Wave-form from unitarity

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Foreword

This talk is based on recent work at two loops in GR and massive $N = 8$ supergravity together with

C. Heissenberg, R. Russo and G. Veneziano, arXiv:2008.12743, arXiv:2101.05772 and arXiv:2104.03256

Plan of the talk

Introduction

- For computing classical quantities as the deflection angle at 3PM one needs to extract the classical part of the two-loop amplitude.
- In particular one needs to compute the integrand of the classical two-loop amplitude that, up to few days ago, was only known for the conservative part of the amplitude: Z. Bern et al, 1901.04424
- Then one must compute the integral.
- This can be done by writing the integral in terms of master integrals that then can be computed with, for instance, the technique of the differential equations.
- The final result for the conservative part of the classical amplitude had the problem that the deflection angle was divergent at high energy.
- This was a consequence of the fact that the real part of the classical amplitude was itself divergent at high energy in contradiction with the results of ACV90 based on unitarity, analyticity and crossing symmetry.

- After infinite discussions on what was the origin of this problem, only an explicit calculation in massive $\mathcal{N} = 8$ supergravity (approximating the integrals in the soft and not in the potential region) convinced everybody that the problem disappears if one adds to the conservative piece also the contribution of radiation reaction [C. Heissenberg, R. Russo, G. Veneziano, PDV, arXiv:2008.12743].
- But then how to compute this extra piece in GR if the classical integrand of the two-loop amplitude was not known?
- From the loss of angular momentum T. Damour computed the radiation reaction contribution to the deflection angle in GR that, added to the conservative part, eliminated the problem with ACV90, T. Damour, 2010.01641.
- And this without knowing the complete classical part of the two-loop amplitude!

- In this talk I will present an alternative approach, based on unitarity, crossing symmetry and analyticity, that allows to compute radiation reaction contribution to the deflection angle again without knowing the complete classical two-loop amplitude.
- It is still not clear from a physical point of view why the two previous approaches give the same result.
- On the other hand, at this point, there is no doubt that they correctly complete the conservative contribution of the classical amplitude, as shown in a recent and beautiful paper by N.E.J. Bjerrum-Bohr, P.H. Damgaard, L. Planté and P. Vanhove, 2105.05218. See Planté's talk.
- They managed to compute in a very simple and intuitive way the complete classical integrand (including both the conservative part and the part due to radiation reaction).
- Then, from it, they derived in a unified way the previous results obtained instead with different techniques.

- In this talk I will briefly describe our way of computing the radiation reaction effects in GR.
- Then I will show that this approach also allows to compute the wave-forms in a rather simple way.
- Our approach is based on the calculation of the 3-particle cut from the unitarity relation.
- This 3-particle cut becomes to be relevant only at 3PM and, having an intermediate graviton exchanged, it naturally provides the radiation reaction contribution.
- Perform the calculation in parallel for $\mathcal{N} = 8$ massive supergravity and GR.

The three-particle cut from unitarity

- The 3-particle cut can be extracted from the unitarity relation:

$$[\mathrm{Im}\, 2A_2]_{3pc} = \int \frac{d^{D-1}k_1}{(2\pi)^{D-1}2k_1^0} \frac{d^{D-1}k_2}{(2\pi)^{D-1}2k_2^0} \frac{d^{D-1}k}{(2\pi)^{D-1}2k^0}$$
$$\times A_5^{MN}(P_1, P_2, K_1, K_2, k) \left[\sum_i \epsilon_{MN}^{(i)} \epsilon_{RS}^{(i)}\right]$$
$$\times A_5^{RS}(P_4, P_3, -K_1, -K_2, -k)(2\pi)^D \delta^{(D)}(p_1 + p_2 + k_1 + k_2 + k)$$

- In $N = 8$ supergravity the indices are 10-dim

$$\sum_i \epsilon_{MN}^{(i)} \epsilon_{RS}^{(i)} = \eta_{MR}\eta_{NS}$$

while in GR they are 4-dim

$$\sum_i \epsilon_{\mu\nu}^{(i)} \epsilon_{\rho\sigma}^{(i)} = \eta_{\mu\rho}\eta_{\nu\sigma} - \frac{1}{D-2}\eta_{\mu\nu}\eta_{\rho\sigma}$$

- No need to symmetrise in $(MN)(RS)$ because amplitudes are symmetric.

- The 5-point classical amplitude is given by

$$A_5^{MN} = (8\pi G)^{\frac{3}{2}} \Bigg\{ \frac{8\left(P_1 k P_2^M - P_2 k P_1^M\right)\left(P_1 k P_2^N - P_2 k P_1^N\right)}{q_1^2 q_2^2}$$

$$+8 P_1 P_2 \left[\frac{P_1^M P_1^N \frac{kP_2}{kP_1} - P_1^{(M} P_2^{N)}}{q_2^2} + \frac{P_2^M P_2^N \frac{kP_1}{kP_2} - P_1^{(M} P_2^{N)}}{q_1^2} \right.$$

$$\left. -2 \frac{P_1 k P_2^{(M} q_1^{N)} - P_2 k P_1^{(M} q_1^{N)}}{q_1^2 q_2^2} \right]$$

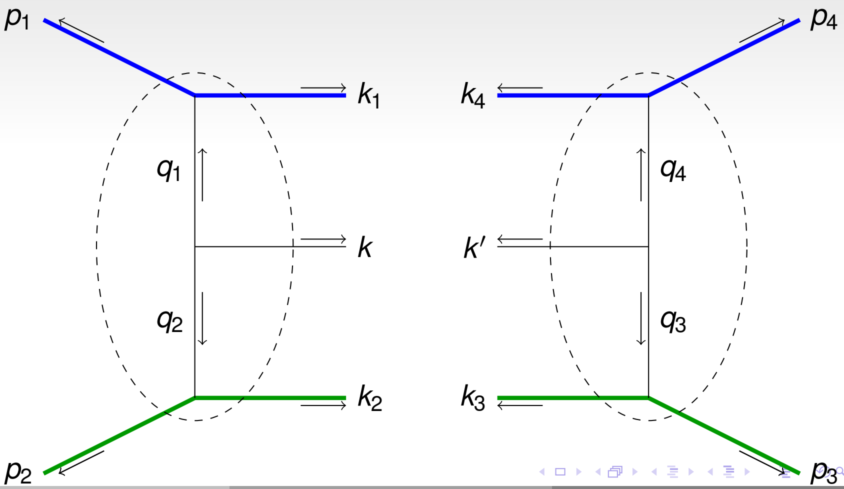
$$+\beta \left[- \frac{P_1^M P_1^N (kq_1)}{(P_1 k)^2 q_2^2} - \frac{P_2^M P_2^N (kq_2)}{(P_2 k)^2 q_1^2} \right.$$

$$\left. +2 \left(\frac{P_1^{(M} q_1^{N)}}{(P_1 k) q_2^2} - \frac{P_2^{(M} q_1^{N)}}{(P_2 k) q_1^2} + \frac{q_1^M q_1^N}{q_1^2 q_2^2} \right) \right] \Bigg\} \; ; \; k_M A_5^{MN} = k_N A_5^{MN} = 0$$

W. Goldberger and A. Ridgeway, 1611.03493
A. Luna, I. Nicholson, D. 'O Connell and C. White, 1711.03901
G. Mogull, J. Plefka and J. Steinhoff, 2010.02865.

- Separate long. and trans. directs wrt the class. dir. of propagation:

$$p_1 = \left(-E_1, \frac{\mathbf{q}}{2}, -\bar{p}\right) \qquad p_4 = \left(E_1, \frac{\mathbf{q}}{2}, \bar{p}\right)$$
$$p_2 = \left(-E_2, -\frac{\mathbf{q}}{2}, \bar{p}\right) \qquad p_3 = \left(E_2, -\frac{\mathbf{q}}{2}, -\bar{p}\right)$$



- In $N = 8$ it is convenient to choose the following 10-dim kinematics:

$$
\begin{aligned}
P_1 &= (p_1; 0,0,0,0,0,m_1) & P_1^2 &= 0 \\
P_2 &= (p_2; 0,0,0,0,m_2,0) & P_2^2 &= 0 \\
K_1 &= (k_1; 0,0,0,0,0,-m_1) & K_1^2 &= 0 \\
K_2 &= (k_2; 0,0,0,0,-m_2,0) & K_2^2 &= 0
\end{aligned}
$$

while in GR all momenta are 4-dim:

$$
\begin{aligned}
P_1 &= (p_1; 0,0,0,0,0,0) & p_1^2 &= -m_1^2 \\
P_2 &= (p_2; 0,0,0,0,0,0) & p_2^2 &= -m_2^2 \\
K_1 &= (k_1; 0,0,0,0,0,0) & k_1^2 &= -m_1^2 \\
K_2 &= (k_2; 0,0,0,0,0,0) & k_2^2 &= -m_2^2
\end{aligned}
$$

and

$$
\beta^{\mathcal{N}=8} = 4m_1^2 m_2^2 \sigma^2 \quad \beta^{GR} = 4m_1^2 m_2^2 \left(\sigma^2 - \frac{1}{D-2}\right) \quad \sigma = -\frac{p_1 p_2}{m_1 m_2}
$$

- Using the momentum conservation δ-functions we can perform the integral over the longitudinal directions of k_1 and k_2.
- The resulting Jacobian cancels the factors $k^0_{1,2}$ and produces an extra factor of $|k_1^0 k_2^L - k_2^0 k_1^L|^{-1}$.
- We are then left with the integrals over q_1 and q_2 only along the remaining $D-2$ transverse directions:

$$[\mathrm{Im}\, 2A_2]_{3pc} = \int \frac{d^{D-2}q_1}{(2\pi)^{D-2}} \int \frac{d^{D-2}q_2}{(2\pi)^{D-2}} \int \frac{d^{D-1}k}{(2\pi)^{D-1} 2k^0}$$
$$\times \frac{(2\pi)^{D-2}\delta^{(D-2)}(k+q_1+q_2)}{4|k_1^0 k_2^L - k_2^0 k_1^L|} A_5^{MN}(P_1, P_2, K_1, K_2, k)$$
$$\times \left[\sum_i \epsilon^{(i)}_{MN} \epsilon^{(i)}_{RS}\right] A_5^{RS}(P_4, P_3, -K_1, -K_2, -k)$$

where we changed variables of integration from $k_{1,2}$ to $q_{1,2}$

$$q_{1,2} = p_{1,2} - k_{1,2} = (q^0_{1,2}, \mathbf{q}_{1,2}, q^L_{1,2}), \qquad k = (k^0, \vec{k}) = (k^0, \mathbf{k}, k^L)$$

- In the classical limit we can safely approximate

$$k_1^L \simeq \bar{p} \simeq p \; ; \; k_2^L \simeq -\bar{p} \simeq -p \; ; \; k_1^0 \simeq E_1 \; ; \; k_2^0 \simeq E_2$$
$$\Longrightarrow 4|k_1^0 k_2^L - k_2^0 k_1^L| \simeq 4Ep \;\; ; \;\; E = E_1 + E_2$$

- In order to treat the two 5-point amplitudes more symmetrically we can introduce q_3 and q_4 such that

$$q_1 + q_4 = P_1 + P_4 = q \;\; ; \;\; q_2 + q_3 = P_2 + P_3 = -q$$

by introducing the two δ-functions:

$$1 = \int d^{D-2} q_4 \, \delta^{(D-2)}(q_1 + q_4 - q) \int d^{D-2} q_3 \, \delta^{(D-2)}(q_3 + q_2 + q)$$

- Then going to impact parameter space we get

$$2\,\mathrm{Im}\,2\delta_2(b,s) = \int \frac{d^{D-2}q}{(2\pi)^{D-2}}\, e^{-ib\cdot q}\, \frac{[\mathrm{Im}\,2A_2]_{3pc}}{4Ep} =$$

$$\int \frac{d^{D-1}k}{(2\pi)^{D-1}2k^0} \sum_i$$

$$\Bigg[\int \frac{d^{D-2}q_1\, d^{D-2}q_2}{(2\pi)^{D-2}} \delta^{(D-2)}(q_1+q_2+k) \frac{e^{-i\frac{b}{2}(q_1-q_2)}}{4Ep}$$

$$\times A_5^{MN}(P_1,P_2,K_1,K_2,k)\epsilon_{MN}^{(i)}\Bigg]$$

$$\times \Bigg[\int \frac{d^{D-2}q_3\, d^{D-2}q_4}{(2\pi)^{D-2}} \delta^{(D-2)}(q_3+q_4-k) \frac{e^{-i\frac{b}{2}(q_4-q_3)}}{4Ep}$$

$$\times A_5^{RS}(P_4,P_3,-K_1,-K_2,-k)\epsilon_{RS}^{(i)}\Bigg]$$

where we have used $q = \frac{1}{2}(q_1 - q_2 + q_4 - q_3)$.

- The previous expression is very powerful because it allows to compute $Im(2\delta_2)$ directly from unitarity without needing to know the complete two-loop amplitude for extracting from it $Im(2\delta_2)$.
- Writing the previous expression in a more compact form we get

$$2\,\mathrm{Im}\,2\delta_2(b,s) = \int \frac{d^3k}{(2\pi)^3 2\omega} \sum_i |\tilde{A}_{5i}(b,\vec{k})|^2$$

  just in terms of the classical tree five-point amplitude in impact parameter space.
- In GR sum over i means a sum over the two graviton polarisations.
- In $\mathcal{N} = 8$ massive sugra is a sum over all massless degrees of freedom (graviton, dilaton...).
- Inserting in the previous relation the double-Regge limit of the 5-point amplitude, ACV90 computed $Im(2\delta_2)$ in the massless case that turned out to be divergent as $\log s$.
- Then, using analyticity and crossing symmetry, ACV90 managed to deduce from it also $Re(2\delta_2)$ that gave a finite deflection angle at high energy.

- We have instead considered the massive case but keeping only the leading divergent term for the momentum of the graviton $k \to 0$.
- In the soft limit for the massless state the five-point amplitude for the graviton drastically simplifies

$$A_5^{\mu\nu} \simeq \kappa$$
$$\times \left[\left(\frac{\bar{p}_1^\mu \bar{p}_1^\nu}{(\bar{p}_1 k)^2} - \frac{\bar{p}_2^\mu \bar{p}_2^\nu}{(\bar{p}_2 k)^2} \right) (qk) - \frac{\bar{p}_1^\mu q^\nu + \bar{p}_1^\nu q^\mu}{(\bar{p}_1 k)} + \frac{\bar{p}_2^\mu q^\nu + \bar{p}_2^\nu q^\mu}{(\bar{p}_2 k)} \right] A_0$$

in terms of a product of a soft factor times the four-point amplitude without the graviton.

- Inserting this simplified amplitude in the previous relation we get for the graviton:

$$(\mathrm{Im}\, 2\delta_2)_{gr}(\sigma, b)$$
$$\simeq -\frac{1}{2\epsilon} \frac{G^3 \beta^2(\sigma)}{\pi b^2 (\sigma^2 - 1)^2} \left[\frac{8 - 5\sigma^2}{3} - \frac{\sigma(3 - 2\sigma^2)}{(\sigma^2 - 1)^{\frac{1}{2}}} \cosh^{-1}(\sigma) \right]$$

- Then, using arguments based again on unitarity, analyticity and crossing symmetry, we argued that the radiation reactions terms should appear in the following combination:

$$\left[1 + \frac{i}{\pi}\left(-\frac{1}{\epsilon} + \log(\sigma^2 - 1)\right)\right] Re(2\delta_2^{(rr)})$$

- More precisely, the two imaginary terms come from the 3-particle cut integrated over ω from zero to $\log(\sigma^2 - 1)$.
- Then real analyticity implies the connection with the real part.
- In this way we extracted $Re(2\delta_2^{(rr)})$ from the divergent part of $Im(2\delta_2)$, finding in GR agreement with T. Damour, 2010.01641.
- For the complete amplitude we have to use again the technique of differential equations and master integrals.

- and we get

$$
\begin{aligned}
\mathrm{Im}\, 2\delta_2^{(gr)} &= \frac{2m_1^2 m_2^2 G^3}{\pi b^2} \frac{(2\sigma^2-1)^2}{(\sigma^2-1)^2} \\
&\times \Bigg\{ -\frac{1}{\epsilon} \left[\frac{8-5\sigma^2}{3} - \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}} \cosh^{-1}(\sigma) \right] \\
&+ \Big(\log(4(\sigma^2-1)) - 3\log(\pi b^2 e^{\gamma_E}) \Big) \\
&\times \left[\frac{8-5\sigma^2}{3} - \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}} \cosh^{-1}(\sigma) \right] \\
&+ (\cosh^{-1}(\sigma))^2 \left[\frac{\sigma\left(3-2\sigma^2\right)}{(\sigma^2-1)^{\frac{1}{2}}} - 2\frac{4\sigma^6-16\sigma^4+9\sigma^2+3}{\left(2\sigma^2-1\right)^2} \right] \\
&+ \cosh^{-1}(\sigma) \left[\frac{\sigma\left(88\sigma^6-240\sigma^4+240\sigma^2-97\right)}{3\left(2\sigma^2-1\right)^2(\sigma^2-1)^{\frac{1}{2}}} \right] \\
&+ \frac{\sigma(3-2\sigma^2)}{(\sigma^2-1)^{\frac{1}{2}}} \mathrm{Li}_2(1-z^2) + \frac{-140\sigma^6+220\sigma^4-127\sigma^2+56}{9\left(2\sigma^2-1\right)^2} \Bigg\}
\end{aligned}
$$

- The divergent term reproduces the one obtained using the leading soft term of the amplitude or using the IR exponentiation in momentum space C. Heissenberg, arXiv:2105.04594.
- The divergent term and the term proportional to $\log(\sigma^2 - 1)$ are related precisely as argued using analyticity and crossing.
- It behaves as $\log s$ at high energy as predicted in ACV90.
- The complete $Re(2\delta_2)$ is then given by

$$\text{Re}\, 2\delta_2^{(gr)} = \frac{4G^3 m_1^2 m_2^2}{b^2} \Bigg\{ \frac{(2\sigma^2-1)^2(8-5\sigma^2)}{6(\sigma^2-1)^2} - \frac{\sigma(14\sigma^2+25)}{3\sqrt{\sigma^2-1}} + \frac{s(12\sigma^4-10\sigma^2+1)}{2m_1 m_2(\sigma^2-1)^{\frac{3}{2}}} + \cosh^{-1}\sigma \times \left[\frac{\sigma(2\sigma^2-1)^2(2\sigma^2-3)}{2(\sigma^2-1)^{\frac{5}{2}}} + \frac{-4\sigma^4+12\sigma^2+3}{\sigma^2-1} \right] \Bigg\}$$

- If we don't integrate over the momentum of the graviton we get the differential spectrum of the number of emitted gravitons according

$$dN_{\rm gr} = \sum_i \left| \tilde{A}_{5,{\rm gr},{\rm i}}(b, \vec{k}) \right|^2 \frac{d^3k}{\hbar(2\pi)^3 2\omega}$$

  that, because of a factor $\frac{1}{\hbar}$, is divergent in the classical limit.
- By multiplying it with $\hbar\omega$ we get the differential spectrum of the energy:

$$dE_{gr} = \hbar\omega dN_{gr} = \frac{1}{2} \sum_i \left| \tilde{A}_{5,{\rm gr},{\rm i}}(b, \vec{k}) \right|^2 \frac{d^3k}{(2\pi)^3}$$

- Integrating over the momentum of the graviton we get the total energy emitted.
- Using the complete classical amplitude we reproduced the results of E. Herrmann, J. Parra-Martinez, M. Ruf, M. Zeng, 2104.03957.

The wave-form for the graviton

- From the previous expression we can extract the wave-form as a function of the frequency and of the angular direction for each massless state:

$$\tilde{A}_{5,i}(b, \vec{k}) = \int \frac{d^{D-2}\Delta}{(2\pi)^{D-2}} \frac{e^{-ib\Delta}}{4Ep} A_{5,i}(P_1, P_2, K_1, K_2, k)$$

where

$$A_{5,i} = A_5^{MN} \epsilon_{MN}^{(i)} \quad ; \quad \Delta \equiv \frac{1}{2}(\mathbf{q}_1 - \mathbf{q}_2) \;\; ; \;\; \mathbf{q}_1 + \mathbf{q}_2 = -\mathbf{k}$$

- The transverse δ-function has been used to perform one of the two integrations.
- In particular, for $i = +, \times$, it will give the two polarisations of the graviton.
- In the following we will discuss in detail the case of the graviton in GR, but the same procedure can be used for other massless states as for instance the dilaton.

- The starting point is the five-point amplitude for graviton emission in the classical limit

$$M_{GR}^{\mu\nu} = (8\pi G_N)^{\frac{3}{2}} \Bigg\{ \beta \Bigg[-\frac{p_1^\mu p_1^\mu (kq_1)}{(p_1 k)^2 q_2^2} - \frac{p_2^\mu p_2^\mu (kq_2)}{(p_2 k)^2 q_1^2}$$
$$+ \frac{p_1^\mu (q_1 - q_2)^\nu + p_1^\nu (q_1 - q_2)^\mu}{2(p_1 k) q_2^2} - \frac{p_2^\mu (q_1 - q_2)^\nu + p_2^\nu (q_1 - q_2)^\mu}{2(p_2 k) q_1^2}$$
$$+ \frac{(q_1 - q_2)^\mu (q_1 - q_2)^\nu}{2 q_1^2 q_2^2} \Bigg] + 8 \frac{\left(p_1 k p_2^\mu - p_2 k p_1^\mu\right)\left(p_1 k p_2^\nu - p_2 k p_1^\nu\right)}{q_1^2 q_2^2}$$
$$+ (2 p_1 p_2) \Bigg(\frac{4 p_1^\mu p_1^\nu \frac{kp_2}{kp_1} - 2(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)}{q_2^2}$$
$$+ \frac{4 p_2^\mu p_2^\nu \frac{kp_1}{kp_2} - 2(p_1^\mu p_2^\nu + p_1^\nu p_2^\mu)}{q_1^2} + \frac{(q_1 - q_2)^\mu \left(-2 p_1 k p_2^\nu + 2 p_2 k p_1^\nu\right)}{q_1^2 q_2^2}$$
$$+ \frac{(q_1 - q_2)^\nu \left(-2 p_1 k p_2^\mu + 2 p_2 k p_1^\mu\right)}{q_1^2 q_2^2} \Bigg) \Bigg\} \; ; \beta = 4 m_1^2 m_2^2 (\sigma^2 - \frac{1}{D-2})$$

- With all outgoing momenta we have

$$p_1^\mu = (-E_1, \frac{\vec{q}}{2}, -\bar{p}) \;\; ; \;\; p_2^\mu = (-E_2, -\frac{\vec{q}}{2}, \bar{p}) \;\; ; \;\; k^\mu = \omega(-1, \vec{k})$$
$$\vec{k} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

- The two polarisations orthogonal to k^μ are

$$e_\phi^\mu = (0, \mathbf{e}_\phi) = (0, \vec{e}_\phi, 0) = (0, -\sin\phi, \cos\phi, 0)$$
$$e_\theta^\mu = (0, \mathbf{e}_\theta) = (0, \cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$$
$$k^\mu e_{\mu\phi} = k^\mu e_{\mu\theta} = 0$$

- We can project A_5 on the two polarisations as follows:

$$A_{5X} = A_5^{\mu\nu} e_{\mu\phi} e_{\nu\theta} \;\; ; \;\; A_{5+} = A_5^{\mu\nu} \frac{1}{2} \left(e_{\mu\theta} e_{\nu\theta} - e_{\mu\phi} e_{\nu\phi} \right)$$

- We need the following equations:

$$q_1^2 = \left(\frac{1}{2}(\mathbf{q_1} - \mathbf{q_2}) + \frac{1}{2}(\mathbf{q_1} + \mathbf{q_2})\right)^2 + c_1^2|\mathbf{k}|^2 = (\Delta - \frac{1}{2}\mathbf{k})^2 + c_1^2|\mathbf{k}|^2$$

$$q_2^2 = \left(-\frac{1}{2}(\mathbf{q_1} - \mathbf{q_2}) + \frac{1}{2}(\mathbf{q_1} + \mathbf{q_2})\right)^2 + c_2^2|\mathbf{k}|^2 = (\Delta + \frac{1}{2}\mathbf{k})^2 + c_2^2|\mathbf{k}|^2$$

- The extra non-transverse term gives a massive propagator

$$\int \frac{d^2\Delta}{(2\pi)^2}\frac{e^{-ib\Delta}}{q_{1,2}^2} = \frac{e^{-ibk/2}}{2\pi}K_0(bc_{1,2}|\mathbf{k}|)$$

  where K_0 is the modified Bessel function.
- We need also

$$\int \frac{d^2\Delta}{(2\pi)^2}e^{-ib\Delta}\frac{kq_1}{q_1^2} = e^{-ikb/2}\frac{\mathbf{k}^2}{2\pi}\left(d_1 K_0(bc_1|\mathbf{k}|) - i\frac{kb}{b|\mathbf{k}|}c_1 K_1(bc_1|\mathbf{k}|)\right)$$

  and the corresponding for $1 \to 2$

- Finally we need also

$$\int \frac{d^2\Delta}{(2\pi)^2} \frac{e^{-ib\Delta}}{q_1^2 q_2^2} = \frac{b}{4\pi} \int_0^1 dx\, e^{i\frac{kb}{2}(1-2x)} \frac{K_1(b|\mathbf{k}|\sqrt{f})}{|\mathbf{k}|\sqrt{f}}$$
$$f = x(1-x) + c_1^2 x + c_2^2 (1-x)$$

- c_1, c_2 and d_1, d_2 are obtained from the conservation of momentum along the time and longitudinal directions:

$$k^0 + q_1^0 + q_2^0 = 0\,, \quad k^L + q_1^L + q_2^L = 0$$

and from the exact mass-shell constraints

$$2p_1 q_1 = q_1^2\,, \qquad 2p_2 q_2 = q_2^2\,, \qquad kq_1 = -kq_2$$

that, in the classical limit, imply:

$$-p_1^0 q_1^0 + p_1^L q_1^L \simeq 0\,, \qquad -p_2^0 q_2^0 + p_2^L q_2^L \simeq 0$$

- Introducing rapidity variables y_1, y_2 and y according to

$$p_1 = \left(-\bar{m}_1 \cosh y_1, \frac{\mathbf{q}}{2}, -\bar{m}_1 \sinh y_1\right)$$
$$p_2 = \left(-\bar{m}_2 \cosh y_2, -\frac{\mathbf{q}}{2}, -\bar{m}_2 \sinh y_2\right)$$
$$k = (|\mathbf{k}| \cosh y, \mathbf{k}, |\mathbf{k}| \sinh y)$$

where

$$y_1 > 0\,, \qquad y_2 < 0 \;\; ; \;\; \bar{m}_{1,2}^2 = m_{1,2}^2 + \frac{\mathbf{q}^2}{4}$$

we get

$$q_1^L = q_1^0 \coth y_1 = |\mathbf{k}| \frac{\sinh y \tanh y_2 - \cosh y}{\tanh y_1 - \tanh y_2}$$
$$q_2^L = q_2^0 \coth y_2 = |\mathbf{k}| \frac{\cosh y - \sinh y \tanh y_1}{\tanh y_1 - \tanh y_2}$$

- We finally arrive at

$$q_1^2 = (\mathbf{q}_1)^2 + c_1^2\mathbf{k}^2 \, , \qquad c_1 = \frac{\cosh(y-y_2)}{\sinh(y_1-y_2)}$$

$$q_2^2 = (\mathbf{q}_2)^2 + c_2^2\mathbf{k}^2 \, , \qquad c_2 = \frac{\cosh(y_1-y)}{\sinh(y_1-y_2)}$$

- We also find

$$kq_1 = \mathbf{k}\mathbf{q}_1 + \mathbf{k}^2 d_1 \, , \qquad d_1 = \frac{\cosh(y-y_2)\sinh(y_1-y)}{\sinh(y_1-y_2)}$$

$$kq_2 = \mathbf{k}\mathbf{q}_2 + \mathbf{k}^2 d_2 \, , \qquad d_2 = \frac{\cosh(y_1-y)\sinh(y-y_2)}{\sinh(y_1-y_2)}$$

where

$$d_1 + d_2 = 1$$

which ensures $k(q_1+q_2) = 0$.

- We can parametrise k^μ also in terms of its frequency and the two angles θ and ϕ in the centre of mass frame:

$$k = \omega(1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

- and write $c_{1,2}$ and $d_{1,2}$ in terms of σ, $\cos\theta$ and $m_{1,2}$

$$|\mathbf{k}|c_1 = \frac{\omega}{\sqrt{s}} \frac{m_2 + m_1\sigma + m_1\sqrt{\sigma^2-1}\cos\theta}{\sqrt{\sigma^2-1}} \,,$$

$$|\mathbf{k}|c_2 = \frac{\omega}{\sqrt{s}} \frac{m_1 + m_2\sigma - m_2\sqrt{\sigma^2-1}\cos\theta}{\sqrt{\sigma^2-1}}$$

$$\mathbf{k}^2 d_1 = \omega^2 \frac{\left(m_2 + m_1\sigma + m_1\cos\theta\sqrt{\sigma^2-1}\right)\left(m_2\sqrt{\sigma^2-1} - (m_1 + m_2\sigma\right.}{s\sqrt{\sigma^2-1}}$$

$$\mathbf{k}^2 d_2 = \omega^2 \frac{(m_1 + m_2\sigma - m_2\cos\theta\sqrt{\sigma^2-1})(m_1\sqrt{\sigma^2-1} + (m_2 + m_1\sigma)}{s\sqrt{\sigma^2-1}}$$

- The previous equations follow from

$$\sinh y_1 = \frac{m_2}{\sqrt{s}}\sqrt{\sigma^2 - 1}\ ;\ \sinh y_2 = -\frac{m_1}{\sqrt{s}}\sqrt{\sigma^2 - 1}\ ;\ \sinh y = \frac{\cos\theta}{\sin\theta}$$

- They are obtained noting that

$$\sigma \simeq \frac{p_1^0 p_2^0 - p_1^L p_2^L}{\bar{m}_1 \bar{m}_2} = \cosh(y_1 - y_2)$$

$$0 = p_1^L + p_2^L \simeq m_1 \sinh y_1 + m_2 \sinh y_2$$

$$\cos\theta = \frac{k^L}{k^0} = \tanh y$$

$$\omega = k^0 = |\mathbf{k}| \cosh y$$

We arrive at the final expression in the frequency domain:

$$\tilde{A}_{5X} = \frac{(8\pi G_N)^{\frac{3}{2}}}{4pE(2\pi)} i(\hat{b}e_\phi)\Bigg\{ \left(4p_1p_2pE\omega\sin\theta + \beta\sin\theta\frac{q_1^L - q_2^L}{2}\right)$$

$$\times \int_0^1 dx\, e^{i\frac{kb}{2}(1-2x)} bK_0(b|\mathbf{k}|\sqrt{f})$$

$$+\beta\Bigg[-\frac{m_2\sin\theta}{\sqrt{s}} e^{ibk/2} K_1(bc_2|\mathbf{k}|) - \frac{m_1\sin\theta}{\sqrt{s}} e^{-ibk/2} K_1(bc_1|\mathbf{k}|)$$

$$+\int_0^1 dx\, e^{i\frac{kb}{2}(1-2x)}$$

$$\times i\left((\hat{b}e_\theta)(b\sqrt{f}|\mathbf{k}|)K_1(b|\mathbf{k}|\sqrt{f}) + i(x - \frac{1}{2})(\mathbf{k}e_\theta)bK_0(b|\mathbf{k}|\sqrt{f})\right)\Bigg]\Bigg\}$$

$$\textit{where } \frac{q_1^L - q_2^L}{2} = -\frac{\omega}{s\sqrt{\sigma^2 - 1}}$$

$$\times \left[(m_1 + m_2\sigma)(m_2 + m_1\sigma) + \frac{m_1^2 - m_2^2}{2}\sqrt{\sigma^2 - 1}\cos\theta \right]$$

- Integrating over the frequency ω one can write the previous amplitude in time-domain.
- One needs the following integral over the Bessel function:

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-iu\omega} K_0(\omega) = \frac{1}{2(1+u^2)^{\frac{1}{2}}}$$

- From it, using that $K_1(x) = -\frac{dK_0(x)}{dx}$, we can derive the others that we need

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega u} K_1(\omega) = \frac{-iu}{2(1+u^2)^{\frac{1}{2}}}$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega u} \omega K_0(\omega) = -\frac{i}{2} \frac{u}{(1+u^2)^{\frac{3}{2}}}$$

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} e^{-i\omega u} \omega K_1(\omega) = -\frac{1}{2(1+u^2)^{\frac{3}{2}}}$$

$$
\int_{-\infty}^{+\infty} \frac{d\omega e^{-i\omega u}}{2\pi} \tilde{A}_{5X} = -\frac{(8\pi G_N)^{\frac{3}{2}}}{4pE(2\pi)} (\hat{b}e_\phi) \Bigg\{ \frac{1}{2} \sin\theta \Bigg(4p_1 p_2 pE
+\beta \frac{q_1^L - q_2^L}{2\omega} \Bigg) \frac{(-1)}{b} \int_0^1 dx \frac{\frac{u}{b} + (\hat{b}\hat{k})(x - \frac{1}{2})}{\left[f \sin^2\theta + \left(\frac{u}{b} + (x - \frac{1}{2})(\hat{k}\hat{b}) \right)^2 \right]^{\frac{3}{2}}}
$$

$$
+\beta \left[\frac{\sin\theta}{2\sqrt{s}} \sum_{i=1,2} \left(m_i \frac{u - (-1)^i (b\hat{k})/2}{(\sin\theta b c_i)^2} \left[1 + \left(\frac{u - (-1)^i \frac{1}{2} (\hat{k} b)}{\sin\theta c_i b} \right)^2 \right]^{-\frac{1}{2}} \right)
$$

$$
+ \frac{(\hat{b}e_\theta)}{2b} \sin^2\theta \int_0^1 dx \frac{(c_1^2 - c_2^2)x + c_2^2 + \frac{1}{4}}{\left[f \sin^2\theta + \left(\frac{u}{b} + (\hat{k}\hat{b})(x - \frac{1}{2}) \right)^2 \right]^{\frac{3}{2}}}
$$

$$
+ \frac{u}{2b^2} \sin\theta \cos\theta \int_0^1 dx \frac{(x - \frac{1}{2})}{\left[f \sin^2\theta + \left(\frac{u}{b} + (x - \frac{1}{2})(\hat{k}\hat{b}) \right)^2 \right]^{\frac{3}{2}}} \Bigg] \Bigg\}
$$

- It remains to perform the integrals over x. We can use the following general expression:

$$\int_0^1 \frac{dx(x+a)}{[x^2+bx+c]^{\frac{3}{2}}} = \frac{a[2b\sqrt{b+c+1}-2(b+2)\sqrt{c}]+b+2c)-4c\sqrt{b+c+1}}{\sqrt{c}(b^2-4c)\sqrt{b+c+1}}$$

- For $u \to \pm\infty$ the leading term of the wave-form $\times$ in time domain goes as a constant and the next to the leading term goes as $\frac{1}{u^2}$.
- The constant term is the memory term that, in frequency domain, comes entirely from the leading soft term for $\omega \to 0$.
- There is no term that goes as $\frac{1}{u}$.

- We need to compare our result with that of other authors who computed the wave-form in the rest frame of one of the particles: G.U. Jakobsen,G. Mogull, J. Steinhoff and J. Plefka, 2101.12688, S. Mougiakakos, M.M. Riva and F. Vernizzi, 2102.08339.
- We have to find out how to go from our centre of mass system to their frame.
- We managed to do that for the memory term finding agreement with the first paper above.
- We can also compare with the papers of A. Sen and collaborators (see A. Sen's talk and References therein).
- The term that goes as a constant is the memory term that reproduces the term $A_{\mu\nu}$ of Sen's talk, while there is no contribution to $B_{\mu\nu}$ and $C_{\mu\nu}$ from the wave-form $\times$.

- Our expression for the memory term can be obtained from $A_{\mu\nu}$ by restricting to the case of two-particle elastic scattering and choosing the momenta as follows:

$$(p_1')^\mu = \bar{p}_1^\mu - \frac{Q^\mu}{2} = -p_1(our) \;\; ; \;\; p_1^\mu = \bar{p}_1^\mu + \frac{Q^\mu}{2} = p_4(our)$$
$$(p_2')^\mu = \bar{p}_1^\mu + \frac{Q^\mu}{2} = -p_2(our) \;\; ; \;\; p_2^\mu = \bar{p}_1^\mu - \frac{Q^\mu}{2} = p_3(our)$$

where

$$Q^\mu = -\frac{\partial(2\delta_0)}{\partial b_\mu} = -\frac{2Gm_1 m_2}{b}\frac{2\sigma^2 - 1}{\sqrt{\sigma^2 - 1}}\hat{b}^\mu$$

Inspiration from Bloch-Nordsieck

- At 3PM the eikonal is not a phase anymore: it gets an imaginary part.
- This follows from the the fact that there is the contribution of the 3-particle cut with the emission of a graviton.
- Although the elastic amplitude has no graviton external state, its imaginary part knows of their existence.
- This strongly asks us to include graviton degrees of freedom in the eikonal.
- Let us see how this problem is solved in the Bloch-Nordsieck model that describes the emission of soft photons from an elastic scattering of two massive particles.
- The photons are soft and therefore their emission does not modify the elastic process.
- I will consider for simplicity the case of a photon, but everything can be extended to gravitons.

- Let us work in the Coulomb gauge where $A_0 = 0$ and $\partial_a A^a = 0$.
- The most general solution of the eq. of motion in the free case $-\partial^2 A^a = 0$ is given by

$$A^a(x,t) = \sum_{i=1}^{2} \int \frac{d^3k}{2k^0(2\pi)^3} \left[a_i(k) e_i^a e^{ikx} + a_i^\dagger(k) e_i^a e^{-ikx} \right]$$

  where a is a space index
- If we have an external current the eq. of motion of A^a is modified to

$$-\partial^2 A^a = j^a \; ; \quad \lim_{t \to \pm\infty} j^a = 0 \Longrightarrow \vec{A}(x) = \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{e}^{ikx} \vec{j}(k)}{\vec{k}^2 - (k^0)^2}$$

- Its solution depends on the choice of boundary conditions.

- In terms of a solution we can introduce the in and out fields:

$$\vec{A}^{in}(x) = \lim_{t \to -\infty} \vec{A}(x) \;\; ; \;\; \vec{A}^{out}(x) = \lim_{t \to +\infty} \vec{A}(x)$$

- We can write our solution in terms of the in and retarded solutions:

$$\vec{A}(x) = \vec{A}^{in}(x) + \vec{A}^{ret}(x) \;\; ; \;\; \lim_{t \to -\infty} \vec{A}^{ret}(x) = 0$$

- The retarded solution is obtained by moving the integration region over k^0 a bit up with respect to the two poles and one gets:

$$\vec{A}^{ret}(x) = i \sum_{i=1}^{2} \int \frac{d^3k}{2\omega(2\pi)^3} \left(\mathrm{e}^{i(\vec{k}\vec{x} - \omega t)} \vec{j}(\vec{k}, \omega) - \mathrm{e}^{-i(\vec{k}\vec{x} - \omega t)} \vec{j}^*(\vec{k}, \omega) \right)$$

- For $t \to \infty$ one gets:

$$\vec{A}^{out}(x) = \vec{A}^{in}(x) + \vec{A}^{ret}(x)$$

- In conclusion, one gets the out field just by adding to the in field a c-number function.
- The S-matrix is then the quantity that satisfies the following equation:

$$\vec{A}^{out}(x) \equiv S^\dagger \vec{A}^{in}(x) S = \vec{A}^{in}(x) + \vec{A}^{ret}(x)$$

- Its solution is:

$$S = \exp\left\{ i \sum_{i=1}^{2} \int \frac{d^3k}{2k^0(2\pi)^3} \left[a_i(k) j_i(k) + a_i^\dagger(k) j_i^*(k) \right] \right\}$$

that is unitary: $SS^\dagger = 1$.

- Using the commutation relations:

$$[a_i(k), a_j^\dagger(p)] = 2k^0(2\pi)^3\delta(\vec{k}-\vec{p})\delta_{ij}$$

it can be rewritten as follows

$$S = \exp\left\{ i\sum_{i=1}^{2}\int \frac{d^3k}{2k^0(2\pi)^3} a_i^\dagger(k) j_i^*(k)\right\}$$
$$\times \exp\left\{ -i\sum_{i=1}^{2}\int \frac{d^3k}{2k^0(2\pi)^3} a_i(k) j_i(k)\right\}$$
$$\times \exp\left\{ -\frac{1}{2}\sum_{i=1}^{2}\int \frac{d^3k}{2k^0(2\pi)^3} |j_i(k)|^2\right\}$$

- Starting from a unitary S-matrix and normal ordering the harmonic oscillators we got a term that has exactly the same form as the imaginary part of the eikonal at 3PM!

- We can compute the probability that in the elastic scattering no photon is emitted:

$$p_0 \equiv |{}_{in}\langle 0|S|0\rangle_{in}|^2 = \exp\left\{ -\sum_{i=1}^{2} \int \frac{d^3k}{2k^0(2\pi)^3} |j_i(k)|^2 \right\}$$

- The probability that one photon is emitted:

$$p_1 = \left[\sum_{i=1}^{2} \int \frac{d^3k}{2k^0(2\pi)^3} |j_i(k)|^2\right] \exp\left\{ -\sum_{i=1}^{2} \int \frac{d^3k}{2k^0(2\pi)^3} |j_i(k)|^2 \right\}$$

- The probability that n photons are emitted:

$$p_n = \frac{1}{n!} \left[\sum_{i=1}^{2} \int \frac{d^3k}{2k^0(2\pi)^3} |j_i(k)|^2\right]^n \exp\left\{ -\sum_{i=1}^{2} \int \frac{d^3k}{2k^0(2\pi)^3} |j_i(k)|^2 \right\}$$

- It is easy to check that

$$\sum_{n=0}^{n} p_n = 1$$

- We get a Poisson distribution:

$$p_n = \frac{\bar{n}}{n!} e^{-\bar{n}} \;\; ; \;\; \bar{n} = \sum_{i=1}^{2} \int \frac{d^3k}{2k^0(2\pi)^3} |j_i(k)|^2$$

  where $\bar{n}$ is the average number of photons emitted.
- All this is valid if the photons are soft and therefore they do not influence the elastic scattering.
- If they start to have enough momentum then they modify the elastic scattering and one cannot treat anymore separately the photons and the other particles.
- How is then the S-matrix modified?
- Going back to our problem: How is the eikonal modified to include the degrees of freedom describing the gravitons when they start to be not so soft?

Conclusions and outlook

- By now at 3PM we have a very good control of the classical amplitude and of the classical observables.
- We would like to understand in a more physical way the contribution of the radiation reaction.
- In particular, why the approaches based on the loss of angular momentum and that based on soft behaviour give the same result for the deflection angle.
- We have to finish the computation of the wave-form and compare with the results by other authors.
- The next step is definitively 4PM.
- We have already the conservative part: Z. Bern et al, 2101.07254.

- How do we get the rest?
- Do we have to go the hard way or can we find some shortcut?