box}} &\sim \frac{G^2}{\hbar^2}\hbar^4 \int d^4\bar{l}\,\frac{1}{\hbar^2\bar{l}^2}\frac{1}{\hbar^2(\bar{l}+\bar{q})^2}\frac{1}{\hbar v\cdot(\bar{l}+\bar{k}_1)}\frac{1}{\hbar v\cdot(\bar{l}+\bar{k}_2)} + \mathcal{O}(\hbar^{-3}) \\ &= \mathcal{O}(\hbar^{-4}). \end{aligned}$$

[Damgaard, Haddad, Helset, 19']

Long-range scattering of two heavy bodies

Long-range scattering has a similar scale hierarchy as HQET

The exchanged graviton momentum
is much smaller than the mass of each object



Restoring hbar into the heavy momentum

$$p^\mu = m_Q v^\mu + \hbar \bar{k}^\mu .$$

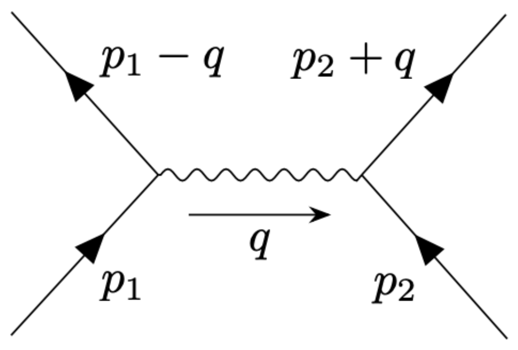
Heavy Black-Hole Effective Theory (HBET)

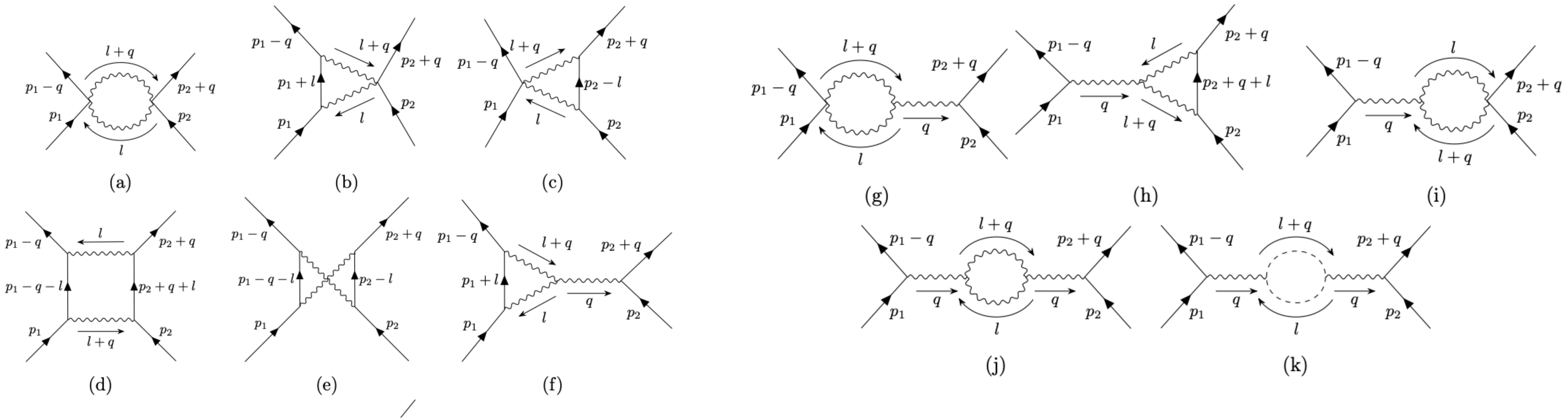
$$p^{\mu} = m_Q v^{\mu} + \hbar \bar{k}^{\mu}.$$

[Damgaard, Haddad, Helset, 19']

Using Lagrangians and Feynman Rules,

classical and leading quantum contributions at 1PM and 2PM for s $\lesssim$ 1/2 were calculated.





Non-trivial job for higher spins…

On-shell Heavy Particle Effective Theory (HPET)

Massive spinor-helicity variables

[Arkani-Hamed, Huang, Huang. 17’]

$$\det(p_{\alpha\dot{\beta}}) = m^2 \rightarrow p_{\alpha\dot{\beta}} = \lambda^I_\alpha \tilde{\lambda}_{\dot{\beta}I} = \lambda^I_\alpha \epsilon_{IJ} \tilde{\lambda}^J_{\dot{\beta}}$$

Rank 2 matrix

I,J are SU(2) little group indices

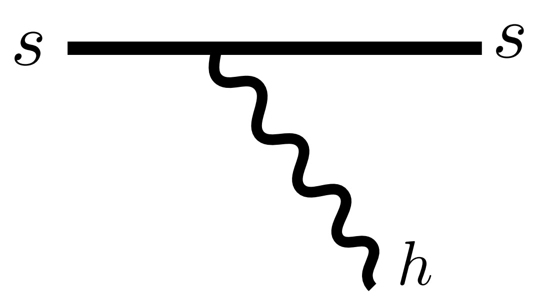
$$\lambda^a_\alpha \leftrightarrow |p^a\rangle_\alpha$$

$$\tilde{\lambda}_{\dot{\beta}a} \leftrightarrow [p_a|_{\dot{\beta}}$$

Dirac spinors and polarization tensors [Ochirov 18’]

$$u^{Aa}_p = \begin{pmatrix} \lambda_{p\alpha}{}^a \\ \tilde{\lambda}^{\dot{\alpha}a}_p \end{pmatrix}, \qquad \bar{u}^a_{pA} = \begin{pmatrix} -\lambda^{\alpha a}_p \\ \tilde{\lambda}_{p\dot{\alpha}}{}^a \end{pmatrix}$$

$$\varepsilon^{ab}_\mu(p) = \frac{i\langle p^{(a}|\sigma_\mu|p^{b)}]}{\sqrt{2}m}$$



e.g: minimal coupling

$$\mathcal{M}^{+|h|,s}_{\text{min}} = (-1)^{2s+h} \frac{g_0 x^{+|h|}}{m^{2s}} \langle \mathbf{21} \rangle^{2s},$$

$$mx\langle q| = [q|p_1$$

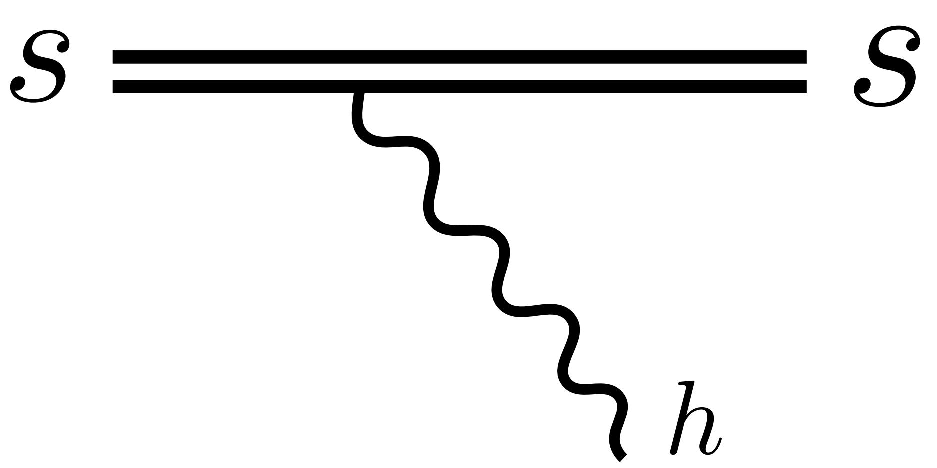
Recover the massless variables in the high-energy limit

$$\lambda_{p\alpha}{}^a \xrightarrow[m\to 0]{} \lambda_{p\alpha}\zeta^a_-, \qquad \tilde{\lambda}^{\dot{\alpha}a}_p \xrightarrow[m\to 0]{} \tilde{\lambda}^{\dot{\alpha}}_p \zeta^a_+,$$

Bold notation = symmetrization over the LG indices

HPET spinor-helicity variables

[Aoude, Haddad, Helset, Jan 20']



$$p^\mu = m v^\mu + k^\mu,$$

We obtain the heavy spinor from the Dirac equation $\not{v} u_v^I = u_v^I$:

$$u_v^I(p) = \left(\frac{\mathbb{I} + \not{v}}{2}\right) u^I(p) = \left(\mathbb{I} - \frac{\not{k}}{2m}\right) u^I(p), \qquad \longrightarrow \qquad \begin{pmatrix} |p_v^I\rangle \\ |p_v^I] \end{pmatrix} = \left(\mathbb{1} - \frac{\not{k}}{2m}\right) \begin{pmatrix} |p^I\rangle \\ |p^I] \end{pmatrix}$$

Momentum associated

$$\not{p}_v = \begin{pmatrix} 0 & |p_v\rangle^I{}_I [p_v| \\ |p_v]_I{}^I \langle p_v| & 0 \end{pmatrix} = m_k \not{v}, \qquad \text{with} \qquad m_k \equiv \left(1 - \frac{k^2}{4m^2}\right) m$$

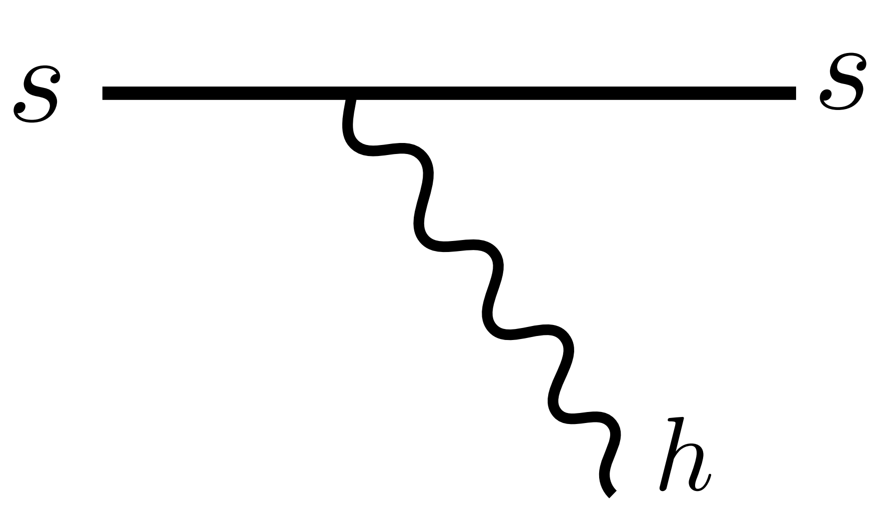
Explicit separation of spinless and spin effects at 3 points

$$\bar{u}_v u_v \sim \langle \mathbf{2}_v \mathbf{1}_v \rangle \sim [\mathbf{2}_v \mathbf{1}_v] \qquad (\bar{u}_v \sigma_{\mu\nu} u_v) q^\mu \epsilon^\nu(q) \sim x \langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle \sim x^{-1} [\mathbf{2}_v q][q \mathbf{1}_v]$$

Massive spinor-helicity variables

[Arkani-Hamed, Huang, Huang. 17’]

General 3-point amplitudes (any spin-s, helicity-h):



Positive helicity in the chiral basis

$$\mathcal{M}^{+|h|,s} = (-1)^{2s+h}\frac{x^{|h|}}{m^{2s}}\left[g_0\langle\mathbf{21}\rangle^{2s} + g_1\langle\mathbf{21}\rangle^{2s-1}\frac{x\langle\mathbf{2}q\rangle\langle q\mathbf{1}\rangle}{m} + \cdots + g_{2s}\frac{(x\langle\mathbf{2}q\rangle\langle q\mathbf{1}\rangle)^{2s}}{m^{2s}}\right]$$

(similar for negative helicity)

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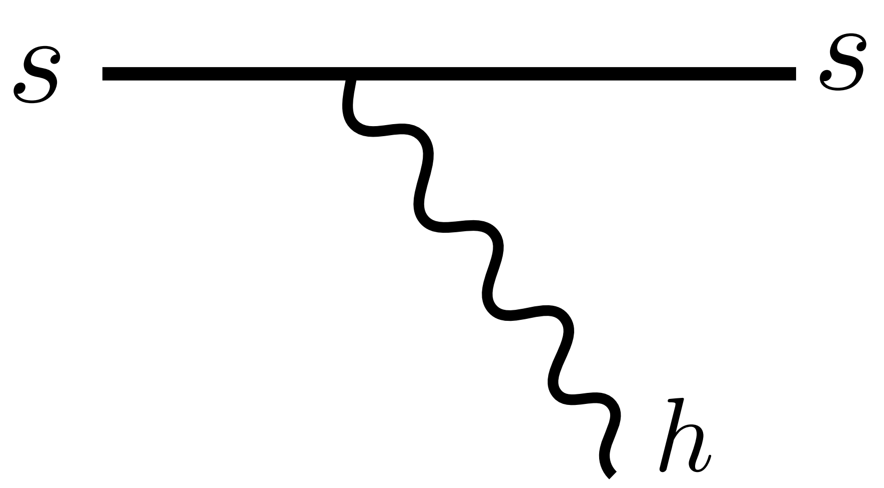

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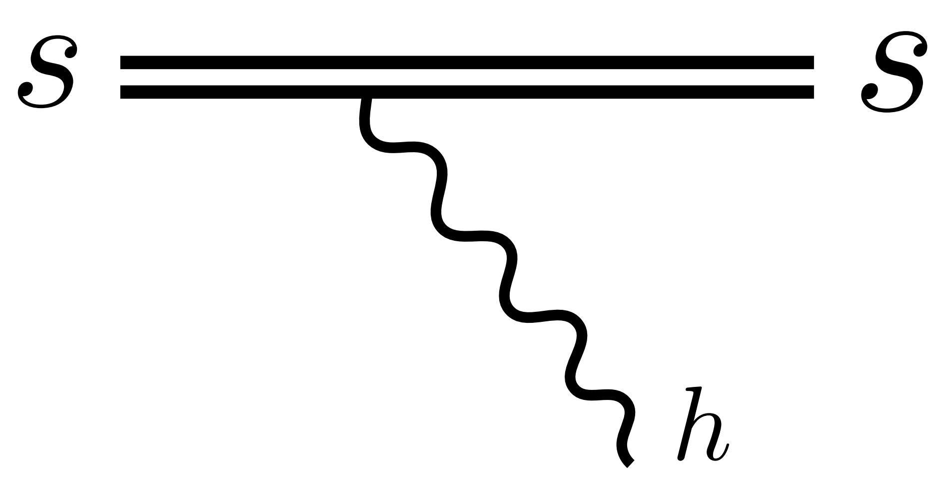
(similar for negative helicity)

Minimal coupling from high-energy limit:

$$\mathcal{M}^{+|h|,s}_{\text{min}} = (-1)^{2s+h}\frac{g_0 x^{+|h|}}{m^{2s}}\langle\mathbf{21}\rangle^{2s},$$

HPET spinor-helicity variables: Spin operator

[Aoude, Haddad, Helset, Jan 20']



$$p^\mu = m v^\mu + k^\mu,$$

Spin-Operator from the Pauli-Lubanski

$$S^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\alpha\beta} p_\nu J_{\alpha\beta} \longrightarrow S^\mu = -\frac{1}{2}\epsilon^{\mu\nu\alpha\beta} v_\nu J_{\alpha\beta}.$$

On-shell version

At three-points

$$(S^\mu)_\alpha{}^\beta = \frac{1}{4}\left[(\sigma^\mu)_{\alpha\dot\alpha} v^{\dot\alpha\beta} - v_{\alpha\dot\alpha}(\bar\sigma^\mu)^{\dot\alpha\beta}\right], \longrightarrow (q\cdot S)_\alpha{}^\beta = \frac{x}{2}|q\rangle\langle q|,$$

Spin vector defined in [Daamgard, Haddad, Helset]

$$S_v^\mu \equiv \frac{1}{2}\bar u_v(p_2)\gamma_5\gamma^\mu u_v(p_1) \longrightarrow S_v^\mu = \bar u_v(p_2) S^\mu u_v(p_1) = -2\langle \mathbf{2}_v|S^\mu|\mathbf{1}_v\rangle = 2[\mathbf{2}_v|S^\mu|\mathbf{1}_v].$$

HPET spinor-helicity variables: Three-point

[Aoude, Haddad, Helset, Jan 20']

General massive 3-point in HPET variables (zero initial residual momentum)

using the relations…

$$\langle \mathbf{2}\mathbf{1}\rangle = \langle \mathbf{2}_v \mathbf{1}_v\rangle + \frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle,$$
$$\langle \mathbf{2}q\rangle\langle q\mathbf{1}\rangle = \langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle.$$

We obtain

$$\mathcal{M}_3^{+|h|,s} = (-1)^{2s+h}\frac{x^{|h|}}{m^{2s}}\sum_{k=0}^{2s} g^{\mathrm{H}}_{s,k}\langle \mathbf{2}_v \mathbf{1}_v\rangle^{2s-k}\left(\frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle\right)^k$$

Purely spinless

Purely spin

Relation between the coeffs.

$$g^{\mathrm{H}}_{s,k} = \sum_{i=0}^{k} g_i \binom{2s-i}{2s-k}$$

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Purely spin

Relation between the coeffs.

$$g^{\mathrm{H}}_{s,k} = \sum_{i=0}^{k} g_i \binom{2s-i}{2s-k}$$

Minimal coupling

$$\mathcal{M}_{\mathrm{HPET,min}}^{+|h|,s=\frac{1}{2}} = (-1)^{1+h}\frac{g_0 x^{|h|}}{m}\left[\langle \mathbf{2}_v \mathbf{1}_v\rangle + \frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q \mathbf{1}_v\rangle\right],$$

Kerr Black Holes as heavy particles

Effective action for spinning gravitating bodies

[Goldberger Rothstein, 06']
Porto 06', Levi Steinhoff 15', …]

$$S = \int d\sigma \left\{ -m\sqrt{u^2} - \frac{1}{2} S_{\mu\nu}\Omega^{\mu\nu} + L_{\mathrm{SI}}[u^\mu, S_{\mu\nu}, g_{\mu\nu}(x^\mu)] \right\}$$

$u^\mu = \frac{dx^\mu}{d\sigma}$	coordinate velocity
$S_{\mu\nu}$	spin operator
$\Omega^{\mu\nu}$	angular velocity

higher spin multipoles

$$L_{\mathrm{SI}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n)!} \frac{C_{S^{2n}}}{m^{2n-1}} D_{\mu_{2n}} \ldots D_{\mu_3} \frac{E_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \ldots S^{\mu_{2n}}$$
$$+ \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{C_{S^{2n+1}}}{m^{2n}} D_{\mu_{2n+1}} \ldots D_{\mu_3} \frac{B_{\mu_1\mu_2}}{\sqrt{u^2}} S^{\mu_1} S^{\mu_2} \ldots S^{\mu_{2n+1}}$$

The WCs contain info about the internal structure of the body

for Kerr BHs

$$C^{\mathrm{Kerr}}_{S^k} = 1 \text{ for all } k.$$

Was matched to the 3-point amplitude

[Chung, Huang, Kim, Lee, 19']

[Chung, Huang, Kim 19']

[Guevara, Ochirov, Vines, 19']

Kerr Black Holes as heavy particles

[Aoude, Haddad, Helset, Jan 20']

In the HPET variables (all incoming) …

$$\mathcal{M}^{+2,s} = \sum_{a+b\leq s} \frac{\kappa m x^2}{2m^{2s}} C_{S^{a+b}} n^s_{a,b} \langle \mathbf{2}_{-v}\mathbf{1}_v\rangle^{s-a} \left(-x\frac{\langle \mathbf{2}_{-v}q\rangle\langle q\mathbf{1}_v\rangle}{2m}\right)^a [\mathbf{2}_{-v}\mathbf{1}_v]^{s-b} \left(x^{-1}\frac{[\mathbf{2}_{-v}q][q\mathbf{1}_v]}{2m}\right)^b, \qquad n^s_{a,b} \equiv \binom{s}{a}\binom{s}{b}$$

converting to the chiral basis

$$\mathcal{M}^{+2,s} = \frac{x^2}{m^{2s}}(-1)^{2s} \sum_{a+b\leq 2s} \frac{\kappa m}{2} C_{S^{a+b}} n^s_{a,b} \langle \mathbf{2}_v\mathbf{1}_v\rangle^{2s-a-b} \left(\frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle\right)^{a+b}$$

Comparing with the general 3-pt HPET

$$\mathcal{M}_3^{+|h|,s} = (-1)^{2s+h}\frac{x^{|h|}}{m^{2s}} \sum_{k=0}^{2s} g^{\mathrm{H}}_{s,k} \langle \mathbf{2}_v\mathbf{1}_v\rangle^{2s-k} \left(\frac{x}{2m}\langle \mathbf{2}_v q\rangle\langle q\mathbf{1}_v\rangle\right)^k$$

$$g^{\mathrm{H}}_{s,k} = \frac{\kappa m}{2} C_{S^k} \sum_{j=0}^{k} n^s_{k-j,j}.$$

Minimal coupling in HPET,
Precisely the multipoles of a Kerr BH

Focusing on the minimal coupling, and normalizing

$$g^{\mathrm{H}}_{s,k} = g_0\binom{2s}{k}$$

$$g_0 = \kappa m/2,$$

$$C^{\min}_{S^k} = \binom{2s}{k}\left[\sum_{j=0}^{k}\binom{s}{k-j}\binom{s}{j}\right]^{-1} = 1.$$

(Obtained first up to O(1/s),
then precisely with Hilbert
space matching)

(no need for boost)

Why is the same spin-multipole?

[Aoude, Haddad, Helset, Jan 20']

Using traditional variables the spinor variables have both spin and spinless effects

$$\langle \mathbf{12} \rangle = -[\mathbf{12}] + \frac{1}{xm}[\mathbf{1}q][q\mathbf{2}]$$

and

$$[\mathbf{12}] = -\langle \mathbf{12} \rangle + \frac{x}{m}\langle \mathbf{1}q \rangle \langle q\mathbf{2} \rangle$$

While for HPET variables (for zero residual momenta) we have clear separation

$$\langle \mathbf{2}_v \mathbf{1}_v \rangle = -[\mathbf{2}_v \mathbf{1}_v]$$

$$\langle \mathbf{2}_v q \rangle \langle q \mathbf{1}_v \rangle = x^{-2}[\mathbf{2}_v q][q \mathbf{1}_v]$$

Although, the final and initial momenta differs by a null vector $p_2 = p_1 - q$

external states are associated with same momenta

$$p_{v,2} = \left(1 - \frac{q^2}{4m^2}\right) m v^\mu = m v^\mu.$$

Compton scattering

[Aoude, Haddad, Helset, Jan 20']

Using BCFW, with a shift on the massless legs. Same helicity case

$$\mathcal{M}(-\mathbf{1}^s, \mathbf{2}^s, q_3^{+2}, q_4^{+2}) = \frac{1}{m^{2s}} \mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{+2}, q_4^{+2}) \langle \mathbf{2}_v|^{2s} \exp\left[\frac{(q_3+q_4)\cdot S}{m_{q_3+q_4}}\right] |\mathbf{1}_v\rangle^{2s},$$

$$\mathcal{M}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{+2}, q_4^{+2}) = \frac{\kappa^2}{8e^4} \frac{\langle 3|p_1|3]\langle 4|p_1|4]}{q_3\cdot q_4} \mathcal{A}(-\mathbf{1}^0, \mathbf{2}^0, q_3^{+1}, q_4^{+1})^2.$$

See also
[Guevara, Ochirov, Vines 20']
[Chung, Huang, Kim, Lee, 19']

Makes immediate the spin-exponentiation.

Clear separation between spinless and spin effects.

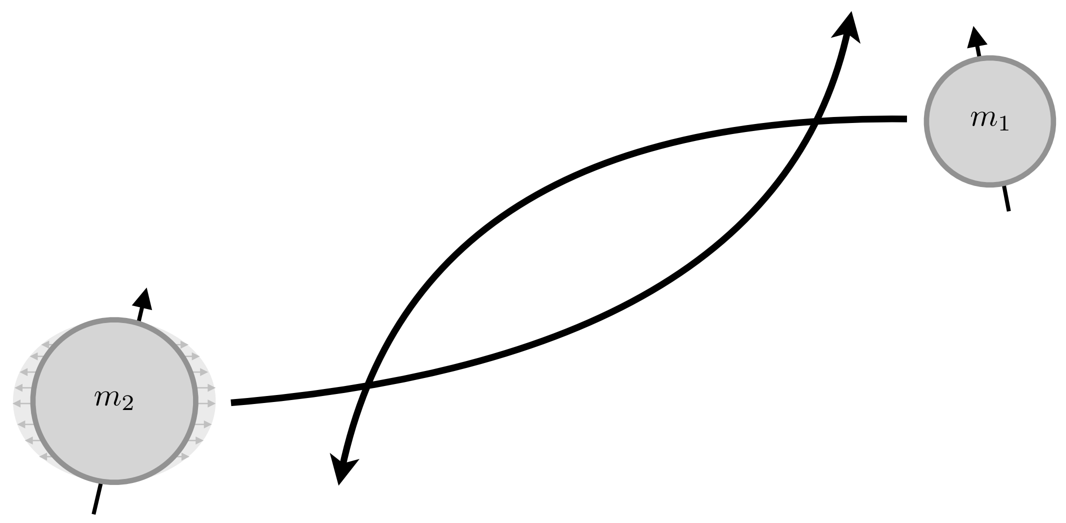
Extended to n-boson (same helicity)

Spin-universality

$$\mathcal{M}^s_{n+2} = \frac{1}{m^{2s}} \mathcal{M}^{s=0}_{n+2} \langle \mathbf{2}_v|^{2s} \exp\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^n q_i\cdot S\right] |\mathbf{1}_v\rangle^{2s}$$

$$= \frac{1}{m^{2s}} \mathcal{M}^{s=0}_{n+2} [\mathbf{2}_v|^{2s} \exp\left[\frac{1}{m_q}\frac{h}{|h|}\sum_{i=1}^n q_i\cdot S\right] |\mathbf{1}_v]^{2s}.$$

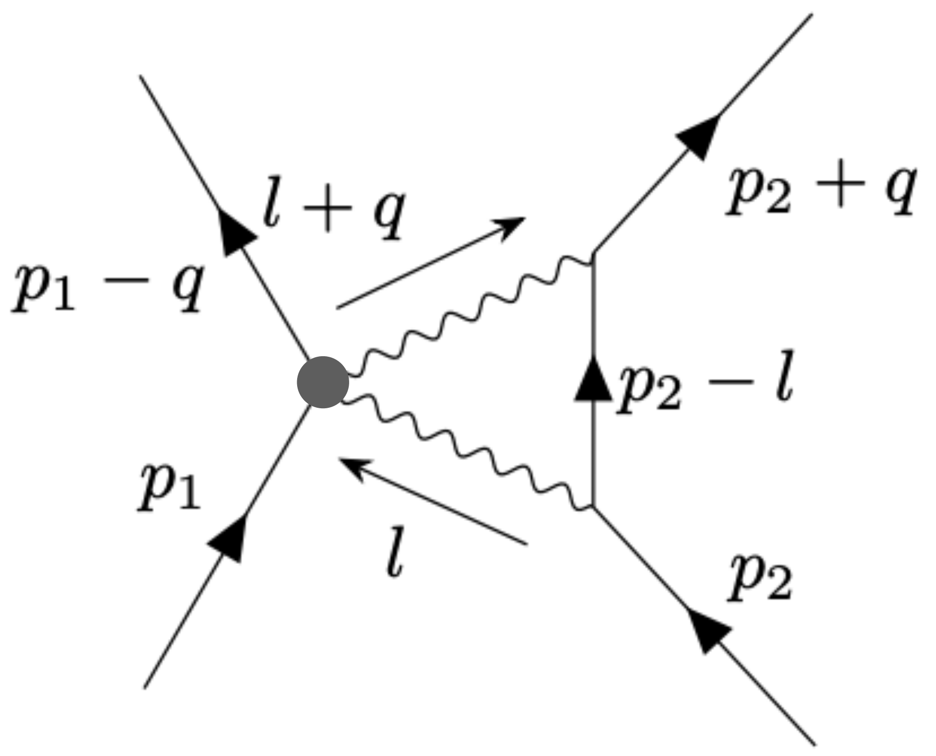
Tidal effects for spinning particles



Tidal effects for spinning particles

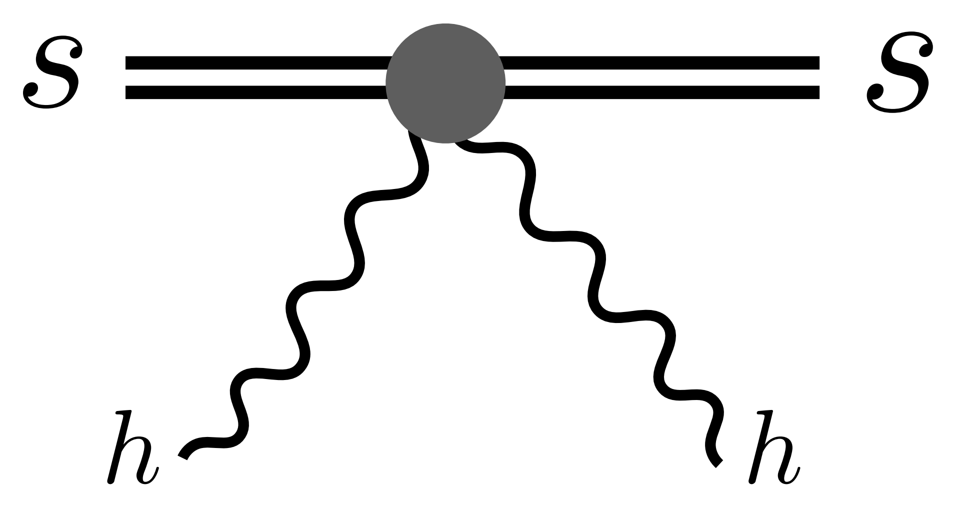
[RA, Kays Haddad and Andreas Helset, Dec 20]

- Hilbert series to enumerate all the number of each operator types
  and construct operator basis
- On-shell amplitudes to guide
- Tidal spin-1/2 scattering amplitude at 2PM
- Classical observables



[based on previous spinless case by Haddad and Helset, 20']

Hilbert series to enumerate all the number of operators



The Hilbert series of 2 fermions-2 graviton

$$\mathcal{H}^{C^2}_{7+2n} = \lfloor n/2+1 \rfloor (C_L^2 + C_R^2)(\psi\psi^c + \psi^{c\dagger}\psi^\dagger)D^{2n} + (n-1)C_L C_R(\psi\psi^c + \psi^{c\dagger}\psi^\dagger)D^{2n}$$
$$+ \frac{1}{2}(1-(-1)^n)(C_L^2\psi\psi^c + C_R^2\psi^{c\dagger}\psi^\dagger)D^{2n}, \qquad (2.6)$$

$$\mathcal{H}^{C^2}_{6+2n} = \lfloor n/2 \rfloor (C_L^2 + C_R^2)(\psi\psi^\dagger + \psi^c\psi^{c\dagger})D^{2n-1} + (n-1)C_L C_R(\psi\psi^\dagger + \psi^c\psi^{c\dagger})D^{2n-1}. \qquad (2.7)$$

Number of operators
of such type

No information on the Lorentz contraction $\longrightarrow$ Need to construct a basis

Hilbert series to enumerate all the number of operators

and construct operator basis

$$\sqrt{-g}\mathcal{L}_{\rm GR} = \sqrt{-g}\left[\bar{\psi}(ie^{\mu}{}_{a}\gamma^{a}D_{\mu} - m)\psi + \Delta\mathcal{L}_{\rm GR}^{\rm odd} + \Delta\mathcal{L}_{\rm GR}^{\rm even}\right],$$

Odd mass dimensions

$$\begin{aligned}
\Delta\mathcal{L}_{\rm GR}^{\rm odd} = &\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} d_1^{(n,k)}\left(\bar{\psi}\overleftrightarrow{D}^{\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C^{\mu\nu\rho\sigma}\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}C_{\mu\nu\rho\sigma}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} d_2^{(n,k)}\left(\bar{\psi}\overleftrightarrow{D}^{\mu\nu\lambda\tau\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C_{\mu\rho\lambda\sigma}\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}C_{\nu\rho\tau\sigma}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} id_3^{(n,k)}\left(\bar{\psi}\gamma_5\overleftrightarrow{D}^{\mu\nu\lambda\tau\alpha_1\ldots\alpha_{2k+1}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C_{\mu\rho\lambda\sigma}\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k+1}}D^{\beta_1\ldots\beta_{n-2k}}\tilde{C}_{\nu\rho\tau\sigma}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} id_4^{(n,k)}\left(\bar{\psi}\gamma_5\overleftrightarrow{D}^{\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C^{\mu\nu\rho\sigma}\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}\tilde{C}_{\mu\nu\rho\sigma}\right)\\
&+\sum_{n=0}^{\infty} ie^{(n)}\left(\bar{\psi}\sigma^{\mu\nu}\overleftrightarrow{D}^{\rho\alpha_1\ldots\alpha_{2n}}\psi\right)\left(C_{\mu\lambda\rho\tau}\overleftrightarrow{D}_{\sigma\alpha_1\ldots\alpha_{2n}}C_{\nu}{}^{\lambda\sigma\tau}\right).
\end{aligned}$$

Hilbert series to enumerate all the number of operators and construct operator basis

$$\sqrt{-g}\mathcal{L}_{\rm GR} = \sqrt{-g}\left[\bar{\psi}(ie^{\mu}{}_{a}\gamma^{a}D_{\mu} - m)\psi + \Delta\mathcal{L}_{\rm GR}^{\rm odd} + \Delta\mathcal{L}_{\rm GR}^{\rm even}\right],$$

Even mass dimensions

$$\begin{aligned}
\Delta\mathcal{L}_{\rm GR}^{\rm even} = &\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} i f_1^{(n,k)}\left(\bar{\psi}\gamma_{\mu}\overleftrightarrow{D}^{\nu\gamma\delta\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C^{\mu\rho\gamma\tau}\overleftrightarrow{D}_{\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}C_{\nu\rho\delta\tau}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} i f_2^{(n,k)}\left(\bar{\psi}\gamma_{\mu}\overleftrightarrow{D}^{\nu\gamma\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C^{\mu\rho\gamma\tau}\overleftrightarrow{D}_{\delta\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}C_{\nu\rho\delta\tau}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} f_3^{(n,k)}\left(\bar{\psi}\gamma_5\gamma_{\mu}\overleftrightarrow{D}^{\nu\gamma\delta\lambda\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C^{\mu\rho\gamma\tau}\overleftrightarrow{D}_{\lambda\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}\tilde{C}_{\nu\rho\delta\tau}\right.\\
&\qquad\left.-D_{\beta_1\ldots\beta_{n-2k}}\tilde{C}^{\mu\rho\gamma\tau}\overleftrightarrow{D}_{\lambda\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}C_{\nu\rho\delta\tau}\right)\\
&+\sum_{n=0}^{\infty}\sum_{k=0}^{\lfloor n/2\rfloor} f_4^{(n,k)}\left(\bar{\psi}\gamma_5\gamma_{\mu}\overleftrightarrow{D}^{\nu\lambda\alpha_1\ldots\alpha_{2k}}\psi\right)\left(D_{\beta_1\ldots\beta_{n-2k}}C^{\mu\rho\gamma\delta}\overleftrightarrow{D}_{\lambda\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}\tilde{C}_{\nu\rho\gamma\delta}\right.\\
&\qquad\left.+D_{\beta_1\ldots\beta_{n-2k}}\tilde{C}^{\mu\rho\gamma\delta}\overleftrightarrow{D}_{\lambda\alpha_1\ldots\alpha_{2k}}D^{\beta_1\ldots\beta_{n-2k}}C_{\nu\rho\gamma\delta}\right).
\end{aligned}$$

On-shell amplitudes to cross check

[Durieux and Machado, 19']

[Durieux, Kitahara, Machado, Shadmi, Weiss , 20']

Amplitude basis for gravity

Helicity	Amplitude
$(++++)$	$[12]^4[\mathbf{34}]x^a y^{2b}, \qquad [12]^3([1\mathbf{4}][2\mathbf{3}]+[1\mathbf{3}][2\mathbf{4}])y^{2b+1}$
$(++--)$	$[12]^4\langle\mathbf{34}\rangle x^a y^{2b}$
$(+-++)$	$[1\|(\mathbf{3}-\mathbf{4})\|2\rangle^4[\mathbf{34}]x^a y^b$
$(+++-)$	$[12]^4[\mathbf{3}\|(1-2)\|\mathbf{4}\rangle x^a y^{2b+1}$
$(++-+)$	$[12]^4\langle\mathbf{3}\|(1-2)\|\mathbf{4}]x^a y^{2b+1}$
$(+-+-)$	$[1\mathbf{3}]\langle 2\mathbf{4}\rangle[1\|(\mathbf{3}-\mathbf{4})\|2\rangle^3 x^a y^b$

where

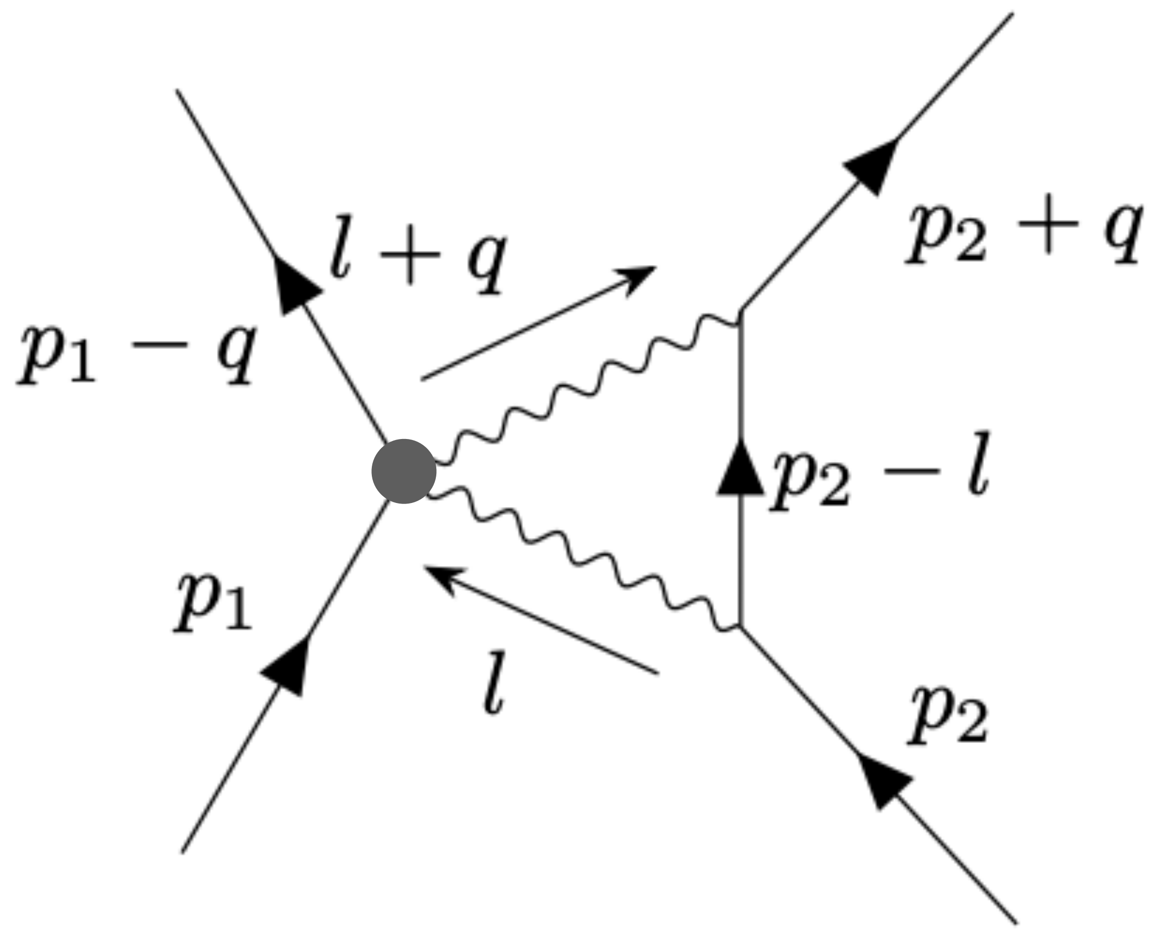
$$x = s_{12} \text{ and } y = s_{13} - s_{23} + s_{24} - s_{14}.$$

Used to check relation between operators
which are not obvious off-shell

Calculate the 2PM tidal spin-1/2 scattering amplitude

$$\Delta\mathcal{M}_2^{s=1/2} = G^2 m_2^2 S \sum_{j=0}^{\infty}\left(-\frac{q^2}{2}\right)^{j+2}\Big[\mathcal{U}_1\mathcal{U}_2 G_j^{(0)} - i\omega\mathcal{E}_1\mathcal{U}_2 G_j^{(1,1)} + i\omega\mathcal{U}_1\mathcal{E}_2 G_j^{(1,2)}$$

$$+ (q\cdot S_1)(q\cdot S_2)G_j^{(2,1)} - q^2(S_1\cdot S_2)G_j^{(2,2)} + \omega q^2 (v_2\cdot S_1)(v_1\cdot S_2)G_j^{(2,3)}\Big]$$



where

$$\mathcal{U}_1 \equiv \bar{u}(p_1-q)u(p_1) \equiv \bar{u}_2 u_1,$$

$$\mathcal{U}_2 \equiv \bar{u}(p_2+q)u(p_2) \equiv \bar{u}_4 u_3,$$

$$\mathcal{E}_i \equiv \epsilon^{\mu\nu\alpha\beta} p_{1\mu} p_{2\nu} \bar{q}_\alpha S_{i\beta},$$

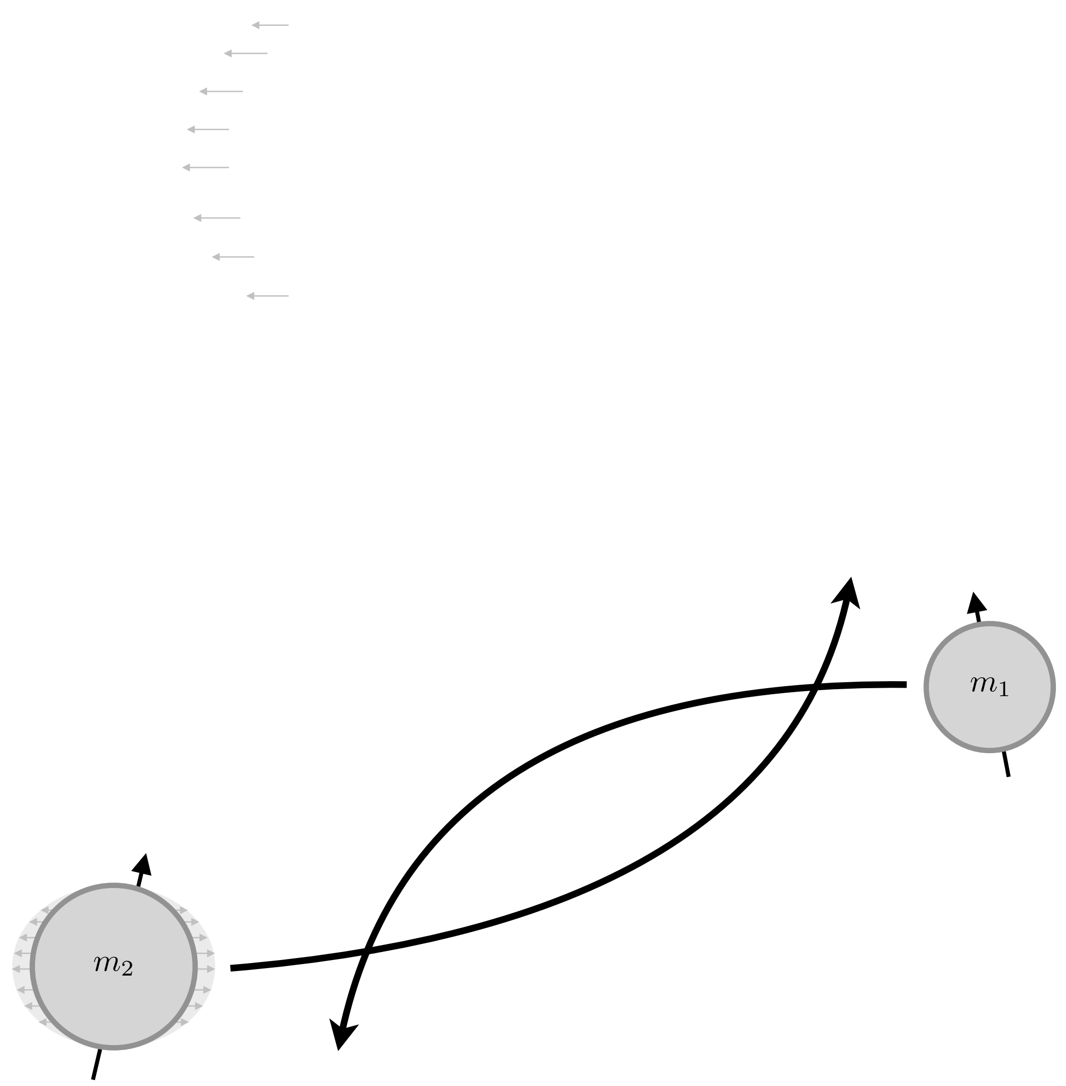
$$\mathcal{E}_i^\mu \equiv \epsilon^{\mu\nu\alpha\beta} p_{i\nu} \bar{q}_\alpha S_{i\beta},$$

$$S_i^\mu \equiv \frac{1}{2}\bar{u}_{2i}\gamma_5\gamma^\mu u_{2i-1},$$

For example

$$\begin{aligned}
G_j^{(1,1)} =& -\frac{e^{(j)}}{m_2}(4m_1)^{2j}(1-\omega^2)^j 16\alpha_{j+1} \\
&+\sum_{k=0}^{j}\frac{f_1^{(j+k,k)}}{m_1 m_2}(4m_1)^{2k+2}(1-\omega^2)^k \\
&\times\left[2(\omega^2-1)(\alpha_{k+2}-2\alpha_{k+1}) + (2\omega^2-1)2\alpha_k + (1-4\omega^2)\alpha_{k+1}\right] \\
&+\sum_{k=0}^{j}\frac{f_3^{(j+k,k)}}{2m_1 m_2}(4m_1)^{2k+4}(1-\omega^2)^k\left[(4\omega^2-3)\alpha_{k+1} - 6(\omega^2-1)\alpha_{k+2}\right] \\
&+\sum_{k=0}^{j}\frac{f_4^{(j+k,k)}}{m_1 m_2}(4m_1)^{2k+2}(1-\omega^2)^k 8\alpha_{k+1},
\end{aligned}$$

Classical observables



- Conservative two-body Hamiltonian

$$H(\boldsymbol{q},\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2+m_1^2}+\sqrt{\boldsymbol{p}^2+m_2^2}+\langle \boldsymbol{n}_1\boldsymbol{n}_2|\left[\hat{V}(\boldsymbol{k}',\boldsymbol{k},\hat{\boldsymbol{S}}_a)+\Delta\hat{V}(\boldsymbol{k}',\boldsymbol{k},\hat{\boldsymbol{S}}_a)\right]|\boldsymbol{n}_1\boldsymbol{n}_2\rangle.$$

[Following Bern, Luna, Roiban, Shen, Zeng, 20']

- Linear and angular impulses (KMOC)

[Following Kosower, Maybee, O'Connell, 18']

- Eikonal phase:
  linear impulse, spin-kick and aligned
  scattering angle

[Following Bern, Luna, Roiban, Shen, Zeng, 20']

Linear and angular impulses (KMOC)

Working with covariant spin

final spinors $\bar{u}(p \pm \hbar\bar{q}) = \bar{u}(p) + \mathcal{O}(q).$

$$\mathcal{U}_i = 2m_i + \mathcal{O}(q^2).$$

$$\begin{aligned}
\Delta p^{\mu}_{1,\mathrm{GR}} =& \frac{-\pi G^2 m_2}{8m_1\sqrt{\omega^2-1}} \sum_{j=0}^{\infty} \left(-\frac{2}{b^2}\right)^{j+3} \frac{\Gamma[7/2+j]}{\Gamma[-3/2-j]} \Bigg[-4m_1 m_2 G^{(0)}_j \frac{b^\mu}{|\boldsymbol{b}|} \\
&+2\omega m_1 m_2^2 \epsilon_{\rho\nu\alpha\beta} v_2^\nu v_1^\alpha \langle S_1^\beta\rangle G_j^{(1,1)} \frac{1}{|\boldsymbol{b}|}\left((7+2j)\frac{b^\mu b^\rho}{b^2} - \Pi^{\mu\rho}\right) \\
&+2\omega m_1^2 m_2 \epsilon_{\rho\nu\alpha\beta} v_1^\nu v_2^\alpha \langle S_2^\beta\rangle G_j^{(1,2)} \frac{1}{|\boldsymbol{b}|}\left((7+2j)\frac{b^\mu b^\rho}{b^2} - \Pi^{\mu\rho}\right) \\
&-\left[\langle S_{1\nu}\rangle\langle S_{2\rho}\rangle G_j^{(2,1)} - \eta_{\nu\rho}\left(\langle S_1\rangle\cdot\langle S_2\rangle\right) G_j^{(2,2)} + \omega\eta_{\nu\rho}\left(v_2\cdot\langle S_1\rangle\right)\left(v_1\cdot\langle S_2\rangle\right) G_j^{(2,3)}\right] \\
&\times\left((9/2+j)\frac{b^\mu b^\nu b^\rho}{b^2} - \frac{3}{2} b^{(\mu}\Pi^{\nu\rho)}\right)\frac{2}{|\boldsymbol{b}|}\left(-\frac{2}{b^2}\right)(7/2+j)\Bigg],
\end{aligned} \tag{5.4}$$

And angular impulse.

$$\Delta S_1^\mu = \left\langle\!\!\left\langle i\int \hat{d}^4 q \hat{\delta}(2p_1\cdot q)\hat{\delta}(2p_2\cdot q) e^{-ib\cdot q}\left(-\frac{p_1^\mu}{m_1^2} q\cdot S_1(p_1)\mathcal{A}(q) + [S_1^\mu(p_1), \mathcal{A}(q)]\right)\right\rangle\!\!\right\rangle,$$

commutator piece

Eikonal phase

$$\chi_i = \frac{1}{4m_1 m_2\sqrt{\omega^2-1}} \int \frac{d^{2-2\epsilon}\boldsymbol{q}}{(2\pi)^{2-2\epsilon}} e^{-i\boldsymbol{q}\cdot\boldsymbol{b}} \mathcal{M}_i'(\boldsymbol{q}),$$

Can use

$$\int \frac{d^2\boldsymbol{q}}{(2\pi)^2} e^{-i\boldsymbol{q}\cdot\boldsymbol{b}} \frac{\pi^2}{|\boldsymbol{q}|} \left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2} \hat{\mathbb{O}}^A(\boldsymbol{p},\boldsymbol{q},\boldsymbol{S}_a) = \hat{\mathcal{O}}^A(\boldsymbol{p},\nabla_{\boldsymbol{b}},\boldsymbol{S}_a) I_{j+2}(\boldsymbol{b}) \frac{\mathcal{N}}{4},$$

Impact parameter operators

$$\hat{\mathcal{O}}^{(0)} = \mathbb{I}, \quad \hat{\mathcal{O}}^{(1,1)} = -(\boldsymbol{S}_1\times\boldsymbol{p})\cdot\nabla_{\boldsymbol{b}}, \quad \hat{\mathcal{O}}^{(1,2)} = -(\boldsymbol{S}_2\times\boldsymbol{p})\cdot\nabla_{\boldsymbol{b}},$$
$$\hat{\mathcal{O}}^{(2,1)} = -(\boldsymbol{S}_1\cdot\nabla_{\boldsymbol{b}})(\boldsymbol{S}_2\cdot\nabla_{\boldsymbol{b}}), \quad \hat{\mathcal{O}}^{(2,2)} = -(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\,\nabla_{\boldsymbol{b}}^2, \quad \hat{\mathcal{O}}^{(2,3)} = -(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2)\nabla_{\boldsymbol{b}}^2.$$

Eikonal operator

$$\Delta\chi_2 = 2G^2 m_2^2 \sum_{j=0}^{\infty} \hat{\mathcal{K}}_j(\omega,\boldsymbol{p},\nabla_{\boldsymbol{b}},\boldsymbol{S}_a)\, I_{j+2}(\boldsymbol{b}), \quad \hat{\mathcal{K}}_j(\omega,\boldsymbol{p},\nabla_{\boldsymbol{b}},\boldsymbol{S}_a) \equiv 4E_1E_2 \sum_A \Delta b_{2,j}^A(\omega)\hat{\mathcal{O}}^A,$$

Eikonal phase

$$\chi_i = \frac{1}{4m_1m_2\sqrt{\omega^2-1}}\int\frac{d^{2-2\epsilon}\boldsymbol{q}}{(2\pi)^{2-2\epsilon}}e^{-i\boldsymbol{q}\cdot\boldsymbol{b}}\mathcal{M}'_i(\boldsymbol{q}),$$

Amplitude in the classical spin-structure basis

$$\hat{\mathbb{O}}^{(0)} = \mathbb{I}, \qquad \hat{\mathbb{O}}^{(1,1)} = \boldsymbol{L}_q\cdot\hat{\boldsymbol{S}}_1, \qquad \hat{\mathbb{O}}^{(1,2)} = \boldsymbol{L}_q\cdot\hat{\boldsymbol{S}}_2,$$
$$\hat{\mathbb{O}}^{(2,1)} = \boldsymbol{q}\cdot\hat{\boldsymbol{S}}_1\boldsymbol{q}\cdot\hat{\boldsymbol{S}}_2, \qquad \hat{\mathbb{O}}^{(2,2)} = \boldsymbol{q}^2\hat{\boldsymbol{S}}_1\cdot\hat{\boldsymbol{S}}_2, \qquad \hat{\mathbb{O}}^{(2,3)} = \boldsymbol{q}^2\boldsymbol{p}\cdot\hat{\boldsymbol{S}}_1\boldsymbol{p}\cdot\hat{\boldsymbol{S}}_2,$$

where $\boldsymbol{L}_q \equiv i(\boldsymbol{p}\times\boldsymbol{q})$.

$$\int\frac{d^2\boldsymbol{q}}{(2\pi)^2}e^{-i\boldsymbol{q}\cdot\boldsymbol{b}}\frac{\pi^2}{|\boldsymbol{q}|}\left(\frac{\boldsymbol{q}^2}{2}\right)^{j+2}\hat{\mathbb{O}}^A(\boldsymbol{p},\boldsymbol{q},\boldsymbol{S}_a) = \hat{\mathcal{O}}^A(\boldsymbol{p},\nabla_{\boldsymbol{b}},\boldsymbol{S}_a)I_{j+2}(\boldsymbol{b})\,\frac{\mathcal{N}}{4},$$

Impact parameter operators

$$\hat{\mathcal{O}}^{(0)} = \mathbb{I}, \quad \hat{\mathcal{O}}^{(1,1)} = -(\boldsymbol{S}_1\times\boldsymbol{p})\cdot\nabla_{\boldsymbol{b}}, \quad \hat{\mathcal{O}}^{(1,2)} = -(\boldsymbol{S}_2\times\boldsymbol{p})\cdot\nabla_{\boldsymbol{b}},$$
$$\hat{\mathcal{O}}^{(2,1)} = -(\boldsymbol{S}_1\cdot\nabla_{\boldsymbol{b}})(\boldsymbol{S}_2\cdot\nabla_{\boldsymbol{b}}), \quad \hat{\mathcal{O}}^{(2,2)} = -(\boldsymbol{S}_1\cdot\boldsymbol{S}_2)\,\nabla^2_{\boldsymbol{b}}, \quad \hat{\mathcal{O}}^{(2,3)} = -(\boldsymbol{p}\cdot\boldsymbol{S}_1)(\boldsymbol{p}\cdot\boldsymbol{S}_2)\nabla^2_{\boldsymbol{b}}.$$

Classical observables from the eikonal phase

Eikonal operator

$$\Delta\chi_2 = 2G^2 m_2^2 \sum_{j=0}^{\infty} \hat{\mathcal{K}}_j(\omega, \boldsymbol{p}, \nabla_{\boldsymbol{b}}, \boldsymbol{S}_a)\, I_{j+2}(\boldsymbol{b}), \qquad \hat{\mathcal{K}}_j(\omega, \boldsymbol{p}, \nabla_{\boldsymbol{b}}, \boldsymbol{S}_a) \equiv 4E_1E_2 \sum_A \Delta b^A_{2,j}(\omega)\hat{\mathcal{O}}^A,$$

Linear impulse and spin-kick $\quad \Delta\boldsymbol{p}_\perp = \nabla_{\boldsymbol{b}}\chi \quad \text{and} \quad \Delta\boldsymbol{S}_a^i = -\epsilon^{ijk}\frac{\partial\chi}{\partial\boldsymbol{S}_a^j}\boldsymbol{S}_a^k,$

$$\Delta\boldsymbol{p}_\perp = -2G^2 m_2^2 \sum_{j=0}^{\infty} \hat{\mathcal{K}}_j(\omega, \boldsymbol{p}, \nabla_{\boldsymbol{b}}, \boldsymbol{S}_a)\left(\nabla_{\boldsymbol{b}} I_{j+2}(\boldsymbol{b})\right),$$

$$\Delta\boldsymbol{S}_a = -2G^2 m_2^2 \sum_{j=0}^{\infty} \left(\frac{\partial\hat{\mathcal{K}}_j(\omega, \boldsymbol{p}, \nabla_{\boldsymbol{b}}, \boldsymbol{S}_a)}{\partial\boldsymbol{S}_a} \times \boldsymbol{S}_a\right) I_{j+2}(\boldsymbol{b}).$$

Matches the commutator piece of the angular impulse

Scattering angle (aligned spin) $\quad \Delta\theta = -\frac{2G^2 m_2^2}{|\boldsymbol{p}|}\sum_{j=0}^{\infty}\frac{2|\boldsymbol{b}|}{(5+2j)}\hat{\mathcal{K}}_j(\omega, \boldsymbol{p}, \nabla_{\boldsymbol{b}}, \boldsymbol{S}_a)\, I_{j+3}(\boldsymbol{b}).$

Conclusions

Introduced HPET variables to calculate classical gravity:

Seems natural; expansion done at operator level from start

Exact matching of the multipole expansion of a Kerr BH

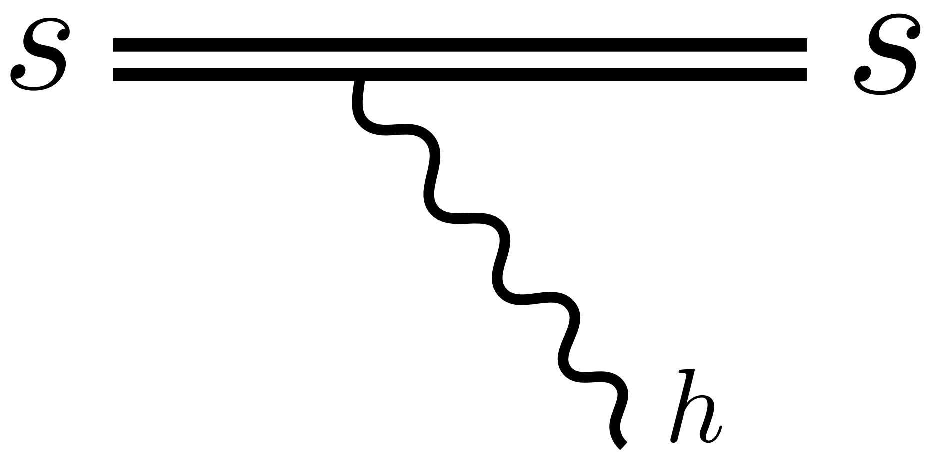
Facilitates classical limit of tree-level amplitudes; Spin-Universality

Used EFT techniques to describe tidal effects for spinning particles

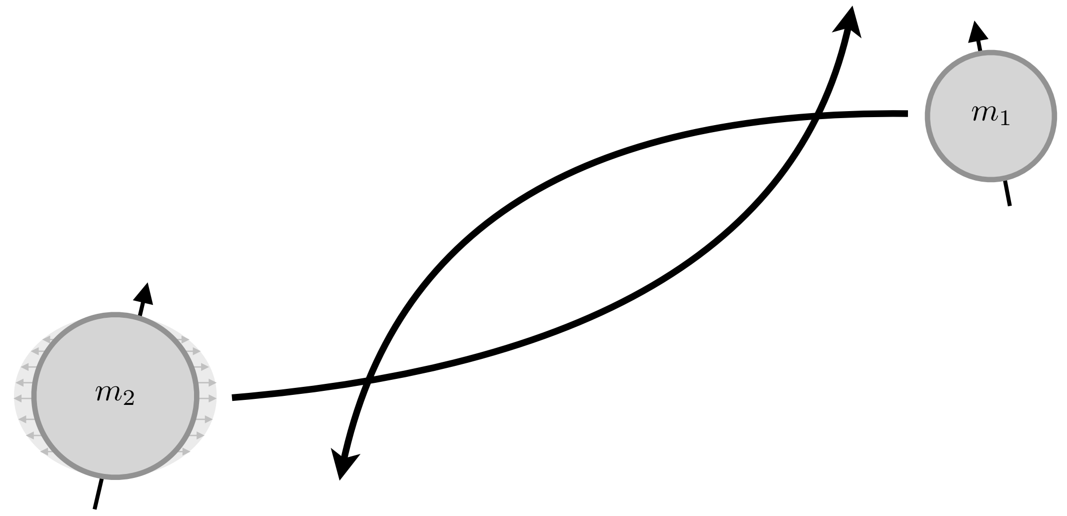
Expanded on recent description for spinless tidal

Used on-shell amplitudes to cross check the tidal action

Calculated several classical observables



Thank you!



Hilbert Series for scalars

Contour integral of the plethystic exponential

$$\mathcal{H} = \int d\mu \frac{1}{P} \mathrm{PE}[\chi_\phi],$$

$$\mathrm{PE}_\phi = \exp\left[\sum_{r=0}^{\infty} z^{r+1} \frac{\phi^r}{r\mathcal{D}^{r\Delta_\phi}} \chi_\phi(x_1^r, \ldots, x_k^r)\right],$$

$$\chi_\phi = \chi_{[1,(0,0)]}(\mathcal{D}; \alpha, \beta),$$
$$\chi_{F_L} = \chi_{[2,(1,0)]}(\mathcal{D}; \alpha, \beta),$$
$$\chi_{F_R} = \chi_{[2,(0,1)]}(\mathcal{D}; \alpha, \beta),$$

$$\chi_{[1,(0,0)]}(\mathcal{D}; \alpha, \beta) = \mathcal{D} P(\mathcal{D}; \alpha, \beta)(1 - \mathcal{D}^2),$$
$$\chi_{[2,(1,0)]}(\mathcal{D}; \alpha, \beta) = \mathcal{D}^2 P(\mathcal{D}; \alpha, \beta)\left[\chi_{(1,0)}(\alpha, \beta) - \mathcal{D}\chi_{(1/2,1/2)}(\alpha, \beta) + \mathcal{D}^2\right]$$
$$\chi_{[2,(0,1)]}(\mathcal{D}; \alpha, \beta) = \mathcal{D}^2 P(\mathcal{D}; \alpha, \beta)\left[\chi_{(0,1)}(\alpha, \beta) - \mathcal{D}\chi_{(1/2,1/2)}(\alpha, \beta) + \mathcal{D}^2\right]$$

$$\chi_{(l_1,l_2)}(\alpha, \beta) = \chi_{l_1}^{SU(2)}(\alpha) \times \chi_{l_2}^{SU(2)}(\beta).$$

$$\chi_0^{SU(2)}(\alpha) = 1,$$
$$\chi_{1/2}^{SU(2)}(\alpha) = \alpha + \frac{1}{\alpha},$$
$$\chi_1^{SU(2)}(\alpha) = \alpha^2 + 1 + \frac{1}{\alpha^2},$$

$$SO(4) \simeq SU(2)_L \times SU(2)_R$$

$$\int d\mu_{\mathrm{Lorentz}} = \left(\frac{1}{2\pi i}\right)^2 \oint_{|\alpha|=1} \frac{d\alpha}{2\alpha}(1 - \alpha^2)\left(1 - \frac{1}{\alpha^2}\right) \oint_{|\beta|=1} \frac{d\beta}{2\beta}(1 - \beta^2)\left(1 - \frac{1}{\beta^2}\right).$$

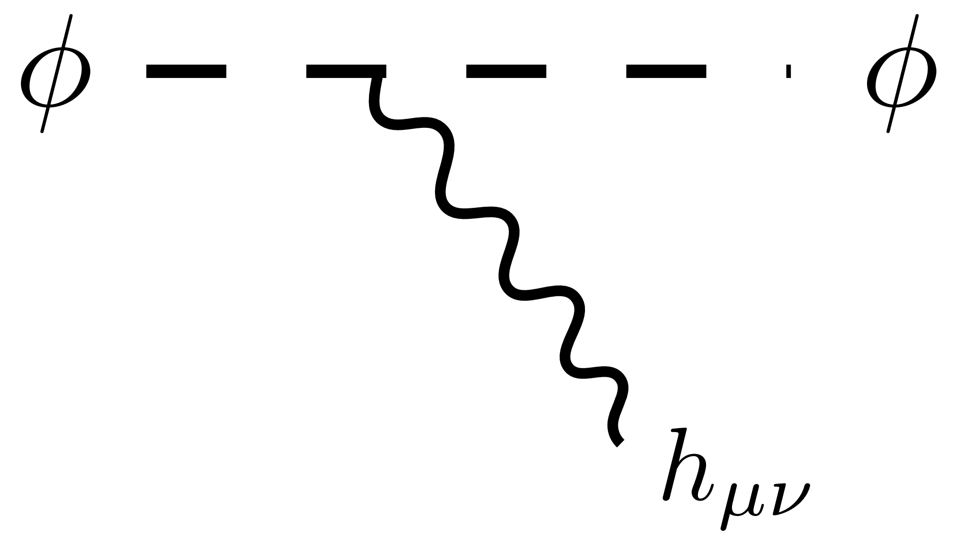
[Damgaard, Haddad, Helset, 19']

Heavy Black-Hole Effective Theory (HBET)

$$p^\mu = m_Q v^\mu + \hbar \bar{k}^\mu.$$

Minimally coupled scalar:

$$\sqrt{-g}\mathcal{L}_{\text{sc-grav}} = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right)$$



Heavy-field limit $\quad \phi \to \frac{1}{\sqrt{2m}}\left(e^{-imv\cdot x}\chi + e^{imv\cdot x}\chi^*\right)$

The Lagrangian

$$\sqrt{-g}\mathcal{L}^{s=0}_{\text{HBET}} = \sqrt{-g}\chi^*\left[g^{\mu\nu}iv_\mu\partial_\nu + \frac{1}{2}m(g^{\mu\nu}v_\mu v_\nu - 1) - \frac{1}{2m}g^{\mu\nu}\partial_\mu\partial_\nu\right]\chi + \mathcal{O}(1/m^2).$$

Linear propagators

$$D_v^{s=0}(k) = \frac{i}{\hbar v\cdot k}$$

Also obtained for s=1/2

Reparametrization invariance

The decomposition of heavy quark momentum is not unique $p^\mu = mv^\mu + k^\mu$

We can always transform

$$(v,k) \to (w,k') \equiv \left(v + \frac{\delta k}{m}, k - \delta k\right),$$

Which effect the variables as

$$|\mathbf{p}_v\rangle = \left(1 - \frac{k'^2}{4m^2}\right)^{-1}\left[\left(1 - \frac{k^2}{4m^2} + \frac{\not{k}\delta\not{k}}{4m^2}\right)|\mathbf{p}_w\rangle - \frac{\delta\not{k}}{2m}|\mathbf{p}_w]\right],$$

$$|\mathbf{p}_v] = \left(1 - \frac{k'^2}{4m^2}\right)^{-1}\left[\left(1 - \frac{k^2}{4m^2} + \frac{\not{k}\delta\not{k}}{4m^2}\right)|\mathbf{p}_w] - \frac{\delta\not{k}}{2m}|\mathbf{p}_w\rangle\right].$$

The S-matrix is reparametrizaion invariant