Gravitational Bremsstrahlung from Spinning Bodies using a Worldline QFT

based on [2010.02865] with Jan Plefka & Jan Steinhoff
& [2101.12688] (PRL) w Gustav Jakobsen, J.P., J.S.
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Worldline QFT

Use a path integral representation: $\kappa^2 = 32\pi G$

$$Z_{WQFT} = \int Dh_{\mu\nu} Dz_i^\mu \exp\Big\{ i \int d^4x \Big(\underbrace{-\tfrac{2}{\kappa^2}\sqrt{-g}R}_{S_{EH}} + \underbrace{(\partial_\nu h^{\mu\nu} - \tfrac{1}{2}\partial^\mu h^\nu{}_\nu)^2}_{S_{gf}} \Big) - i \sum_{i=1}^{2} \underbrace{\int d\tau_i \frac{m_i}{2} g_{\mu\nu} \dot{x}_i^\mu \dot{x}_i^\nu}_{S_{pm}^{(i)}} \Big\}$$

where $x_i^\mu(\tau_i) = b_i^\mu + \tau_i v_i^\mu + z_i^\mu(\tau_i)$,
$g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x)$

equivalent to $m_i \int d\tau_i$

BASIC IDEA: compute physical quantities as expectation values of operators in the WQFT...

$$\langle O(h,\{z_i\})\rangle_{WQFT} := \frac{1}{Z_{WQFT}} \int Dh_{\mu\nu} Dz_i^\mu \, O(h,\{z_i\}) \, e^{i(S_{EH} + S_{gf} + \sum_i S_{pm}^{(i)})}$$

In practice work in Fourier space (energy, momentum) and draw Feynman diagrams. The classical limit is identified with tree-level diagrams.

Here we focus on the unbound 2-body problem. Mappings to bound orbits by e.g. [Kälin, Porto].

WQFT Feynman Rules

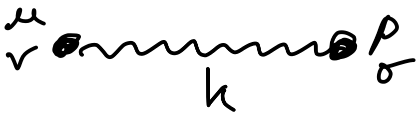
Treat $z_i^\mu(\tau_i)$ as a propagating degree of freedom:

$$S = -\frac{2}{\kappa^2}\int d^4x\sqrt{-g}R - \sum_i \frac{m_i}{2}\int d\tau_i\, g_{\mu\nu}\dot{x}_i^\mu \dot{x}_i^\nu \quad \left.\begin{array}{l} g_{\mu\nu}(x) = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x) \\ x_i^\mu(\tau_i) = b_i^\mu + \tau_i v_i^\mu + z_i^\mu(\tau_i) \end{array}\right.$$

Feynman rules from bulk action are standard; on the worldline w/ $z_i^\mu(\omega)$ only energy ω is conserved. We insert

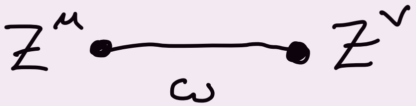
$$h_{\mu\nu}(x) = \int_k e^{-ik\cdot x} h_{\mu\nu}(k) \qquad z_i^\mu(\tau_i) = \int_\omega e^{-i\omega\tau_i} z_i^\mu(\omega_i)$$

Graviton propagator in usual de Donder gauge, $\partial^\mu h_{\mu\nu} = \frac{1}{2}\partial_\nu h^\mu{}_\mu$



$$= i\,\frac{P_{\mu\nu;\rho\sigma}}{k^2} \qquad P_{\mu\nu;\rho\sigma} = \eta_{\mu(\rho}\eta_{\sigma)\nu} - \frac{1}{2}\eta_{\mu\nu}\eta_{\rho\sigma}$$

Worldline propagator:



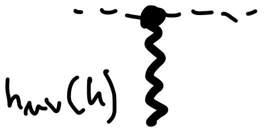
$$= -i\,\frac{\eta^{\mu\nu}}{m(\omega + i\varepsilon)^2} \quad \left.\begin{array}{l} +i\varepsilon \text{ prescription} \Rightarrow \text{retarded propagator,} \\ z^\mu(-\infty) = 0 \end{array}\right.$$

Worldline Interactions

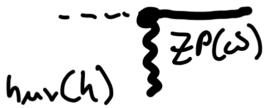
$$h_{\mu\nu}(x(\tau)) = \int_k e^{ik\cdot(b+\tau v+z(\tau))}\, h_{\mu\nu}(-k) = \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_k e^{ik\cdot(b+\tau v)}\,(k\cdot z(\tau))^n\, h_{\mu\nu}(-k)$$

$$= \sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{k,\omega_1,\dots,\omega_n} e^{ik\cdot b}\, e^{i(k\cdot v+\sum_i\omega_i)\tau}\left(\prod_{i=1}^{n} k\cdot z(-\omega_i)\right) h_{\mu\nu}(-k) \quad \Big\}\; z^\mu(\tau) = \int_\omega e^{-i\omega\tau} z^\mu(\omega)$$

Feed this into the interacting part of the action, read off e.g.



$$= -i\,\frac{m}{2m_{Pl}}\, e^{ik\cdot b}\,\delta(k\cdot v)\, v^\mu v^\nu \quad \Big\}\ \text{source for } h_{\mu\nu}(k)$$



$$= \frac{m}{2m_{Pl}}\, e^{ik\cdot b}\,\delta(k\cdot v+\omega)\left(2\omega v^{(\mu}\delta^{\nu)}_{\rho} + v^\mu v^\nu k_\rho\right) \quad \Big\}\ \text{mixing with } z^\rho(\omega)$$

Vertices couple linearly to $h_{\mu\nu}(k)$; known n-point expression:

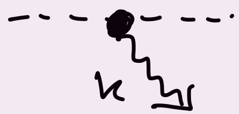
$$V^{WL,\mu\nu}_{\rho_1\rho_2\cdots\rho_n}(k;\omega_1,\dots,\omega_n) = i^{n-1}\frac{m}{m_{Pl}}\, e^{ik\cdot b}\,\delta\!\left(k\cdot v+\sum_i\omega_i\right)\times$$

$$\left(\frac{1}{2}\left(\prod_i k_{\rho_i}\right)v^\mu v^\nu + \sum_{i=1}^{n}\omega_i\left(\prod_{j\neq i}^{n} k_{\rho_j}\right)v^{(\mu}\delta^{\nu)}_{\rho_i} + \sum_{i<j}^{n}\omega_i\omega_j\left(\prod_{l\neq i,j}^{n} k_{\rho_l}\right)\delta^{(\mu}_{\rho_i}\delta^{\nu)}_{\rho_j}\right)$$

Radiation

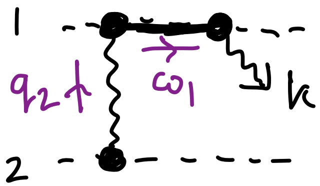
Compute $\langle h_{\mu\nu}(k) \rangle_{\text{WQFT}}$ by drawing graphs with a single outgoing graviton line. E.g. 1-body leading order:

$$-ik^2 \langle h^{\mu\nu}(k) \rangle_{\text{WQFT}} =$$

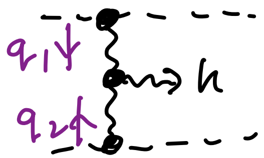


$$= -i\kappa m e^{ik\cdot b} \delta(k\cdot v) v^\mu v^\nu$$

Leading 2-body radiation (2PM) consists of 3 diagrams:



$$= -\frac{\kappa^3 m_1 m_2}{8} \int_{q_2,\omega_1} e^{i(k-q_2)\cdot b_1} e^{iq_2\cdot b_2} \delta(q_2\cdot v_2)\, \delta(\omega_1 - q_2\cdot v_1)\, \delta(\omega_1 - k\cdot v_1)$$
$$\frac{(2\omega_1 v_1^{(\mu} \delta_\rho^{\nu)} - v_1^\mu v_1^\nu k_\rho)(2\omega_1 v_1^{(\sigma} \eta^{\lambda)\rho} - v_1^\sigma v_1^\lambda q_2^\rho)}{(\omega_1 + i\varepsilon)^2} \frac{P_{\sigma\lambda;\alpha\beta}}{q_2^2} v_2^\alpha v_2^\beta$$



$$= -\frac{\kappa^3 m_1 m_2}{8} \int_{q_1, q_2} e^{i(q_1\cdot b_1 + q_2\cdot b_2)} \delta(q_1\cdot v_1)\, \delta(q_2\cdot v_2)\, \delta(k - q_1 - q_2)$$
$$V_3^{(\mu\nu)(\rho\sigma)(\lambda\tau)} \frac{P_{\rho\sigma;\alpha\beta}}{q_1^2} \frac{P_{\lambda\tau;\gamma\delta}}{q_2^2} v_1^\alpha v_1^\beta v_2^\gamma v_2^\delta$$

And the $(1 \leftrightarrow 2)$ version of the first diagram. Integral measures are brought into a common form.

Classical EoMs

We can calculate $x_i^\mu(\tau_i)$ and $h_{\mu\nu}(x)$ from the EoMs: $\quad \kappa^2 = 32\pi G$

$$\ddot{x}_i^\mu = -\Gamma^\mu{}_{\nu\rho}\dot{x}_i^\mu \dot{x}_i^\nu, \qquad \Box h_{\mu\nu} = -\frac{\kappa}{2} S_{\mu\nu}, \qquad S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^\rho{}_\rho$$

extended stress-energy tensor

Solve these iteratively as power series in G:

$$x_i^\mu(\tau_i) = b_i^\mu + \tau_i v_i^\mu + G x_i^{(1)\mu}(\tau_i) + G^2 x_i^{(2)\mu}(\tau_i) + \mathcal{O}(G^3)$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + G h^{(1)}_{\mu\nu}(x) + G^2 h^{(2)}_{\mu\nu}(x) + \mathcal{O}(G^3)$$

For instance... to $\mathcal{O}(G)$ from the geodesic equation:

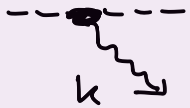
$$\ddot{x}_i^{(1)\mu}(\tau_i) = -\frac{1}{2}\left(2h^{(1)\mu}{}_{\nu,\rho} - h^{(1)}_{\nu\rho,}{}^{\mu}\right) v_i^\nu v_i^\rho$$

$$\Rightarrow -\omega^2 x_i^{(1)\mu}(\omega) = \frac{i}{2\kappa}\int e^{ik\cdot b}\,\delta(k\cdot v+\omega)\left(2\omega v_i^{(\nu}\eta^{\rho)\mu} + v_i^\nu v_i^\rho k^\mu\right) h^{(1)}_{\nu\rho}(-k)$$

Path integral localizes on the EoM in the tree-level (steepest descent) approximation.

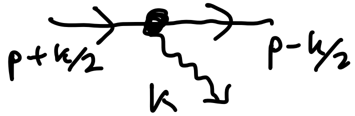
Connection to scattering amplitudes

Recall that for a single black hole:

$$-ik^2\langle h^{\mu\nu}(k)\rangle_{WQFT} =$$ 

$$= -i\,\frac{m}{2m_{Pl}}\,e^{ik\cdot b}\,\delta(k\cdot v)\,v^\mu v^\nu$$

This we compare with the $\langle\phi\bar{\phi}h\rangle$ vertex:



$$= -iK\Big[p^\mu p^\nu - \underbrace{\tfrac{1}{4}\left(\eta^{\mu\nu}k^2 - k^\mu k^\nu\right)}_{\text{suppressed in }\hbar\to0\text{ limit},}\Big] \qquad k = \hbar\bar{k}$$

There is a match, and we identify

$$k^2\langle h^{\mu\nu}(k)\rangle_{WQFT} = \frac{i}{2m}\,e^{ik\cdot b}\,\delta(k\cdot v)\,\lim_{\hbar\to0}\mathcal{M}^{\mu\nu}_{GR}$$

[Kosower, Maybee, O'Connell]

Similar relationships exist in conservative scattering, e.g. (in $\hbar\to0$ limit)

$$Z_{WQFT} = e^{i\chi} = \frac{1}{4m_1m_2}\int\frac{d^Dq}{(2\pi)^{D-2}}\,\delta(q\cdot v_1)\,\delta(q\cdot v_2)\,e^{iq\cdot b}\,\langle\phi_1\phi_2|S|\phi_1\phi_2\rangle$$

Formal understanding from Feynman-Schwinger dressed scalar propagator.

2-Body Radiation

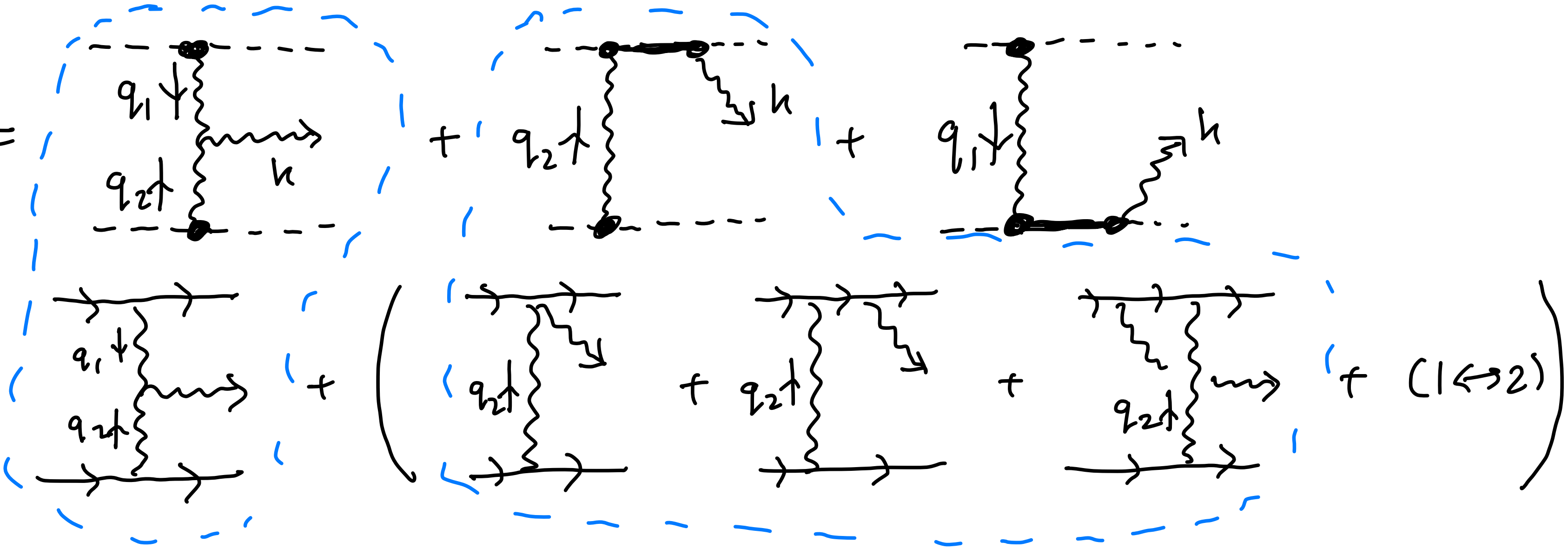
[Luna, Nicholson, O'Connell, White]

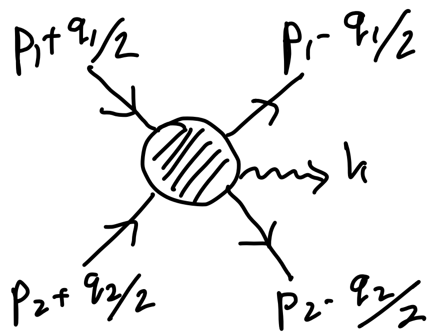
For a 5-point amplitude the explicit connection is

$$k^2\langle h^{\mu\nu}(k)\rangle_{\mathrm{WQFT}} = \frac{i}{4m_1m_2}\int_{q_1,q_2} e^{i(q_1\cdot b_1+q_2\cdot b_2)}\,\delta(q_1\cdot v_1)\,\delta(q_2\cdot v_2)\,\delta(k-q_1-q_2)\lim_{\hbar\to 0}\mathcal{M}^{\mu\nu}_{\mathrm{GR}}$$

Compare WQFT with 5-point amplitude directly at level of diagrams:

$k^2\langle h^{\mu\nu}(k)\rangle_{\mathrm{WQFT}} =$





$$\frac{i}{(p_1+q_1/2+q_2)^2-m_1^2} = \frac{i}{2p_1\cdot q_2+q_2\cdot k} = \frac{i}{2p_1\cdot q_2}\left(1-\frac{q_2\cdot k}{2p_1\cdot q_2}+\dots\right)$$

$$\frac{i}{(p_1-q_1/2-q_2)^2-m_1^2} = \frac{-i}{2p_1\cdot q_2-q_2\cdot k} = \frac{i}{2p_1\cdot q_2}\left(-1-\frac{q_2\cdot k}{2p_1\cdot q_2}+\dots\right)$$

$\Big\}\ \hbar\to 0$ expansion

Leading Waveform

Go back and consider the Einstein eq^n :

$T_{\mu\nu}$ = extended energy-momentum tensor

$$\boxed{k^2\langle h_{\mu\nu}(k)\rangle_{\mathrm{WQFT}} = \frac{\kappa}{2} S_{\mu\nu}(k)} \quad \Bigg\} \quad S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^{\rho}{}_{\rho}\,, \quad \kappa^2 = 32\pi G$$

How does $h_{\mu\nu}$ then look in position space? For a fixed energy Ω, in the wave zone :

$$\boxed{\kappa h_{\mu\nu}(t, \underline{x}) = \frac{4G}{r} S_{\mu\nu}(k)\, e^{-ik\cdot x} + \text{c.c.}} \quad \Bigg\} \quad \underbrace{r = |\underline{x}| \gg \left\{|b_i|, \Omega^{-1}, \Omega|b_i|^2\right\}}_{\text{wave zone}}$$

$\hookrightarrow$ studied by [Mougiakakos, Riva, Vernizzi '21]

where $\quad \boxed{k^\mu = \Omega\rho^\mu = \Omega(1, \hat{\underline{x}})}$

TT = transverse-traceless

$$\boxed{\kappa h_{ij}^{TT} = \frac{f_{ij}(u,\theta,\phi)}{r} = \frac{4G}{r}\int_\Omega e^{-ik\cdot x}\, S_{ij}^{TT}(k)} \quad \Bigg\} \quad k\cdot x = \Omega\underbrace{(t-r)}_{\text{retarded time, } u} = \Omega u$$

The time-domain waveform is $\quad \boxed{f_{ij} = \sum_n G^n f_{ij}^{(n)}}$

$$\boxed{f_{ij} = f_+(e_+)_{ij} + f_\times(e_\times)_{ij}} \quad \Longleftrightarrow \quad \boxed{f_{+,\times} = \frac{1}{2} f_{ij}(e_{+,\times})_{ij}}$$

Performing the integrals

$p=(1,\hat{\underline{x}})$

$\Rightarrow p\cdot v_2 = 1 - v\cos\theta$

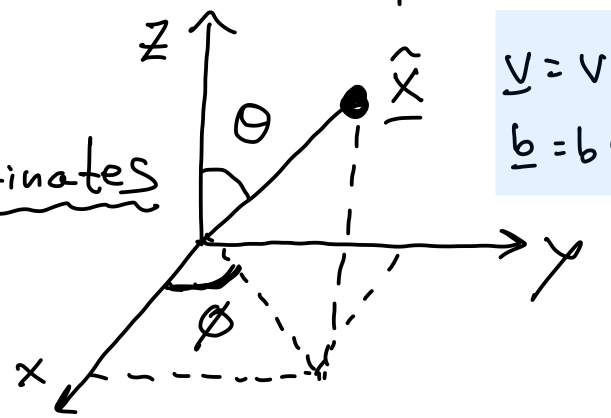
We now have an extra energy integral. So...

$$\int_{\Omega,q_1,q_2} e^{i(q_1\cdot b_1+q_2\cdot b_2-k\cdot x)}\,\delta(q_1\cdot v_1)\,\delta(q_2\cdot v_2)\,\delta(k-q_1-q_2) = \frac{1}{p\cdot v_2}\int_{\underline{q}} e^{i\underline{q}\cdot\underline{\tilde{b}}}$$

Delta f^{n}s evaluate all energy components. To make it simple, use the rest frame of the first black hole:

$$v_1^\mu = (1,\underline{0}) \qquad b_1^\mu = (0,\underline{0})$$
$$v_2^\mu = \gamma(1,\underline{v}) \qquad b_2^\mu = (0,\underline{b})$$

} Spherical polar coordinates



$\underline{v} = v\hat{\underline{e}}_1$

$\underline{b} = b\hat{\underline{e}}_2$

$$q_2 = k - q_1\,,\quad q_1 = (0,\underline{q})\,,\quad \Omega = -\frac{\gamma v}{p\cdot v_2}\,\underline{q}\cdot\hat{\underline{e}}_1$$

$$\hat{\underline{x}} = \hat{\underline{e}}_1\cos\theta + \sin\theta(\hat{\underline{e}}_2\cos\phi + \hat{\underline{e}}_3\sin\phi)$$

$\hat{\underline{e}}_i$ = Cartesian basis vecs

The shifted impact parameter:

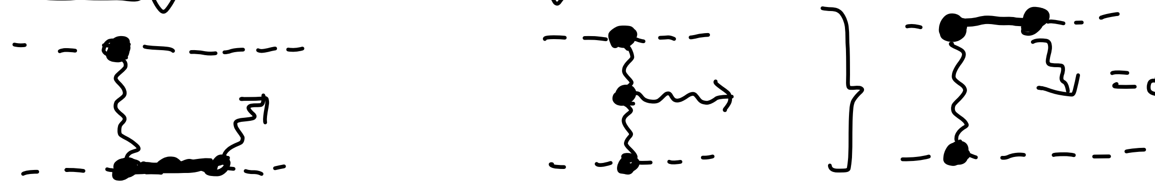
$$\tilde{b}^\mu = b^\mu + u_2 v_2^\mu - u_1 v_1^\mu \quad \Big\}\quad u_i = \frac{p\cdot(x-b_i)}{p\cdot v_i}$$

$$\Rightarrow \underline{\tilde{b}} = \underline{b} + \gamma u_2 \underline{v}\,,\qquad u_1 = u\,,\quad u_2 = \frac{1}{p\cdot v_2}(u + \underline{b}\cdot\hat{\underline{x}})$$

Integrated waveform

[Kovacs, Thorne '77]

$$\frac{f^{(2)}}{m_1 m_2} = 4\pi \int_q e^{i q\cdot \tilde{b}} \left(\underbrace{\frac{N^i q^i}{q^2 (q\cdot \hat{e}_1 - i\varepsilon)}} + \underbrace{\frac{M^{ij} q^i q^j}{q^2 \, q\cdot L \cdot q}} \right) \qquad L^{ij} = \delta^{ij} + 2\frac{\gamma v}{p\cdot v_2} \hat{e}_1^{(i} \hat{x}^{j)}$$



$= 0$ in this frame

Now perform these integrals, one for each diagram ... we get

$$\frac{f^{(2)}_{+,\times}}{m_1 m_2} = \frac{\alpha_{+,\times}}{|\tilde{b}|} + \alpha'_{+,\times} + \frac{1}{(\tilde{b})^2}\left(\frac{\beta_{+,\times}}{|\tilde{b}|} + \frac{\beta'_{+,\times}}{|\tilde{b}'|} \right)$$

$$|\tilde{b}| = \sqrt{b^2 + \gamma^2 v^2 u_2^2}$$
$$|\tilde{b}'| = \sqrt{b^2 + \gamma^2 v^2 u_1^2}$$

where $\alpha_{+,\times}$, $\alpha'_{+,\times}$, $\beta_{+,\times}$, $\beta'_{+,\times}$ are functions of $b = |\underline{b}|$, $v = |\underline{v}|$, θ, ϕ, and u, e.g.:

$$\alpha'_+ = \frac{2(2\gamma^2-1)(1-(p\cdot v_2)^2)\cos\phi \sin\theta}{b\gamma v\, (p\cdot v_2)^2}, \quad \alpha'_\times = -\frac{4(2\gamma^2-1)\sin\theta \sin\phi}{b(p\cdot v_2)} \qquad p\cdot v_2 = 1 - v\cos\theta$$

This result agrees precisely with Kovacs & Thorne. We have a visualization of the waveform...

Radiated angular momentum

[Damour '20]

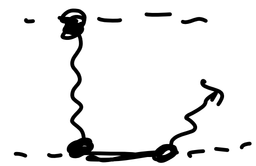
Using the waveform we can check the total radiated angular momentum.

$$J^{\text{rad}}_{ij} = \frac{1}{8\pi G}\int du\, d\sigma \left(f_{k[i}\dot f_{j]k} - \frac{1}{2} x_{[i}\partial_{j]} f_{kl}\dot f_{kl} \right)$$

$\dot f_{ij} := \partial_u f_{ij}$
$d\sigma := \sin\theta \, d\theta\, d\phi$

Focus on the z-direction. The leading-order $f^{(1)}_{ij}(\theta,\phi)$ does not depend on retarded time u. So all we need from $f^{(2)}(u,\theta,\phi)$ is the wave memory:

$$\Delta f_{ij} := \left[f_{ij}\right]^{u=+\infty}_{u=-\infty} = -2G^2 m_1 m_2 \frac{b^{i} N^{i}}{b^2} + O(G^3)$$



only

Perform the spherical integral, and we get

⇒ rad. ang. momentum not dependent on bulk interactions

$$\frac{J^{\text{rad}}_{xy}}{J^{\text{init}}_{xy}} = \frac{4G^2 m_1 m_2}{b^2}\frac{(2\gamma^2-1)}{\sqrt{\gamma^2-1}} I(v) + O(G^3)$$

$$I(v) = -\frac{8}{3} + \frac{1}{v^2} + \frac{(3v^2-1)}{v^3}\operatorname{arctanh}(v)$$

Agrees with Damour, even in CoM frame!

where $J^{\text{init}}_{xy} = m_2 |\underline{v}_2||\underline{b}| = m_2 \gamma v b$ (see also [Mougiakakos, Riva, Vernizzi '21])

Radiated Energy

$$P^\mu_{rad} = \frac{1}{32\pi G}\int du\, d\sigma \left[\dot{f}_{ij}\right]^2 \rho^\mu \quad \Big\}\ \rho^\mu = (1, \hat{x})$$

We seek the radiated energy in the frame of BH1:

$$E_{rad} = V_1 \cdot P_{rad} \Rightarrow \frac{dE_{rad}}{d\sigma} = \frac{1}{32\pi G}\int du \left[\dot{f}_{ij}\right]^2$$

$$\frac{dE_{rad}}{d\sigma} = \frac{G^3 m_1^2 m_2^2 V}{512\, b^3}$$

leading PN order, see paper for up to V^3

$$\times\left[45(\cos^2\theta\cos^2\phi + \sin^2\phi)^2 + 109\sin^4\theta + 630\sin^2\theta\sin^2\phi + 354\sin^2\theta\cos^2\theta\cos^2\phi\right]$$

$$\Rightarrow \frac{E_{rad}}{\pi} = \frac{G^3 m_1^2 m_2^2 V}{b^3}\left(\frac{37}{15} + \frac{2393}{840}V^2 + \frac{61703}{10080}V^4\right) + O(V^7)$$

agrees with [Herrmann, Parra-Martinez, Ruf, Zeng '21] 3PM calculation using KMOC "in-in" formalism

$\hookrightarrow$ [Kosower, Maybee, O'Connell]

SUSY in the Sky

[Gibbons, Rietdijk, van Holten '93]

Dirac theory of a spin-½ fermion described by an $\mathcal{N}=1$ theory in $\hbar \to 0$ limit:

$$\boxed{S = -m\int d\tau \left[\tfrac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \tfrac{i}{2}\psi_a \frac{D\psi^a}{D\tau}\right]} \quad \left.\right\} \frac{D\psi^a}{D\tau} = \psi^a{}_{;\mu}\dot{x}^\mu = \dot{\psi}^a - \dot{x}^\mu \omega_{\mu,}{}^a{}_b \psi^b$$

where $\boxed{g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}}$, $\boxed{\psi^\mu = e^\mu_a \psi^a}$ } $\psi^a(\tau)$ is a real, Grassmann-valued spinor

Theory is invariant under
$$\boxed{\begin{aligned} \delta x^\mu &= i\varepsilon\psi^\mu \\ \delta\psi^a &= -\varepsilon e^a_\mu \dot{x}^\mu - \delta x^\mu \omega_{\mu,}{}^a{}_b \psi^b \end{aligned}}$$

Spinor we can think of as "square root" of spin tensor $\boxed{S^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} p_\rho a_\sigma}$

$$\boxed{S^{\mu\nu} = -i\psi^\mu\psi^\nu} \quad \left.\right\} H = \frac{m}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{m}{2}, \quad Q = \dot{x}_\mu \psi^\mu = 0 \Rightarrow \boxed{S^{\mu\nu} p_\nu = 0} \quad \text{(SSC)}$$

$$\frac{D^2 x^\mu}{D\tau^2} = \frac{i}{2}\psi^a\psi^b R_{ab,}{}^\mu{}_\nu \dot{x}^\nu \qquad \Longrightarrow \qquad \frac{D^2 x^\mu}{D\tau^2} = -\frac{1}{2} S^{ab} R_{ab,}{}^\mu{}_\nu \dot{x}^\nu$$

$$\frac{D\psi^a}{D\tau} = 0 \qquad\qquad \frac{DS^{\mu\nu}}{D\tau} = 0$$

(Mathisson-Papapetrou equations @ $\mathcal{O}(S)$)

The $\mathcal{N}=2$ theory

[Bastianelli, Benincasa, Giombi '05]

Generalize to a complex-valued spinor $\psi^a(\tau)$:

$$\boxed{S = -m\int d\tau \left[\frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + i\bar{\psi}_a \frac{D\psi^a}{D\tau} + \frac{1}{2} R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d\right]} \Bigg\} \text{ valid up to } O(S^2)$$

We now identify the spin tensor $S^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} p_\rho a_\sigma$ as

$$\boxed{S^{\mu\nu} = -2i\,\bar{\psi}^{[\mu}\psi^{\nu]}} \quad \Bigg\} \quad \{\bar{\psi}^\mu, \psi^\nu\}_{P.B.} = -i\eta^{\mu\nu} \quad (1^{\text{st}}\text{-order formalism})$$

$$\Rightarrow \{S^{\mu\nu}, S^{\rho\sigma}\}_{P.B.} = \eta^{\mu\rho}S^{\nu\sigma} + \eta^{\nu\sigma}S^{\mu\rho} - \eta^{\nu\rho}S^{\mu\sigma} - \eta^{\mu\sigma}S^{\nu\rho}$$

$\mathcal{N}=2$ SUSY implies four conserved supercharges:

$$\underbrace{\boxed{g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 1 - \frac{1}{2}R_{abcd}\bar{\psi}^a\psi^b\bar{\psi}^c\psi^d}}_{\text{energy conservation}}, \quad \underbrace{\boxed{\dot{x}_\mu\psi^\mu = \dot{x}_\mu\bar{\psi}^\mu = 0}}_{\text{SSC}}, \quad \underbrace{\boxed{\bar{\psi}\cdot\psi = \text{const.}}}_{S^{\mu\nu}S_{\mu\nu} = \text{const.}}$$

Again, classical EoMs $\Rightarrow$ Mathisson-Papapetrou eq$^{\text{ns}}$ @ $O(S^2)$

The Spinning WQFT

The partition function now becomes

$$Z_{WQFT} := \text{const.} \times \int Dh_{\mu\nu} \left(\prod_{i=1}^{2} DZ_i^\mu D\psi'^a_i\right) \exp\left\{ i\left(S_{EH} + S_{gf} + \sum_{i=1}^{2} S^{(i)} + S^{(i)}_{ES^2}\right)\right\}$$

We can also include spin-induced multipoles:

$$\left.\boxed{S_{ES^2} = -m \int d\tau\, C_E E_{ab} \bar{\psi}^a \psi^b \bar{\psi}\cdot\psi}\ \right\} \quad E_{ab} = R_{a\mu b\nu} \dot{X}^\mu \dot{X}^\nu$$

"electric" component of Weyl tensor

$$\boxed{\mathcal{S}^{\mu\nu} = -2i\, \bar{\Psi}^{[\mu} \Psi^{\nu]}}$$

$$x^\mu(\tau) = b^\mu + \tau v^\mu + z^\mu(\tau), \quad \psi^a(\tau) = \Psi^a + \psi'^a(\tau), \quad S^{\mu\nu}(\tau) = \mathcal{S}^{\mu\nu} + S'^{\mu\nu}(\tau)$$

$$\left. e_\mu^a(x) = \eta^{a\nu}\left(\eta_{\mu\nu} + \frac{\kappa}{2} h_{\mu\nu} - \frac{\kappa^2}{8} h_{\mu\rho} h^\rho{}_\nu + \mathcal{O}(\kappa^3)\right) \right\} \text{Vierbein expansion}$$

We compute physical quantities as expectation values, as before:

$$\boxed{\langle \mathcal{O}(h_{\mu\nu}, \{z_i, \psi'_i\}) \rangle_{WQFT} := \frac{1}{Z_{WQFT}} \int Dh_{\mu\nu} \left(\prod_{i=1}^{2} DZ_i^\mu D\psi'^a_i\right) \mathcal{O}\ e^{i\left(S_{EH} + S_{gf} + \sum_i S^{(i)} + S^{(i)}_{ES^2}\right)}}$$

Proceed in momentum space with Feynman diagrams, Feynman rules

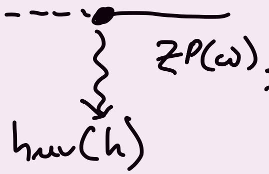
WQFT Feynman rules with spin

Feynman rules now also include $S^{\mu\nu} = -2i\bar{\Psi}^{[\mu}\Psi^{\nu]}$:

exponential structure … c.f [Guevara, Ochirov, Vines]

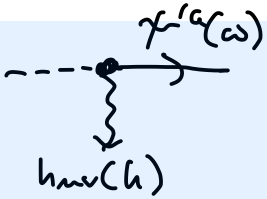
$$= -i\frac{\kappa m}{2} e^{ik\cdot b}\delta(k\cdot v)\left(v^\mu v^\nu + i k_\rho S^{\rho(\mu}v^{\nu)} + \frac{1}{2}k_\rho k_\sigma S^{\rho\mu}S^{\nu\sigma}\right)$$

(ignoring C_E corrections)



$$= \frac{\kappa m}{2} e^{ik\cdot b}\delta(k\cdot v+\omega)\left(2\omega v^{(\mu}\delta^{\nu)}_\rho + v^\mu v^\nu k_\rho + i k_\lambda S^{\lambda(\mu}\left(k_\rho v^{\nu)} + \omega\delta^{\nu)}_\rho\right) + \frac{1}{2}k_\rho k_\sigma k_\lambda S^{\sigma\mu}S^{\nu\lambda}\right)$$

Now also include

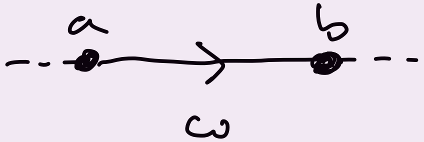


$$= -im\kappa e^{ik\cdot b}\delta(k\cdot v+\omega)\, k_{[a}\delta_{b]}^{(\mu}\left(v^{\nu)} - iS^{\nu)\rho}k_\rho\right)\bar{\Psi}^b$$

$\neq 0$

The relevant retarded propagators are

$$= -i\frac{\eta^{\mu\nu}}{m(\omega+i\epsilon)^2}$$

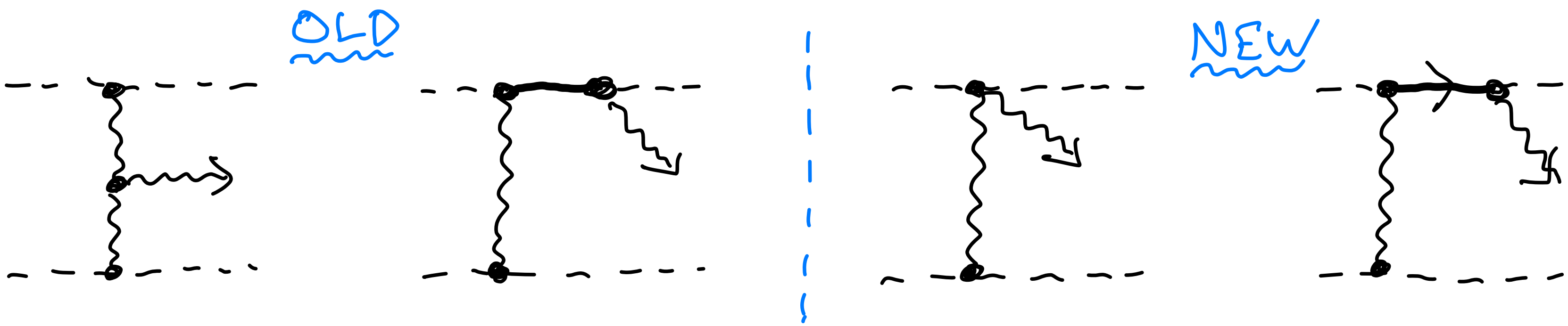


$$= -i\frac{\eta^{ab}}{m(\omega+i\epsilon)}$$

note: direction of arrow does not always follow $i\epsilon$ prescription

Bremsstrahlung @ $\mathcal{O}(s^2)$

Integrand similar to the non-spinning case, but with 2 new diagrams:



The integrand also follows a similar pattern:

$$\frac{f^{(2)}_{t,x}}{m_1 m_2} = 4\pi \int_q e^{i q\cdot \tilde{b}} \left(\frac{\mathcal{N}_{t,x}(q)}{q^2 (q\cdot \hat{e}_1 - i\varepsilon)} + \frac{\mathcal{M}_{t,x}(q)}{q^2 (q\cdot L \cdot q)} \right)$$

Use derivatives w.r.t. $\tilde{b}$ to derive higher-rank integrals from lower-rank counterparts

where

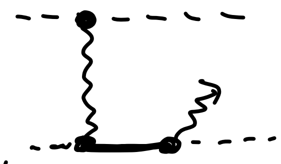
$$\begin{aligned} \mathcal{N}_{t,x}(q) &= \mathcal{N}^{i}_{t,x} q^i + \mathcal{N}^{ij}_{t,x} q^i q^j + \mathcal{N}^{ijk}_{t,x} q^i q^j q^k \\ \mathcal{M}_{t,x}(q) &= \underbrace{\mathcal{M}^{ij}_{t,x} q^i q^j}_{\mathcal{O}(s^0)} + \underbrace{\mathcal{M}^{ijk}_{t,x} q^i q^j q^k}_{\mathcal{O}(s)} + \underbrace{\mathcal{M}^{ijkl}_{t,x} q^i q^j q^k q^l}_{\mathcal{O}(s^2)} \end{aligned}$$

$$\frac{f^{(2)}_{t,x}}{m_1 m_2} = \sum_{s=0}^{2} \left[\frac{\alpha^{(s)}_{t,x}}{|\tilde{b}|^{2s+1}} + \frac{\alpha'^{(s)}_{t,x}}{|\tilde{b}|^{2s}} + \frac{1}{(\tilde{b}^2)^{s+1}} \left(\frac{\beta^{(s)}_{t,x}}{|\tilde{b}|^{2s+1}} + \frac{\beta'^{(s)}_{t,x}}{|\tilde{b}'|^{2s+1}} \right) \right]$$

Radiated angular momentum

Use the same formula as the non-spinning case:

$$J_{ij}^{\mathrm{rad}} = \frac{1}{8\pi G}\int du\, d\sigma \left(f_{k[i}\,\dot f_{j]k} - \frac{1}{2} x_{[i}\partial_{j]} f_{kl}\,\dot f_{kl} \right)$$



is the only diagram

The two non-zero components @ $O(G^2)$ are

$$J_{xy}^{\mathrm{rad}} + i J_{zx}^{\mathrm{rad}} = \frac{1}{8\pi}\int d\sigma\, e^{-i\phi}\left[i\,\frac{\delta f_+^{(1)}\,\Delta f_\times}{\sin\theta} - \partial_\theta\, \delta f_+^{(1)}\,\frac{\Delta f_+}{2}\right] + O(G^3)$$

After integration we find that

$$J_{xy}^{\mathrm{rad}} + i J_{zx}^{\mathrm{rad}} = \frac{4G^2 m_1 m_2^2}{b^3}(2\gamma^2-1)\, I(v)\left(b^2 - \frac{2ibv\,\mathbf{a}_3\cdot\boldsymbol{\ell}}{1+v^2} - (\mathbf{a}_3\cdot\boldsymbol{\ell})^2 + \sum_{i=1}^{2} C_{E,i}(\mathbf{a}_i\cdot\boldsymbol{\ell})^2\right)$$

$$I(v) = -\frac{8}{3} + \frac{1}{v^2} + \frac{(3v^2-1)}{v^3}\,\mathrm{arctanh}(v)$$

where $\mathbf{a}_3 = \mathbf{a}_1 + \mathbf{a}_2$, $\boldsymbol{\ell} = \hat{\mathbf{e}}_2 + i\hat{\mathbf{e}}_3$, $a_1^\mu = \begin{pmatrix} 0 \\ \mathbf{a}_1 \end{pmatrix}$, $a_2^\mu = \begin{pmatrix} \gamma(\mathbf{v}\cdot\mathbf{a}_2) \\ \mathbf{a}_2 + \frac{\gamma^2}{1+\gamma}(\mathbf{v}\cdot\mathbf{a}_2)\mathbf{v} \end{pmatrix}$

Conclusions

- WQFT is a powerful tool for classical 2-body calculations, with a natural SUSY extension.

- WQFT path integrals include $Z_i^\mu(\tau_i)$, $h_{\mu\nu}(x)$, and $\psi_i^a(\tau_i)$ when spin is involved. Feynman rules on the worldline conserve only energy, so allow "Loop integrals" from tree-level.

- We've successfully obtained the leading-order (2PM) waveform produced from a scattering of 2 non-spinning BHs, reproducing [Kovacs & Thorne], radiated angular momentum [Damour], and radiated energy spectra.

- Now also generalized to $O(S^2)$, including radiated ang. momentum.

- Next steps:

- Investigate spinning WQFT further with other observables: deflection, spin kick, eikonal c.f. [Liu, Porto, Yang '21], [Kosmopoulos, Luna '21]

- Further exploration of 1D supergravities, higher spin, connection to amplitudes?

Thanks for listening!

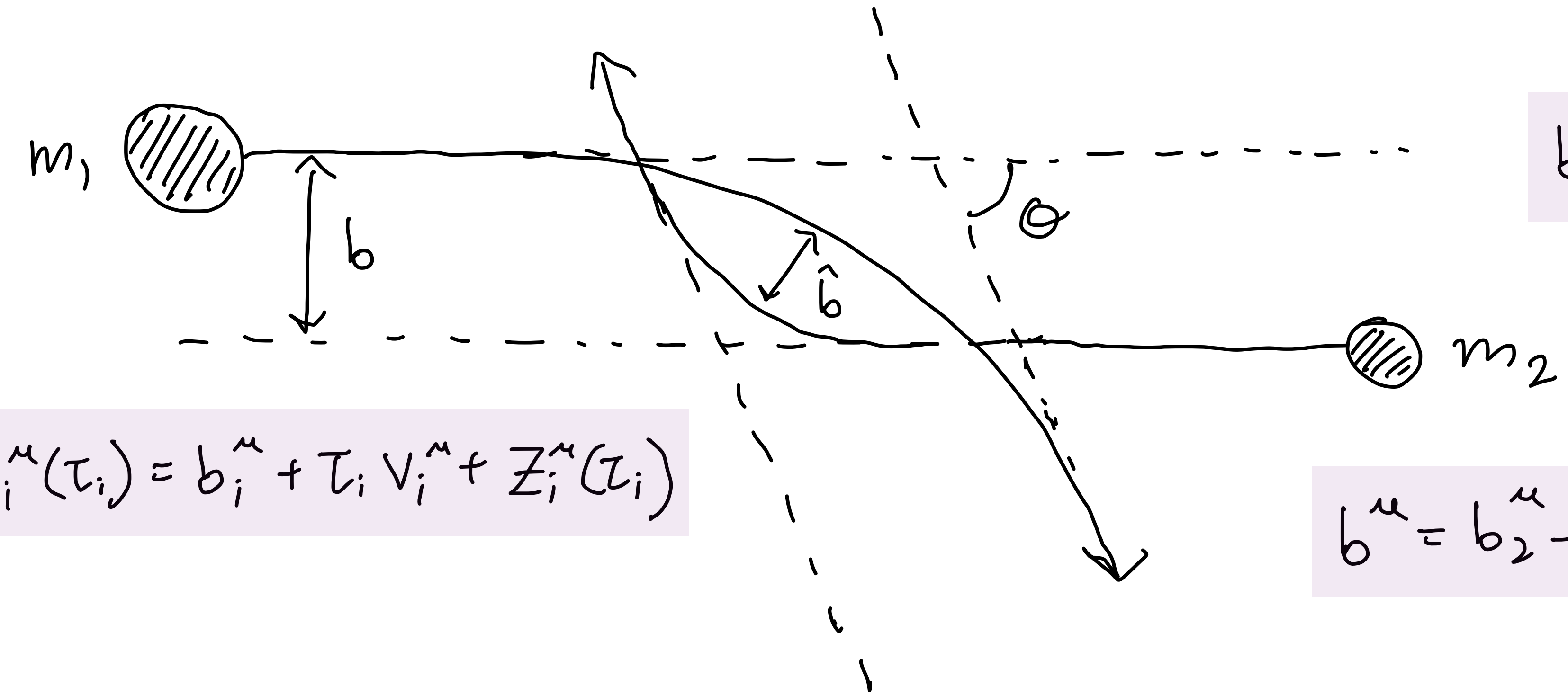
Worldline propagator

Pay attention to the $i\varepsilon$ prescription:

$$Z^\mu \underset{\omega}{\bullet\!\!-\!\!-\!\!-\!\!\bullet} Z^\nu = -i\,\frac{\eta^{\mu\nu}}{m(\omega \pm i\varepsilon)^2}$$

Different signs give retarded/advanced propagators:

- retarded $\Rightarrow Z^\mu(-\infty)=0 \Rightarrow$ identify b_i^μ, v_i^μ with incoming states
- advanced $\Rightarrow Z^\mu(+\infty)=0 \Rightarrow$ " " " outgoing states
- time-symmetric $\Rightarrow Z^\mu(0)=0 \Rightarrow$ " " " intermediate states



$$b = \hat{b}\cos\left(\frac{\theta}{2}\right)$$

$$X_i^\mu(\tau_i) = b_i^\mu + \tau_i v_i^\mu + Z_i^\mu(\tau_i)$$

$$b^\mu = b_2^\mu - b_1^\mu$$

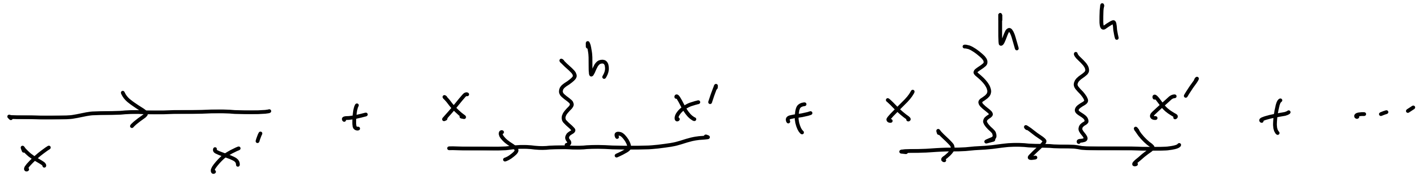
Graviton-dressed scalar propagator

Massive complex scalar $\phi(x)$ interacting with gravity:

$$S = \int d^D x \sqrt{-g} \left(g^{\mu\nu} \partial_\mu \phi^\dagger \partial_\nu \phi - m^2 \phi^\dagger \phi - \xi R \phi^\dagger \phi \right)$$

$$\left(\nabla_\mu \nabla^\mu + m^2 + \xi R \right) G(x,x') = \sqrt{-g}\, \delta^{(D)}(x-x')$$

Weak-field approximation

$G(x,x') =$ 

This gravitationally-dressed Green's function has a worldline path integral form [deWitt, Bekenstein, Parker]:

$$G(x,x') \sim \int_0^\infty ds\, e^{-ism^2} \int_{x(0)=x}^{x(s)=x'} \mathcal{D}x \exp\left[-i \int_0^s d\sigma \left(\frac{1}{4} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \underbrace{\left(\xi - \frac{1}{4}\right) R}_{\text{suppressed in } \hbar \to 0 \text{ limit}} \right) \right]$$

This provides a link : QFT $\longleftrightarrow$ worldline .

From the S-matrix to the worldline

$$G(x,x') = \int_0^\infty ds\, e^{-ism^2} \int_{x(0)=x}^{x(s)=x'} Dx\, \exp\left[-i\int_0^s d\sigma \left(\frac{1}{4} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \left(\xi - \frac{1}{4}\right)R\right)\right]$$

How does the dressed propagator help us analyze S-matrices? Use the 2-point function:

$$G_i(x,x') = \int D\phi_i\, \phi_i(x)\, \phi_i^\dagger(x')\, e^{iS_i}$$

$$S_{EH} = -\frac{2}{\kappa^2}\int d^Dx \sqrt{-g}\, R$$

Insert into a 5-point time-ordered correlator:

$$\begin{aligned}
&\langle \Omega | T\{h_{\mu\nu}(x)\phi_1(x_1)\phi_1^\dagger(x_1')\phi_2(x_2)\phi_2^\dagger(x_2')\} | \Omega \rangle \\
&= \int D[h_{\mu\nu}, \phi_1, \phi_2]\, h_{\mu\nu}(x)\phi_1(x_1)\phi_1^\dagger(x_1')\,\phi_2(x_2)\,\phi_2^\dagger(x_2')\, e^{i(S_{EH} + S_1 + S_2)} \\
&= \int Dh_{\mu\nu}\, h_{\mu\nu}(x)\, G_1(x_1,x_1')\, G_2(x_2,x_2')\, e^{iS_{EH}} \quad \text{“}=\text{''} \quad \langle h_{\mu\nu}(x)\rangle_{WQFT}
\end{aligned}$$

Neglect virtual scalar loops using classical $\hbar \to 0$ limit. We should also consider LSZ reduction, which gives $e^{iq\cdot b}$ & $\delta(q\cdot v)$ factors.