Classical Gravity From Loop Amplitudes

Based on [2104.04510] and [2105.05218], in collaboration with NEJ. BJERRUM-BOHR, PH. DAMGAARD and P. VANHOVE

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18/05/2021 - Gravitational scattering, inspiral, and radiation (workshop)

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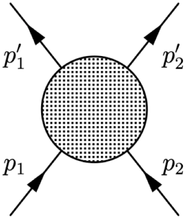
Motivations

Main objectives of this talk :

- Show how the scattering amplitude computation can be simplified, by assembling the integrands, so that
  1. delta functions ("velocity cuts") appear, reducing the dimensionality of the integrals ;
  2. we can clearly separate the iterations of sub-loop integrals that are cancelled in the matching procedure vs the integrals really relevant for the 3PM observables ;
- Point what is exactly the difference between the potential and the soft region computations ;
- Identify how we can make contact between the amplitude and the world-line formalism and how the eikonal matching procedure works at the integral level.

Definitions (1/2)

2-body interaction in both supergravity and Einstein gravity :



and

$$s \equiv (p_1 + p_2)^2 = (p_1' + p_2')^2 = m_1^2 + m_2^2 + 2m_1 m_2 \sigma, \quad \sigma \equiv \frac{p_1 \cdot p_2}{m_1 m_2},$$

$$t \equiv (p_1 - p_1')^2 = (p_2' - p_2)^2 \equiv q^2 = -\vec{q}^{\,2},$$

$$u \equiv (p_1 - p_2')^2 = (p_1' - p_2)^2,$$

Soft expansion (small q expansion and Laurent expansion in $\hbar$) of the Feynman integrals implies a classification of the different corrections in order of classicality.

Definitions (2/2)

Going from q-space to b-space :

$$\widetilde{\mathcal{M}}_2(\sigma,b)=\frac{1}{4E_{\text{c.m.}}P}\int_{\mathbb{R}^{D-2}}\frac{d^{D-2}\vec{q}}{(2\pi)^{D-2}}\mathcal{M}_2(p_1,p_2,p_1',p_2')e^{i\vec{q}\cdot\vec{b}},$$

where $4E_{\text{c.m.}}P=4m_1m_2\sqrt{\sigma^2-1}$ and $E_{\text{c.m.}}=\sqrt{s}$.

Eikonal in b-space (see also Carlo Heissenberg talk) :

$$1+i\sum_{L\geq 0}\widetilde{\mathcal{M}}_L(\sigma,b)=(1+2i\Delta(\sigma,b))\exp\left(\frac{2i}{\hbar}\sum_{L\geq 0}\delta_L(\sigma,b)\right).$$

$$\tilde{\mathcal{M}}_0^{(-1)}=\frac{2}{\hbar}\delta_0 \tag{1}$$

$$\tilde{\mathcal{M}}_1^{(-2)}=\frac{i}{2}(\tilde{\mathcal{M}}_0^{(-1)})^2,\ \tilde{\mathcal{M}}_1^{(-1)}=\frac{2}{\hbar}\delta_1,\ \tilde{\mathcal{M}}_1^{(0)}=2\Delta_1 \tag{2}$$

$$\tilde{\mathcal{M}}_2^{(-3)}=-\frac{1}{3!}(\tilde{\mathcal{M}}_0^{(-1)})^3,\ \tilde{\mathcal{M}}_2^{(-2)}=i\tilde{\mathcal{M}}_0^{(-1)}\tilde{\mathcal{M}}_1^{(-1)},\ \tilde{\mathcal{M}}_2^{(-1)}=i\tilde{\mathcal{M}}_0^{(-1)}\tilde{\mathcal{M}}_1^{(0)}+\frac{2}{\hbar}\delta_2 \tag{3}$$

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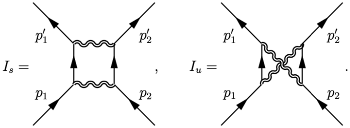
The box integral

$$\mathcal{M}_{\square} = c_{\square}(I_s + I_u) + \mathcal{O}(|\vec{q}|^2 I_s, |\vec{q}|^2 I_u) \tag{4}$$



To perform the soft expansion we rescale $l = |\vec{q}|l$, $q = |\vec{q}|u_q$ and $|\vec{q}| = \hbar|\underline{\vec{q}}|$

$$\mathcal{M}_{\square} = \frac{c_{\square}|\underline{\vec{q}}|^{D-6}}{8\hbar^2}\int \frac{d^D l}{(2\pi)^D}\left(\frac{1}{p_1'.l - i\epsilon} - \frac{1}{p_1.l + i\epsilon}\right)\left(\frac{1}{p_2.l - i\epsilon} - \frac{1}{p_2'.l + i\epsilon}\right)\frac{1}{l^2(l+u_q)^2} + \mathcal{O}(1) \tag{5}$$

$$p_1 = \bar{p}_1 + \frac{\hbar|\underline{\vec{q}}|}{2}u_q,\ p_1' = \bar{p}_1 - \frac{\hbar|\underline{\vec{q}}|}{2}u_q,\ p_2 = \bar{p}_2 - \frac{\hbar|\underline{\vec{q}}|}{2}u_q,\ p_2' = \bar{p}_2 + \frac{\hbar|\underline{\vec{q}}|}{2}u_q.$$

$$\mathcal{M}_{\square} = -\frac{c_{\square}|\underline{\vec{q}}|^{D-6}}{8\hbar^2}\int \frac{d^D l}{(2\pi)^{D-2}}\frac{\delta(\bar{p}_1.l)\delta(\bar{p}_2.l)}{l^2(l+u_q)^2}$$
$$+ \frac{ic_{\square}|\underline{\vec{q}}|^{D-5}}{16\hbar}\int \frac{d^D l}{(2\pi)^{D-1}}\left(\frac{\delta(\bar{p}_2.l)}{(\bar{p}_1.l)^2} + \frac{\delta(\bar{p}_1.l)}{(\bar{p}_2.l)^2}\right)\frac{1}{l^2(l+u_q)^2} + \mathcal{O}(1) \tag{6}$$

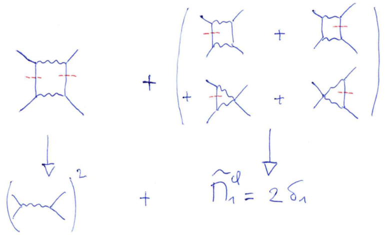
The one-loop amplitude

$$\mathcal{M}_1 = \frac{c_{SC}}{\hbar^2}\int \delta(\bar{p}_1.l)\delta(\bar{p}_2.l) + \frac{c_C}{\hbar}\left(\int \delta(\bar{p}_1.l) + \int \delta(\bar{p}_2.l)\right) + \mathcal{O}(|\vec{q}|^{D-4}) \quad (7)$$

identification in b-space (after Fourier transform) :

$$\tilde{\mathcal{M}}_1 = \frac{i}{2}(\tilde{\mathcal{M}}_0)^2 + \tilde{\mathcal{M}}_1^{Cl} = \frac{i}{2}(\tilde{\mathcal{M}}_0)^2 + \frac{2}{\hbar}\delta_1 \quad (8)$$

interpretation with velocity cuts :



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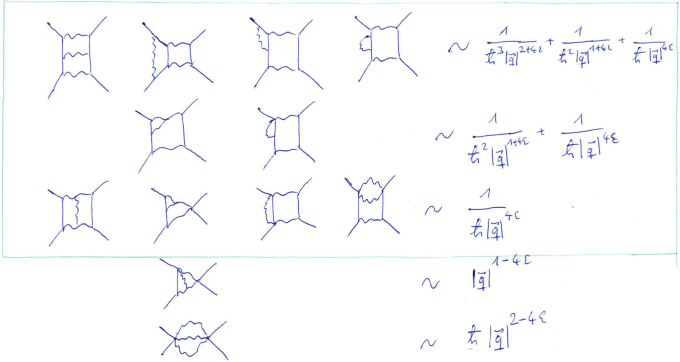
The two loop topologies

Laurent expansion in $\hbar$:



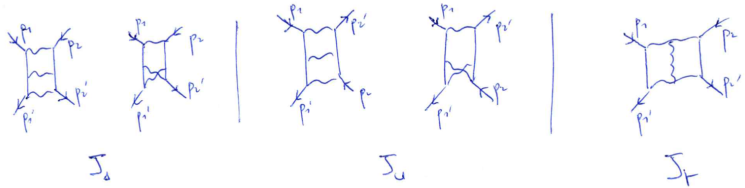
Amplitude decomposition :

$$\mathcal{M}_2 = \mathcal{M}_2^{\square\square} + \mathcal{M}_2^{\triangleleft\square} + \mathcal{M}_2^{\square\triangleright} + \mathcal{M}_2^{\triangleright\triangleright} + \mathcal{M}_2^{\triangleleft\triangleleft} + \mathcal{M}_2^{H} + \mathcal{M}_2^{\square\circ} + \mathcal{M}_2^{SE} \qquad (9)$$

Exploiting the symmetry of the integrand - the $N = 8$ supergravity case

$$\mathcal{M}^{2-\text{loop}}(p_1, p_2, p_1', p_2') = (8\pi G_N)^3 \left(4m_1^2 m_2^2 \sigma^2 (J_s + J_u) + 2\hbar^2 m_1 m_2 |\underline{\vec{q}}|^2 \sigma J_u + \hbar^4 |\underline{q}|^4 J_t\right)$$

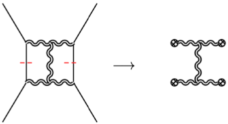
We only need to compute (i) J_t and J_u at the first order in $|\vec{q}|$, (ii) $J_s + J_u$ up to the third order in $|\vec{q}|$.



J_t integral

$$J_t = \frac{-|\underline{q}|^{2D-12}}{2\hbar^5}\int \frac{d^D l_1 d^D l_2}{(2\pi)^{2D}}\left(\frac{1}{2p_1\cdot l_1 + i\varepsilon} - \frac{1}{2p_1'\cdot l_1 - i\varepsilon}\right)\left(\frac{1}{2p_2\cdot l_2 - i\varepsilon} - \frac{1}{2p_2'\cdot l_2 + i\varepsilon}\right)$$
$$\times \frac{1}{((l_1+u_q)^2+i\varepsilon)((l_2+u_q)^2+i\varepsilon)(l_1^2+i\varepsilon)(l_2^2+i\varepsilon)((l_1+l_2+u_q)^2+i\varepsilon)}.$$

contributes at leading order in $|\vec{q}|$, each matter line being reduced to two external sources :



$$J_t = -\frac{|\vec{q}|^{2D-12}}{8\hbar^5}\int \frac{\delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_2)}{l_1^2 l_2^2 (l_1-u_q)^2(l_2+u_q)^2(l_1+l_2)^2}$$

Final result :

$$J_t = -\frac{1}{256 m_1 m_2 \hbar^5 \pi^3 \epsilon}\frac{(4\pi e^{-\gamma_E})^{2\epsilon}}{|\underline{q}|^{4+4\epsilon}}\frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}$$
$$\times\left(\pi + i\left(\frac{-1}{4(\sigma^2-1)}\right)^{\epsilon}\operatorname{arccosh}(\sigma) + \mathcal{O}(\epsilon)\right) + O(\hbar|\underline{q}|).$$

$J_s + J_u$ integral

The $J_s + J_u$ integral

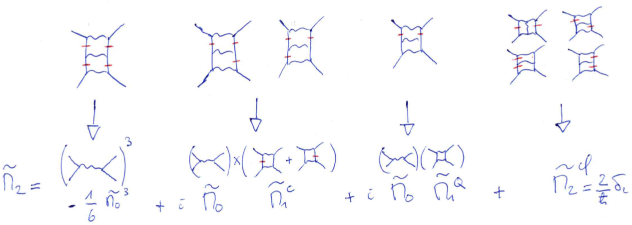
$$J_s + J_u = \frac{|q|^{2D-10}}{96\hbar^3} \int \frac{d^D l_1 d^D l_2}{(2\pi)^{2D}} \frac{1}{(l_1^2 + i\varepsilon)(l_2^2 + i\varepsilon)((l_1 + l_2 + u_q)^2 + i\varepsilon)}$$
$$\sum_{1\leq i\neq j\leq 3} \sum_{1\leq k\neq n\leq 3} \frac{1}{\left(\bar{p_1}\cdot l_i + |\vec{q}|\frac{u_q\cdot l_i}{2} + i\epsilon\right)\left(\bar{p_1}\cdot l_j - |\vec{q}|\frac{u_q\cdot l_j}{2} - i\epsilon\right)}$$
$$\times \frac{1}{\left(\bar{p_2}\cdot l_k - |\vec{q}|\frac{u_q\cdot l_k}{2} - i\epsilon\right)\left(\bar{p_2}\cdot l_n + |\vec{q}|\frac{u_q\cdot l_n}{2} + i\epsilon\right)}.$$

Expansion in $|\vec{q}|$, and identification of delta functions gives

$$\begin{aligned} J_s + J_u = {} & \frac{|\vec{q}|^{2D-10}}{\hbar^3} \int \delta(\bar{p}_1.l_1)\delta(\bar{p}_1.l_2)\delta(\bar{p}_2.l_1)\delta(\bar{p}_2.l_2) \\ & + \frac{|\vec{q}|^{2D-9}}{\hbar^2}\left(\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_1)\delta(\bar{p}_1.l_2) + \int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_1)\delta(\bar{p}_2.l_2)\right) \\ & + \frac{|\vec{q}|^{2D-8}}{\hbar}\left(\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_1) + \int \delta(\bar{p}_1.l_1)\delta(\bar{p}_1.l_2) + \int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_2) + \int \delta(\bar{p}_2.l_1)\delta(\bar{p}_2.l_2)\right) \end{aligned} \quad (10)$$

Velocity cuts and eikonal matching in b-space

One-to-one match between velocity cuts and the eikonal matching decomposition :



$$\tilde{\mathcal{M}}_2^{(-3)} = \frac{1}{\hbar^3}\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_1.l_2)\delta(\bar{p}_2.l_1)\delta(\bar{p}_2.l_2) = -\frac{1}{3!}(\tilde{\mathcal{M}}_0^{(-1)})^3$$

$$\tilde{\mathcal{M}}_2^{(-2)} = \frac{1}{\hbar^2}\left(\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_1)\delta(\bar{p}_1.l_2) + \int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_1)\delta(\bar{p}_2.l_2)\right) = i\tilde{\mathcal{M}}_0^{(-1)}\tilde{\mathcal{M}}_1^{(-1)}$$

$$\frac{1}{\hbar}\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_1) = i\tilde{\mathcal{M}}_0^{(-1)}\tilde{\mathcal{M}}_1^{(0)} = 2i\tilde{\mathcal{M}}_0^{(-1)}\Delta_1$$

$$\frac{1}{\hbar}\left(\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_1.l_2) + \int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_2) + \int \delta(\bar{p}_2.l_1)\delta(\bar{p}_2.l_2)\right) = \frac{2}{\hbar}\delta_2 \quad (11)$$

The final value for the deviation angle

Deviation angle at 3PM in $N = 8$:

$$\chi_{3PM} = -\frac{16m_1^3 m_2^3 G_N^3 \sigma^6}{3J^3(\sigma^2-1)^{\frac{3}{2}}} - \frac{32m_1^4 m_2^4 \sigma^4 G_N^3 \operatorname{arccosh}(\sigma)}{J^3 s} + \frac{32m_1^4 m_2^4 G_N^3 \sigma^4}{J^3 s(4(\sigma^2-1))^{\epsilon}}\left(\frac{\sigma(\sigma^2-2)\operatorname{arccosh}(\sigma)}{(\sigma^2-1)^{\frac{3}{2}}} + \frac{\sigma^2}{\sigma^2-1}\right) \quad (12)$$

where

$$\chi_{3PM}^{Rad} = \frac{4m_1^4 m_2^4 G_N^3 (2\sigma^2)^2}{J^3 s(4(\sigma^2-1))^{\epsilon}} \frac{d}{d\sigma}\left(\frac{2\sigma^2 \operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right) \quad (13)$$

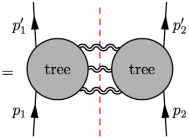
Match with Di Vecchia, Heissenberg, Russo and Veneziano.

The case of Einstein gravity

Natural question : Does the method work in Einstein gravity ? The answer is yes !

Main piece is the 3-graviton cut (see also Radu Roiban talk)

$\mathcal{M}^2_{3-\mathrm{cut}}(\sigma, q^2) =$



The integrand can then be decomposed as

$$\mathcal{M}_2^{3-cut} = \mathcal{M}_2^{\Box\Box} + \mathcal{M}_2^{\triangleleft\Box} + \mathcal{M}_2^{\Box\triangleright} + \mathcal{M}_2^{\triangleright\triangleright} + \mathcal{M}_2^{\triangleleft\triangleleft} + \mathcal{M}_2^{H} + \mathcal{M}_2^{\Box\circ} \quad (14)$$

The double box contribution

$$\mathcal{M}_2^{\square\square}(\sigma, q^2) = 4096\pi^3 G_N^3 m_1^5 m_2^5 (2\sigma^2-1)^2 \Big(m_1 m_2 (2\sigma^2-1)(J_s + J_u) - 6\sigma\hbar^2 |\underline{q}|^2 J_u + 8\sigma\hbar^2 |\underline{q}|^2 J_{\square\square}^{NP} \Big).$$

Only one additional integral compared to $N = 8$, sum of non planar integrals, also written in terms of delta functions

$$J_{\square\square}^{NP} = -\frac{|\underline{q}|^{2D-10}}{16\hbar^3} \int \frac{d^D l_1 d^D l_2}{(2\pi)^{2D-2}} \frac{l_2 \cdot l_3}{l_1^2 l_2^2 (l_1 + l_2 + u_q)^2} \left(\frac{\delta(\bar{p}_1 \cdot l_1)}{\bar{p}_1 \cdot l_2 + i\varepsilon} - \frac{\delta(\bar{p}_1 \cdot l_2)}{\bar{p}_1 \cdot l_1 + i\varepsilon} \right) \times \left(\frac{\delta(\bar{p}_2 \cdot l_3)}{\bar{p}_2 \cdot l_1 + i\varepsilon} - \frac{\delta(\bar{p}_2 \cdot l_1)}{\bar{p}_2 \cdot l_3 + i\varepsilon} \right).$$

Velocity cuts and eikonal matching in b-space - double box

As in $N = 8$, one-to-one match between the velocity cuts insertion and the eikonal matching procedure :

$$\tilde{\mathcal{M}}_2^{\square\square(-3)} = \frac{1}{\hbar^3}\left(\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_1.l_2)\delta(\bar{p}_2.l_1)\delta(\bar{p}_2.l_2)\right) = -\frac{1}{3!}(\tilde{\mathcal{M}}_0^{(-1)})^3$$

$$\tilde{\mathcal{M}}_2^{\square\square(-2)} = \frac{1}{\hbar^2}\left(\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_1.l_2)\delta(\bar{p}_2.l_2) + \int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_1)\delta(\bar{p}_2.l_2)\right) = i\tilde{\mathcal{M}}_0^{(-1)}\tilde{\mathcal{M}}_1^{\square(-1)}$$

$$\tilde{\mathcal{M}}_2^{\square\square(-1)} = \frac{1}{\hbar}\left(\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_1)\right) + \frac{1}{\hbar}\left(\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_2) + \int \delta(\bar{p}_1.l_1)\delta(\bar{p}_1.l_2) + \int \delta(\bar{p}_2.l_1)\delta(\bar{p}_2.l_2)\right)$$

$$= i\tilde{\mathcal{M}}_0\tilde{\mathcal{M}}_1^{\square(0)} - \frac{2i(2\sigma^2-1)^2 G_N^3 m_1^2 m_2^2 (\pi b^2 e^{\gamma_E})^{3\epsilon}}{\hbar\epsilon\pi b^2(\sigma^2-1)}\left(\frac{-1}{4(\sigma^2-1)}\right)^{\epsilon}\frac{d}{d\sigma}\left(\frac{(2\sigma^2-1)\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)$$

$$= 2i\tilde{\mathcal{M}}_0^{(-1)}\Delta_1^{\square} + \frac{2}{\hbar}\delta_2^{\square\square} \tag{15}$$

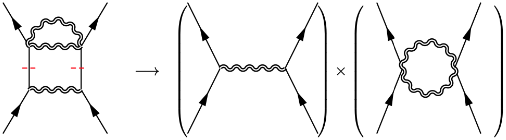
The box triangle and the box bubble topologies

Same thing for the box triangle and the box bubble topologies :

$$\tilde{\mathcal{M}}_2^{\Box\triangleright(-2)} = \frac{1}{\hbar^2}\Big(\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_1.l_2)\delta(\bar{p}_2.l_2)\Big) = i\tilde{\mathcal{M}}_0^{(-1)}\tilde{\mathcal{M}}_1^{\triangleright(-1)} \quad (16)$$

$$\tilde{\mathcal{M}}_2^{\Box\triangleright(-1)} = \frac{1}{\hbar}\Big(\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_1)\Big) + \frac{1}{\hbar}\Big(\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_2) + \int \delta(\bar{p}_1.l_1)\delta(\bar{p}_1.l_2)\Big)$$
$$= i\tilde{\mathcal{M}}_0^{(-1)}\tilde{\mathcal{M}}_1^{\triangleright(0)} + \frac{G_N^3 m_1 m_2 (\pi b^2 e^{\gamma_E})^{3\epsilon}}{\hbar b^2\sqrt{\sigma^2-1}}\Big(\frac{3(2\sigma^2-1)(1-5\sigma^2)(m_1+m_2\sigma)}{2(\sigma^2-1)} + \frac{(2\sigma^2-1)m_1}{2} + \frac{m_2\sigma(55+2\sigma^2)}{6}\Big)$$
$$= 2i\tilde{\mathcal{M}}_0^{(-1)}\Delta_1^{\triangleright} + \frac{2}{\hbar}\delta_2^{\Box\triangleright} \quad (17)$$

$$\tilde{\mathcal{M}}_2^{\Box\circ(-1)} = \frac{1}{\hbar}\Big(\int \delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_1)\Big) = i\tilde{\mathcal{M}}_0\tilde{\mathcal{M}}_1^{\circ(0)} = 2i\tilde{\mathcal{M}}_0^{(-1)}\Delta_1^{\circ} \quad (18)$$



The 3PM scattering angle in Einstein gravity

Decomposition at the level of the amplitude, from the match at each topology level :

$$\tilde{\mathcal{M}}_2^{(-3)} = \tilde{\mathcal{M}}_2^{\Box\Box(-3)} = -\frac{1}{3!}(\tilde{\mathcal{M}}_0^{(-1)})^3$$

$$\tilde{\mathcal{M}}_2^{(-2)} = \tilde{\mathcal{M}}_2^{\Box\Box(-2)} + \tilde{\mathcal{M}}_2^{\triangleleft\Box(-2)} + \tilde{\mathcal{M}}_2^{\Box\triangleright(-2)} = i\tilde{\mathcal{M}}_0^{(-1)}\left(\tilde{\mathcal{M}}_1^{\Box(-1)} + \tilde{\mathcal{M}}_1^{\triangleright(-1)} + \tilde{\mathcal{M}}_1^{\triangleleft(-1)}\right) = i\tilde{\mathcal{M}}_0^{(-1)}\tilde{\mathcal{M}}_1^{(-1)}$$

$$\tilde{\mathcal{M}}_2^{(-1)} = \tilde{\mathcal{M}}_2^{\Box\Box(-1)} + \tilde{\mathcal{M}}_2^{\Box\triangleright(-1)} + \tilde{\mathcal{M}}_2^{\triangleleft\Box(-1)} + \tilde{\mathcal{M}}_2^{\Box\circ(-1)} + \tilde{\mathcal{M}}_2^{H(-1)} + \tilde{\mathcal{M}}_2^{\triangleright\triangleright(-1)} + + \tilde{\mathcal{M}}_2^{\triangleleft\triangleleft(-1)} + \tilde{\mathcal{M}}_2^{SE(-1)}$$
$$= 2i\tilde{\mathcal{M}}_0^{(-1)}\left(\Delta_1^{\Box} + \Delta_1^{\triangleleft} + \Delta_1^{\triangleright} + \Delta_1^{\circ}\right) + \frac{2}{\hbar}\left(\delta_2^{\Box\Box} + \delta_2^{\triangleleft\Box} + \delta_2^{\Box\triangleright} + \delta_2^{H} + \delta_2^{\triangleleft\triangleleft} + \delta_2^{\triangleright\triangleright} + \delta_2^{SE}\right) = 2i\tilde{\mathcal{M}}_0^{(-1)}\Delta_1 + \frac{2}{\hbar}\delta_2 \tag{19}$$

$$\delta_2 = \frac{G^3 m_1 m_2 (\pi b^2 e^{\gamma_E})^{3\epsilon}}{2b^2\sqrt{\sigma^2-1}}\Bigg(-\frac{3(2\sigma^2-1)(1-5\sigma^2)s}{2(\sigma^2-1)}$$
$$+\frac{1}{2}(m_1^2+m_2^2)(-1+18\sigma^2) - \frac{1}{3}m_1 m_2\sigma(103+2\sigma^2) + \frac{4m_1 m_2(3+12\sigma^2-4\sigma^4)\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}$$
$$-\frac{2im_1 m_2(2\sigma^2-1)^2}{\pi\epsilon\sqrt{\sigma^2-1}}\left(\frac{-1}{4(\sigma^2-1)}\right)^{\epsilon}\left(-\frac{11}{3}+\frac{d}{d\sigma}\left(\frac{(2\sigma^2-1)\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right)\right)\Bigg) \tag{20}$$

Match with Bern, Cheung, Roiban, Shen, Solon and Zeng for the potential contribution. Match with Damour and Di Vecchia, Heissenberg, Russo and Veneziano for the radiation-reaction part.

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The relevant basis for 3PM observables

Set of integrals for relevant classical terms

$$\int \frac{1}{(\bar{p}_1.l_2 \pm i\epsilon)^{n_1}(\bar{p}_2.l_1 \pm i\epsilon)^{n_2}((u_q - l_1)^2)^{n_3}((u_q + l_2)^2)^{n_4}(l_1^2)^{n_5}(l_2^2)^{n_6}} \frac{\delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_2)}{((l_1 + l_2)^2)^{n_7}} \tag{21}$$

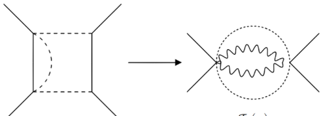
It reduces the dimensionality of the integrals

$$\int \frac{1}{(k.l_1 \mp i\epsilon)^{n_1}(k.l_2 \mp i\epsilon)^{n_2}((u_q - l_1)^2)^{n_3}((u_q + l_2)^2)^{n_4}((l_1)^2)^{n_5}(l_2^2)^{n_6}} \frac{1}{((l_1 + l_2)^2 - 2(\sigma - 1)k.l_1 k.l_2)^{n_7}} \tag{22}$$

Then determine a set of master integrals (we find 9) $\mathcal{I}_n$, and an ODE system

$$\frac{d}{d\sigma}\vec{\mathcal{I}}(\sigma) = \epsilon \mathrm{M}\vec{\mathcal{I}}(\sigma) \tag{23}$$

Potential vs soft : The $\mathcal{I}_5$ integral (1/2)



1/ Velocity $(\sigma - 1)$ expansion :

$$\mathcal{I}_5 \sim \int \frac{1}{l_1^2(l_1 - u_q)^2((l_1 + l_2)^2 - 2(\sigma - 1)k.l_1 k.l_2)} = \sum_{n=0}^{\infty} \int \frac{(2(\sigma - 1)(k.l_1)(k.l_2))^n}{l_1^2(l_1 - u_q)^2(l_1 + l_2)^{2(n+1)}} = 0 \quad (24)$$

2/ Direct computation :

$$\mathcal{I}_5 \sim m_1 m_2 \int \frac{\delta(\bar{p}_1.l_1)\delta(\bar{p}_2.l_2)}{l_1^2(l_1 - u_q)^2(l_1 + l_2)^2} = -\frac{i\epsilon(4\pi e^{-\gamma_E})^{2\epsilon}}{32\pi^3}\left(\frac{-1}{4(\sigma^2 - 1)}\right)^{\epsilon} + \mathcal{O}(\epsilon^2) \quad (25)$$

What is the problem ? (24) is only valid when $2(\sigma - 1)(k.l_1)(k.l_2) << (l_1 + l_2)^2$, i.e. in the potential region.

Potential vs soft : The $\mathcal{I}_5$ integral (2/2)

Differential equation method

$$\frac{d\mathcal{I}_5}{d\sigma} = -\epsilon \frac{2\sigma}{\sigma^2 - 1}\mathcal{I}_5 \tag{26}$$

1/ Expansion in ϵ, $\mathcal{I}_5 = \sum_{n=N}^{\infty} \epsilon^n \mathcal{I}_{5,n}$, with boundary condition $\mathcal{I}_5(\sigma = 1) = 0$:

$$\mathcal{I}_5 = 0 \tag{27}$$

2/ Direct computation :

$$\mathcal{I}_5 = c_5(\sigma^2 - 1)^{-\epsilon} \tag{28}$$

where c_5 is a constant of integration, that cannot be determined by the $\sigma = 1$ limit.

What is the problem ? Non-commutativity of limits, $\lim_{\sigma\to 1} \lim_{\epsilon\to 0} \mathcal{I}_5 = c_5$ whereas $\lim_{\epsilon\to 0} \lim_{\sigma\to 1} \mathcal{I}_5 = 0$.

Potential vs soft : General solution for the integrals

Same method for all integrals of the basis, that can be written

$$\mathcal{I}_n = \mathcal{I}_n^P + \left(\frac{-1}{4(\sigma^2-1)}\right)^{\epsilon} \mathcal{I}_n^R \tag{29}$$

Potential constants of integration at $\sigma = 1$:

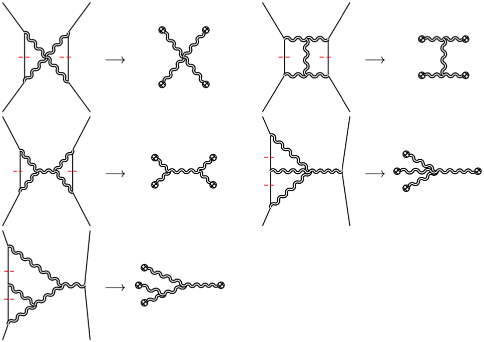
$$\lim_{\sigma\to 1} \mathcal{I}_n = \lim_{\sigma\to 1} \mathcal{I}_n^P \tag{30}$$

$\mathcal{I}_n^R$ are the radiation-reaction terms. The result for $\mathcal{I}_5$ is sufficient to determine entirely the value of all $\mathcal{I}_n^R$ terms (no need for additional constants of integration).

Relation with the world line approach

Exact correspondence between the basis we use here and the basis used in the world line approach (Kalin, Liu and Porto). It implies that radiation-reaction should be present also in the world line formalism.

This correspondence tends also to prove that the basis with two velocity cuts that we use here is in fact the minimal basis necessary to generate the 3PM observables.



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Conclusion and outlook

We have been able to develop a new method to compute the classical terms of the 2-body amplitudes in both $N = 8$ supergravity and Einstein gravity, up to 3PM. The general idea is to exploit symmetries of the amplitude, and sum parts of the integrand that generate velocity cuts when the soft expansion is performed. This method has several advantages :

- It decreases both the dimension of the integrals that have to be performed and the number of master integrals that have to be computed to get the physical observables ;
- It is performed in a covariant formalism and do not need any velocity expansion (that may have problems because of the non commutativity of $\epsilon \to 0$ and $\sigma \to 1$ limits) ;
- It points exactly the differences between the soft and the potential regions, showing also that the full soft region computation of the integrals is not more complicated to achieve than the potential region one. No new integral is needed to get the radiation-reaction terms, compared to the conservative result ;
- There is a one-to-one correspondence between the classification in terms of velocity cuts and the terms appearing in the eikonal matching, so that at higher loop orders, it opens the way to more efficient computations, focusing directly on the terms $\int \delta(\bar{p}.l_1)...\delta(\bar{p}.l_n)$ contributing to the eikonal phase δ_n.