Waveforms from the KMOC formalism and coherent states

Riccardo Gonzo based on
work in progress with A.Cristofoli, D.Kosower and D.O'Connell

SAGEX
Scattering Amplitudes: from Geometry to Experiment

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Content

Motivation (I)

- Scattering amplitudes are useful for the conservative regime of the inspiral phase of the BBH/BNS/BHNS system[1], using a suitable analytic continuation between bound and unbound orbits[Kälin,Porto].
  Can we extend this to the dissipative case?

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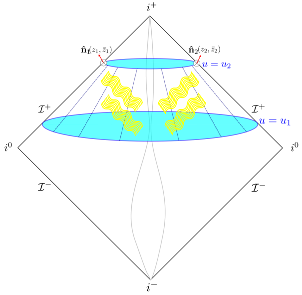
How is the classical gravitational radiation produced in the two-body scattering problem represented at the quantum level?
What is the expression of the waveform in terms of scattering amplitudes?



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- Fresh look to an old subject: how does photon quantum scattering amplitude contain information about the classical light ray scattering?

What are
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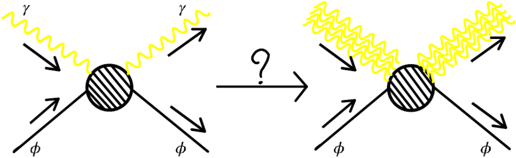
Bonus: Toy model for gravity!



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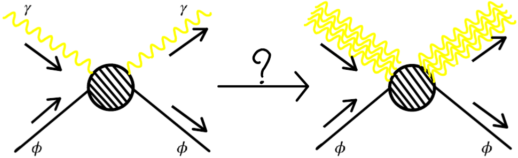
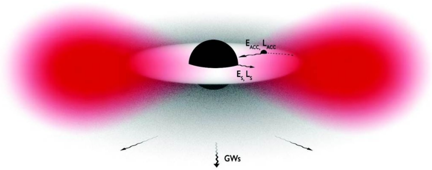
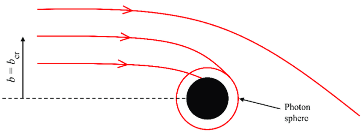


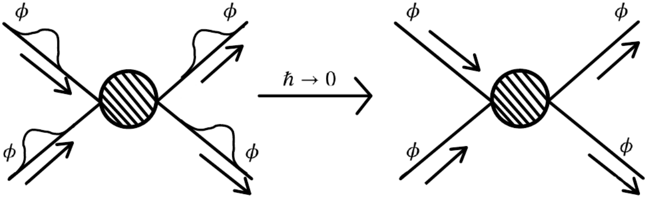
- Phenomenologically relevant for gravitational physics: Light deflection around a classical object, superradiance, ...





Classical limit for massive particles

The classical limit for external massive particles can be achieved by using appropriate minimum-uncertainty wavefunctions which are localized on the classical trajectory [Kosower,Maybee,O'Connell; Krivitski,Tsytovich]



$$\phi(p) = \mathcal{N} m^{-1} \exp\left[-\frac{p \cdot u}{\hbar \ell_c / \ell_w^2}\right] \overset{\text{rest frame}}{\rightarrow} \mathcal{N}' \exp\left(-\frac{p^2}{2m^2 \ell_c^2/\ell_w^2}\right)$$

where p is the 4-momentum, ℓ_c is the Compton wavelength and ℓ_w is related to the intrinsic spread of the wavefunction.

Classical limit for external massless particles?

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Solution:

Realize that a single massless particle is *never* classical: only an infinite superposition of them can be! ($\to$ Coherent state)

Why coherent states?(I)

Toy Model: $\mathcal{L}^{(S)} = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - \lambda j(x)\phi(x)$

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Toy Model: $\mathcal{L}^{(S)} = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - \lambda j(x)\phi(x)$
Equation of motion $\Box\phi(x) = \lambda j(x)\phi(x)$:

$$\phi(x) = \phi_{\text{in}}(x) + \lambda \int d^4x' \, G_{\text{ret}}\,(x - x')j(x')$$
$$= \phi_{\text{out}}(x) + \lambda \int d^4x' \, G_{\text{adv}}(x - x')j(x')$$

where $G_{\text{ret}}\,(x)/G_{\text{adv}}(x)$ are the retarded/advanced propagators and $\phi_{\text{in}}(x)$ (resp. $\phi_{\text{out}}(x)$) is the field configuration at $t \to -\infty$ (resp.$t \to +\infty$). We have then

$$\phi_{\text{out}}(x) = \phi_{\text{in}}(x) + \lambda \int d^4x' \, G_{\text{F}}(x - x')j(x')$$

where $G_{\text{F}}(x)$ is the Feynman propagator.

Why coherent states?(II)

There must exist a unitary transformation which connects in and out states

$$\phi_{\text{out}}(x) = S^\dagger \phi_{\text{in}}(x) S \qquad |out\rangle = S\,|in\rangle$$

Solution: $$\boxed{S = \exp\left(-i\lambda \int d^4x \phi_{\text{in}}(x) j(x)\right)}$$

This means that if we start at $t \to -\infty$ with a vacuum state (i.e. no incoming wavepacket) we will create a coherent state

$$|out\rangle = \exp\left(-i\lambda \int d^4x \phi_{\text{in}}(x) j(x)\right)|0\rangle =$$
$$= \exp\left(-\frac{1}{2}\int d\Phi(k)\,\lambda^2 |j(k)|^2\right) \underbrace{\exp\left(-i\lambda \int d\Phi(k)\, j(k) a^\dagger(\text{k})\right)}_{\text{Infinite superposition of massless modes}} |0\rangle$$

Glauber Sudarshan P-representation and S-matrix

- Theorem(Glauber,1963): Every quantum state of radiation in QFT (i.e. every density matrix) can be written as a superposition of coherent states

$$\boxed{\hat{\rho}_{\text{radiation}} = \int P(\alpha) \left|\alpha\right\rangle \left\langle\alpha\right| d\Re(\alpha) d\Im(\alpha)}$$

where for the classical case $P(\alpha) \geq 0$.

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$$P(\alpha) = f(\alpha) + \sum_{j=1}^{+\infty} c_j \delta^2(\alpha - \alpha_j)$$

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- (Hillery,1985) The superposition of pure states is trivial, i.e. a classical pure state can be written as a single coherent state

$$P(\alpha) = c_\star \delta^2(\alpha - \alpha_\star)$$

Example: Coherent states from soft expansion

- At leading order in the soft expansion, one can study the interaction hamiltonian of massive scalar particles minimally coupled with gravity in asymptotic limit $|t| \to +\infty$ [Faddeev,Kulish; Ware,Saotome,Akhoury]

$$H(t) = H_0 + V^{\text{asy}}(t) = H_0 - \int \mathrm{d}^3x \, h^{\mu\nu}(t, \vec{x}) \, T^{\text{asy}}_{\mu\nu}$$

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$T^{\text{asy}}_{\mu\nu} \to$ point-particle stress tensor for a straight-line trajectory

- Solving the asymptotic evolution equation for the potential $V^{\text{asy}}(t)$ gives an evolution operator which generates a coherent state of soft gravitons

$$\boxed{\mathbb{C}_{\alpha,\sigma} |0\rangle = \left|\alpha_p^\sigma\right\rangle = \exp\left[\int d\Phi(p) \left(\alpha_p(p) a^\dagger_\sigma(p) - \text{h.c.}\right)\right] |0\rangle}$$

See also [Addazi,Bianchi,Veneziano;Ciafaloni,Colferai,Veneziano; Monteiro,O'Connell,Veiga,Sergola]

Classical light beam and coherent states (I)

- Let's choose the waveshape $\alpha(k)$ in such a way to have a localized beam of light moving along the z-direction, and symmetrical around the z-axis

$$\alpha(k) = \frac{1}{\hbar^3}|\vec{k}|(2\pi)^3 A_0 \sqrt{2\hbar}\, \delta_{\sigma_\parallel}(\omega - k^z/\hbar)\delta_{\sigma_\perp}(k^x/\hbar)\delta_{\sigma_\perp}(k^y/\hbar)$$

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- The expectation value of the number operator in a coherent state

$$N_\gamma = \langle\alpha^+|\,\mathbb{N}_\gamma\,|\alpha^-\rangle = \int d\Phi(k)|\alpha(k)|^2 \overset{k=\hbar\bar{k},\bar{\alpha}=\alpha\hbar^{\frac{3}{2}}}{=} \frac{1}{\hbar}\int d\Phi(\bar{k})|\bar{\alpha}(\bar{k})|^2 \,.$$

is large in the classical limit $\boxed{N_\gamma \gg 1}$.

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- The momentum carried by the classical wave

$$k_\gamma^\mu = \langle\alpha^+|\,\mathbb{K}^\mu\,|\alpha^-\rangle = \int d\Phi(k)|\alpha(k)|^2\,k^\mu \overset{k=\hbar\bar{k},\bar{\alpha}=\alpha\hbar^{\frac{3}{2}}}{=} \int d\Phi(\bar{k})|\bar{\alpha}(\bar{k})|^2\,\bar{k}^\mu\,.$$

is finite in the classical limit.

Classical light beam and coherent states (II)

- The expectation value of the field is

$$\langle \alpha^+ | \mathbb{A}_\mu(x) | \alpha^- \rangle = \frac{1}{\sqrt{\hbar}} \int d\Phi(k) \left[\alpha(k) \varepsilon_\mu^{+,*}(k) e^{-ik\cdot x/\hbar} + \alpha^*(k) \varepsilon_\mu^{+}(k) e^{+ik\cdot x/\hbar} \right]$$
$$\stackrel{\sigma_\parallel \to 0}{=} \sqrt{2} A_0 \Re \left[e^{-i\omega(t-z)} \varepsilon_\mu^{+,*}(\bar{k}_0)\, e^{-(x^2+y^2)/(4\ell_\perp^2)} \right]$$

where we require that the transverse momentum components should be subdominant (analogue of the Goldilocks conditions for a localized wave)

$$\boxed{\omega = \lambda^{-1} \gg \sigma_\perp = \ell_\perp^{-1}}$$

Factorization and Glauber complete coherence condition

- For a single coherent state, Glauber's complete coherence condition should hold

$$\boxed{\langle\alpha^+|\,\mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\rho\sigma}(y)\,|\alpha^-\rangle \simeq \langle\alpha^+|\,\mathbb{F}^{\mu\nu}(x)\,|\alpha^-\rangle\,\langle\alpha^+|\,\mathbb{F}^{\rho\sigma}(y)\,|\alpha^-\rangle}$$

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- Check: for a constant waveshape ($\sigma_{\|} \to 0, \sigma_{\perp} \to 0$) we have

$$\begin{aligned}\left\langle\alpha^+\right| \mathbb{F}^{\mu\nu}(x)\mathbb{F}^{\rho\sigma}(y)\left|\alpha^-\right\rangle = &\left\langle\alpha^+\right| \mathbb{F}^{\mu\nu}(x)\left|\alpha^-\right\rangle \left\langle\alpha^+\right| \mathbb{F}^{\rho\sigma}(y)\left|\alpha^-\right\rangle \\ &+ 4\hbar\partial^{[\mu}\eta^{\nu][\sigma}\partial^{\rho]} \int d\Phi(\bar{k})\, e^{-i\bar{k}\cdot(x-y)},\end{aligned}$$

so that asking the factorization implies exactly $\hbar \to 0$.

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- Classical impulse for the eikonal wave $\langle \text{out} | \, \Delta \mathbb{P}_{\gamma}^{\mu} \, | \text{out} \rangle$
- Wave deflection angle $\Delta\Theta_{\gamma}$
- Energy event shapes $\boxed{\mathcal{E}(\hat{n}) = \int_{-\infty}^{+\infty} \mathrm{d}u \lim_{r \to \infty} r^2 \, T_{uu}(u, r, \hat{n})}$

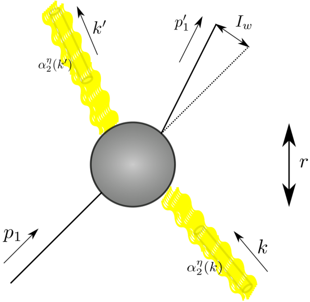
  [Belitsky,Korchemsky,Sterman;G.,Pokraka]
  $\to$ connected to the amplitude of the waveform!

Classical impulse

Theory: Scalar QED

- Using the KMOC formalism we can write the impulse for the massive charged particle as

$$\langle \Delta p_1^\mu \rangle = \langle \psi_w | \, i[\mathbb{P}_1^\mu, T] \, |\psi_w\rangle + \langle \psi_w | \, T^\dagger [\mathbb{P}_1^\mu, T] \, |\psi_w\rangle = I^\mu_{w(1)} + I^\mu_{w(2)}$$



where

$$|\psi_w\rangle_{\text{in}} = \int d\Phi(p_1)\, \phi_1(p_1)\, e^{ib\cdot p_1/\hbar} |p_1 \, \alpha_2^\eta\rangle_{\text{in}} \, .$$

Classical impulse in wave scattering (II)

- The properties of the coherent state as a displacement operator

$$\mathbb{C}^{\dagger}_{\alpha,\eta} a_{\rho}(k) \mathbb{C}_{\alpha,\eta} = a_{\eta}(k) + \delta_{\eta\rho}\alpha(k), \quad \mathbb{C}^{\dagger}_{\alpha,\eta} a^{\dagger}_{\rho}(k) \mathbb{C}_{\alpha,\eta} = a^{\dagger}_{\eta}(k) + \delta_{\eta\rho}\alpha(k)$$

allow us to formulate the problem as an expansion around the background

$$\langle \Delta p_1^{\mu} \rangle := \int d\Phi(p_1) d\Phi(p_1') \, \phi_1(p_1) \phi_1^*(p_1') e^{-i\frac{q\cdot b}{\hbar}} \langle p_1' | \, i[\mathbb{P}_1^{\mu}, T(A + A_{\sigma}^{\text{class}})] \, | p_1 \rangle +$$

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where $q := p_1' - p_1$ and $T(A + A_{\sigma}^{\text{class}})$ is a short notation for the background field expansion at the level of the interaction lagrangian.

- Note: $I^{\mu}_{w(1)}$ is always proportional to the two point function in the classical background $\alpha(k)$!

Classical field strength in Thomson scattering

- Classically, we solve perturbatively the equations of motion:

$$\frac{dp^\mu}{d\tau} = g\, F^{\mu\nu}(x(\tau))u_\nu(\tau) \qquad x(0) = u(0)\tau$$

$$\partial^\mu F_{\mu\nu} = \int d\tau\, g\, u_\nu \delta^4(x - x(\tau))$$

where $F_{\mu\nu} = \partial_{[\mu} A_{\nu]}$ and $A^{\mu,(-)}(x) = \int d\Phi(k_1)\, \alpha(k_1)(\varepsilon_\mu^{-,*}(k_1) e^{-ik_1 \cdot x} + h.c.)$.

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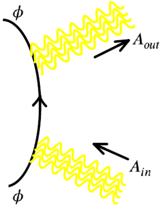
- The solution at order g^2 is

$$A^\nu(x) = -\frac{g^2}{m}\int d\Phi(k_1)\,\alpha(k_1)\frac{\delta(k_1\cdot u - p_1\cdot u)}{k_1^2 + i\epsilon}\Big[-\varepsilon^\nu(\mathrm{k}_1) + \frac{(u\cdot\varepsilon(\mathrm{k}_1))\,p_1^\nu + (p_1\cdot\varepsilon(\mathrm{k}_1))\,u^\nu}{p_1\cdot u + i\varepsilon} - \frac{u^\nu\,[(p_1\cdot k_1)\,(\varepsilon(k_1)\cdot u)]}{(p_1\cdot u + i\varepsilon)^2}\Big]$$



Classical field strength in Compton scattering (I)

- The incoming state is

$$|\psi\rangle_{\text{in}} = \int d\Phi(p_1)\phi_1(p_1)e^{ib\cdot p_1/\hbar}\left|p_1\alpha^-\right\rangle$$

and using the KMOC formalism

$$\boxed{\langle \Delta F^{\mu\nu(+)}(x)\rangle = i\langle\psi|[\mathbb{F}^{\mu\nu(+)},T]|\psi\rangle + \langle\psi|T^\dagger[\mathbb{F}^{\mu\nu(+)},T]|\psi\rangle}$$

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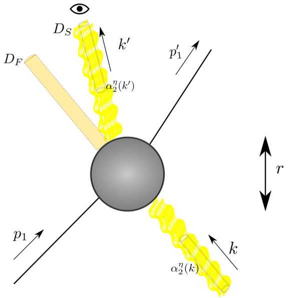
- we get two types of terms up to g^2

$$\langle \Delta F^{\mu\nu(+)}(x)\rangle_1 = \frac{1}{\hbar^{3/2}}2\Re\left\{\int d\Phi(k)d\Phi(p_1)d\Phi(p_1')\phi(p_1)\phi^*(p_1')\; e^{-i\frac{b\cdot q}{\hbar}}\right.$$
$$\left.\times\left[k^{[\mu}\varepsilon^{(+)\nu]*}\langle p_1'\,\alpha_2^+|a_+^\dagger(k)\;T|p_1\,\alpha_2^-\rangle e^{-ik\cdot x/\hbar}\right]\right\}$$

Classical field strength in Compton scattering (II)

We cannot observe a forward scattered photon [Frantz;Moncrief]: therefore we should restrict the domain of $\mathbb{F}^{\mu\nu}$ to the scattered region!



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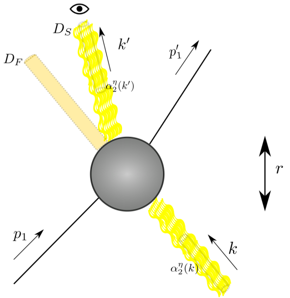

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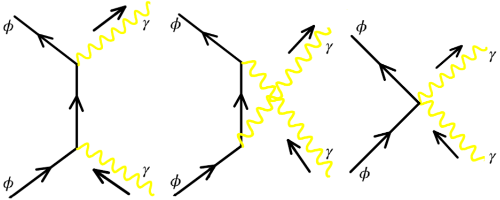
Using the Compton scattering amplitude for the evaluation of $\langle \Delta F^{\mu\nu(+)}(x) \rangle_{1,D_S}$



Agreement in the classical limit with the Thomson scattering!

Waveform and localized detectors(I)

- In the original KMOC paper *global* observables were introduced

$$R^\mu \equiv \langle k^\mu \rangle =_{\text{in}} \langle \psi | \, S^\dagger \mathbb{K}^\mu S \, | \psi \rangle_{\text{in}}$$

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- The radiation's spectral function $f(\omega,\vec{n})$ can be obtained from the waveform via a Fourier transform,

$$f(\omega,\vec{n};\vec{x}) = \int_{-\infty}^{+\infty} dt \; W(t,\vec{n};\vec{x}) \, e^{i\omega t}$$

Waveform and localized detectors(II)

- Let's consider two massive charged particles scattering each other

$$|\psi\rangle_{\text{in}} = \int d\Phi(p_1) d\Phi(p_2) \phi_1(p_1) \phi_2(p_2) e^{i\frac{b \cdot p_1}{\hbar}} |p_1 p_2\rangle$$

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- From $\langle F_{\mu\nu}(x)\rangle \equiv_{\rm in} \left\langle \psi \left| S^\dagger \mathbb{F}_{\mu\nu}(x) S \right| \psi \right\rangle_{\rm in}$ we have, considering the linear terms

$$\boxed{\langle F_{\mu\nu}(x)\rangle = \frac{1}{\hbar^{\frac{3}{2}}} 2\Re \int d\Phi(k)\left[k_{[\mu}\tilde{j}_{\nu]}(k)e^{-ik\cdot x}\right]}$$

where

$$\tilde{j}_\mu(k) \equiv \sum_\sigma \int d\Phi\left(p_1'\right)d\Phi\left(p_2'\right)d\Phi\left(p_1\right)d\Phi\left(p_2\right)\phi_1^*\left(p_1'\right)\phi_2^*\left(p_2'\right)\phi_1\left(p_1\right)\phi_2\left(p_2\right)$$

$$\times e^{-i\frac{b\cdot\left(p_1'-p_1\right)}{\hbar}}\varepsilon_\mu^{*,(\sigma)}(k)\left\langle p_1' p_2' k^\sigma\right|T\left|p_1 p_2\right\rangle$$

Waveform and localized detectors(III)

- Assuming that the measurement distance is much larger than the impact parameter, so that there is a unique and well-defined direction, and using

$$G_{\text{ret}}(x) = i\theta\left(x^0\right)\int d\Phi(k)\left(e^{-ik\cdot x} - e^{ik\cdot x}\right) = \frac{1}{4\pi|\vec{x}|}\delta\left(x^0 - |\vec{x}|\right)$$

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- we get for $x^0 > 0$

$$\langle F_{\mu\nu}(x)\rangle = \int d^4y G_{\mathrm{ret}}(x-y)\left[\frac{\partial}{\partial y^\mu}j_\nu(y) - \frac{\partial}{\partial y^\nu}j_\mu(y)\right] =$$
$$= -\frac{i}{4\pi|\vec{x}|}\int \hat{d}\omega e^{-i\omega u} k_{[\mu}\tilde{j}_{\nu]}(k)\Big|_{k=(\omega,\omega\hat{x})}$$

where $u = x^0 - |\vec{x}|$ is the retarded time($\rightarrow$ Spectral function).
This is the structure of the waveform written in terms of scattering amplitudes! (see also [Maggiore;Jakobsen,Mogull,Plefka,Steinhoff; Mougiakakos,Maria Riva,Vernizzi] for a complementary approach)

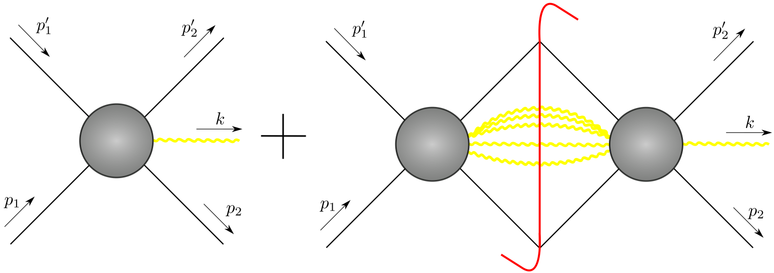
Newman-Penrose Φ_2

- Contracting the field strength with an appropriate tetrad at infinity

$$K = \frac{k}{\omega} = (1, \hat{n}) \quad M = \varepsilon_+ \quad \bar{M} = (\varepsilon_+)^* = \varepsilon_- \quad K \cdot N = 1 = -M \cdot \bar{M}$$

we can define $\Phi_2(\omega, r, \hat{n}) = F_{\mu\nu}(\omega, r, \hat{n}) \bar{M}^\mu N^\nu$. Schematically

$\Phi_2 =$



EM memory effect (I)

- The leading EM memory effect is related with the Weinberg soft factor

$$\tilde{j}_\mu(k) \overset{\omega\to 0}{\simeq} \sum_\sigma \int d\Phi\left(p_1\right) d\Phi\left(p_2\right) |\phi_1\left(p_1\right)|^2 |\phi_2\left(p_2\right)|^2$$

$$\times \int \widehat{d^4 q_1} \widehat{d^4 q_2} \hat{\delta}(2p_1 \cdot q_1) \hat{\delta}(2p_2 \cdot q_2) e^{-i\frac{b \cdot q_1}{\hbar}} \varepsilon_\mu^{*,(\sigma)}(k) \hat{\delta}^4(k - q_1 - q_2)$$

$$\times\, g \left[\frac{\varepsilon^{*,(\sigma)}(k) \cdot (p_1 - q_1)}{k \cdot (p_1 - q_1)} - \frac{\varepsilon^{*,(\sigma)}(k) \cdot p_1}{k \cdot p_1} + (1 \leftrightarrow 2) \right] \langle p_1' p_2' | T | p_1 p_2 \rangle$$

- Expanding the soft factor in the classical limit $q_1 = \hbar \bar{q}_1$ we get

$$g \left[\underbrace{q_1^\mu}_{\text{Impulse!}} \left(-\frac{1}{k \cdot p_1} \varepsilon_\mu^{*,(\sigma)}(k) + \frac{\varepsilon^{*,(\sigma)}(k) \cdot p_1}{(k \cdot p_1)^2} k_\mu \right) \right]$$

EM memory effect (II)

- We can then compute the leading expectation value of $\langle \Phi_2 \rangle$

$$\langle \Phi_2(\omega \simeq 0, r, \hat{n}) \rangle = \frac{g}{4\pi r} \Bigg[\left(\frac{1}{k \cdot p_1} \right) I_{(1)} \cdot \varepsilon_\mu^{*,(\sigma)}(k) - \left(\frac{\varepsilon^{*,(\sigma)}(k) \cdot p_1}{(k \cdot p_1)^2} \right) I_{(1)} \cdot k + (1 \leftrightarrow 2) \Bigg]$$

which is exactly the electromagnetic memory effect (we checked that it agrees with the purely classical prediction, see also [Guevara, Bautista]).

Summary and future directions

- Since massless particles have no rest frame and due to Newton-Wigner theorem, the classical limit requires to work with coherent states.
- We can scatter coherent states with a definite waveshape and build a notion of impulse and of field strength from KMOC formalism which connects smoothly to the classical wave scattering observables. Interesting IR safe observables defined for localized observers like the energy event shape.
- The waveform and its spectral function have a natural expression in terms of scattering amplitudes; the memory effect is naturally included in our framework.
- For the future: focus purely on the gravitational case. Understand better the role of coherent states, classical exponentiation to calculate more efficiently our observables in terms of on-shell data.