Classical scattering of spinning black holes from quantum amplitudes

based on 1812.06895, 1906.10071
with Alfredo Guevara and Justin Vines

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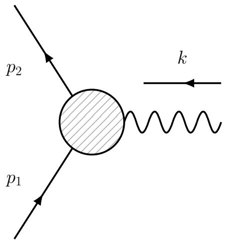
Introduction

- **This workshop:**
  amazing results for classical binaries using ampl. methods
- **This talk:** classical scattering of spinning black holes
- **Focus:** connect 3pt amplitude to 1PM scattering — Guevara, AO, Vines '19

Arkani-Hamed, Huang, Huang '17

$$\mathcal{M}_3^{(s,+)} = -\frac{\kappa}{2}\frac{\langle 12\rangle^{\odot 2s}}{m^{2s-2}}x^2$$

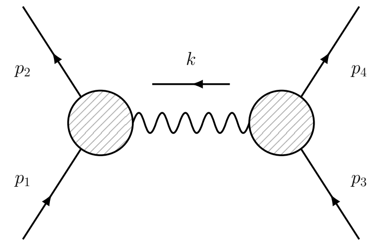


Vines '17

$$\Delta p_{\mathrm{a}}^\mu = G m_{\mathrm{a}} m_{\mathrm{b}} \Re Z^\mu$$

$$\Delta a_{\mathrm{a}}^\mu = -\frac{G m_{\mathrm{b}}}{m_{\mathrm{a}}}\left[p_{\mathrm{a}}^\mu (a_{\mathrm{a}}\cdot \Re Z) + \epsilon^{\mu\nu\rho\sigma}(\Im Z_\nu) p_{\mathrm{a}\rho} a_{\mathrm{a}\sigma}\right]$$

$$Z^\mu = \frac{\gamma}{v}\sum_{\pm}\left(1\pm v\right)^2\left[\eta^{\mu\nu} \mp i(*w)^{\mu\nu}\right]\frac{(b\mp w * a_0)_\nu}{(b \mp w * a_0)^2}$$



- **Aim:** elucidate spin aspects of classical limit

Kosower, Maybee, O'Connell '18
Maybee, O'Connell, Vines '19
Bern, Luna, Roiban, Shen, Zeng '20
de la Cruz, Maybee, O'Connell, Ross '20

Outline

1. Building blocks: amplitudes
2. Recognizing spin operators: spin exponentiation
3. Classical limit: spin coherent states
4. 1PM for general spins
5. 2PM for aligned spins

6. Summary & outlook

Building blocks: amplitudes

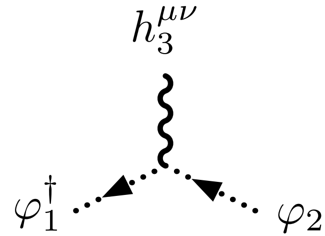
3-pt gravitational vertices*

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \qquad \Rightarrow \qquad g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu} + \mathcal{O}(\kappa^2)$$

Spin 0:

$$\mathcal{L}_{\text{scalar}} = g^{\mu\nu}(\partial_\mu\varphi)^\dagger(\partial_\nu\varphi) - m^2\varphi^\dagger\varphi$$
$$\mathcal{L}_{\varphi\varphi h} = -\kappa h^{\mu\nu}(\partial_\mu\varphi^\dagger)(\partial_\nu\varphi)$$

$\Rightarrow$

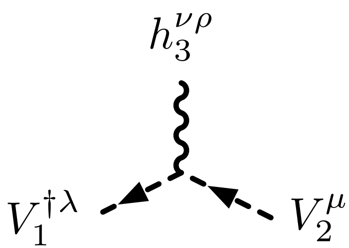


$$\simeq i\kappa p_1^{(\mu}p_2^{\nu)}$$

Spin 1:

$$\mathcal{L}_{\text{Proca}} = -\frac{1}{2}V^\dagger_{\mu\nu}V^{\mu\nu} + m^2 V^\dagger_\mu V^\mu$$
$$\mathcal{L}_{VVh} = \kappa h^{\mu\nu}\left(V^\dagger_{\mu\sigma}V_\nu{}^\sigma - m^2 V^\dagger_\mu V_\nu\right)$$

$\Rightarrow$



$$\simeq -i\kappa\Big[\big((p_1\cdot p_2) + m^2\big)\eta^{\lambda(\nu}\eta^{\rho)\mu} + \eta^{\lambda\mu}p_1^{(\nu}p_2^{\rho)} - \eta^{\lambda(\nu}p_2^{\rho)}p_1^\mu - p_2^\lambda p_1^{(\nu}\eta^{\rho)\mu}\Big]$$

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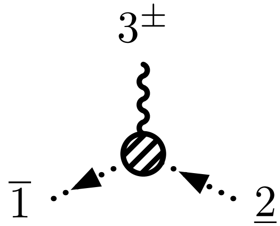
*Disclaimer: all momenta incoming

3-pt gravitational amplitudes

Spin 0:

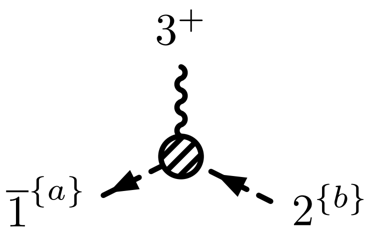


$$= -i\kappa(p_1 \cdot \varepsilon_3)^2 = -\frac{i\kappa}{2}m^2 x_{\pm}^2$$

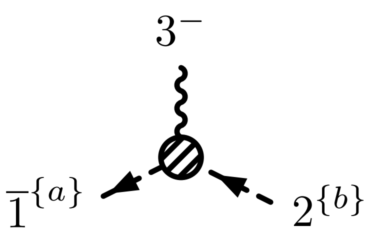
where $x = \sqrt{2}\,\frac{p_1 \cdot \varepsilon_3}{m}$

Spin 1:

using $\varepsilon_{p\mu}^{\{a_1,a_2\}} = \frac{i\langle p^{(a_1}|\sigma_\mu|p^{a_2)}]}{\sqrt{2}m}$



$$= -\frac{i\kappa}{2}x_{+}^2\langle 1^{(a_1}\,2^{(b_1}\rangle\langle 1^{a_2)}\,\overline{2}^{b_2)}\rangle$$



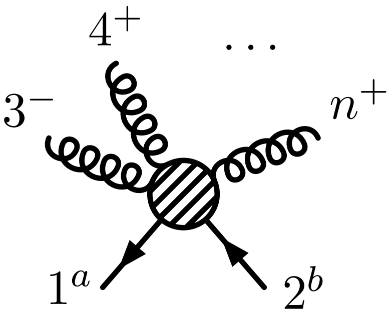
$$= -\frac{i\kappa}{2}x_{-}^2[1^{(a_1}\,2^{(b_1}][1^{a_2)}\,\overline{2}^{b_2)}]$$

Why does spinor helicity help?

Consider QFT amplitude $\mathcal{A}(\underline{1}^a, 3^-, 4^+, \ldots, n^+, \overline{2}^b)$

Feynman rules give function of

- momenta p_i^μ
- pol. tensors $\varepsilon_\pm^\mu(p_i)$, $\varepsilon_\pm^{\mu\nu}(p_i)$ — gauge-dep.!
- external spinors $\bar{v}^a(p_1)$, $u^b(p_2)$



But all vector, spinor indices must be contracted

Remaining indices ⇔ physical quantum numbers:

- helicities $\pm$ ⇔ spins $\{\pm 1/2\}_p$, $\{\pm 1\}_p$, etc.
- $\mathrm{SU}(2)$ labels a, b ⇔ spins $\{\pm 1/2\}_q$, $\{\pm 1, 0\}_q$, etc.

Crucial on-shell notion — LITTLE GROUP

Little groups

- Quantum fields $\Leftarrow$ reps of $\mathrm{SO}(1,3)$
- Quantum states $\Leftarrow$ reps of LITTLE GROUP
  - massless states $\Leftarrow$ $\mathrm{SO}(2)$
  - massive states $\Leftarrow$ $\mathrm{SO}(3)$

Little groups

- Quantum fields $\Leftarrow$ reps of $\mathrm{SO}(1,3)$ $\subset$ $\mathrm{SL}(2,\mathbb{C})$
- Quantum states $\Leftarrow$ reps of LITTLE GROUP's dbl cover
  - massless states $\Leftarrow$ $\mathrm{SO}(2)$ $\subset$ $\mathbf{U(1)}$
  - massive states $\Leftarrow$ $\mathrm{SO}(3)$ $\subset$ $\mathbf{SU(2)}$

Minor complication: spinorial reps use groups' double covers

$\mathrm{U}(1)$ and $\mathrm{SU}(2)$ arise naturally in spinor helicity

Massless vs massive spinor helicity

Arkani-Hamed, Huang, Huang '17

Spinor map: $p_{\alpha\dot\beta} = p_\mu \sigma^\mu_{\alpha\dot\beta}$

MASSLESS	MASSIVE
$\det\{p_{\alpha\dot\beta}\} = 0$	$\det\{p_{\alpha\dot\beta}\} = m^2$
$p_{\alpha\dot\beta} = \lambda_{p\alpha}\tilde\lambda_{p\dot\beta} \equiv \|p\rangle_\alpha [p\|_{\dot\beta}$	$p_{\alpha\dot\beta} = \lambda_{p\alpha}^{\ a}\epsilon_{ab}\tilde\lambda_{p\dot\beta}^{\ b} \equiv \|p^a\rangle_\alpha [p_a\|_{\dot\beta}$ $\det\{\lambda_{p\alpha}^{\ a}\} = \det\{\tilde\lambda_{p\dot\alpha}^{\ a}\} = m$
$p^\mu = \frac{1}{2}\langle p\|\sigma^\mu\|p]$	$p^\mu = \frac{1}{2}\langle p^a\|\sigma^\mu\|p_a]$
$p_{\alpha\dot\beta}\tilde\lambda_p^{\dot\beta} = 0$	$p_{\alpha\dot\beta}\tilde\lambda_p^{a\dot\beta} = m\lambda_{p\alpha}^{\ a}$
$\langle p\,q\rangle = -\langle q\,p\rangle \quad \Rightarrow \quad \langle p\,p\rangle = 0$ $[p\,q] = -[q\,p] \quad \Rightarrow \quad [p\,p] = 0$ $\langle p\,q\rangle[q\,p] = 2p\cdot q$	$\langle p^a q^b\rangle = -\langle q^b p^a\rangle$ e.g. $\langle p^a p^b\rangle = -m\epsilon^{ab}$ $[p^a q^b] = -[q^b p^a]$ e.g. $[p^a p^b] = m\epsilon^{ab}$ $\langle p^a q^b\rangle[q_b\,p_a] = 2p\cdot q$

SU(2) $a = 1,2$ spin s $(a_1 \ldots a_{2s})$

Wavefunctions from helicity spinors

Massless:

$$\varepsilon^{\mu}_{p+} = \frac{\langle q|\sigma^{\mu}|p]}{\sqrt{2}\langle q\,p\rangle} \qquad \varepsilon^{\mu}_{p-} = \frac{\langle p|\sigma^{\mu}|q]}{\sqrt{2}[p\,q]} \quad \Rightarrow \quad \begin{cases} \varepsilon_p^{\pm}\cdot p = \varepsilon_p^{\pm}\cdot q = 0 \\ \varepsilon^{\mu}_{p+}\varepsilon^{\nu}_{p-} + \varepsilon^{\mu}_{p-}\varepsilon^{\nu}_{p+} = -\eta^{\mu\nu} + \dfrac{p^{\mu}q^{\nu} + q^{\mu}p^{\nu}}{p\cdot q} \\ \varepsilon_p^{h_1}\cdot\varepsilon_p^{h_2} = -\delta^{h_1(-h_2)} \end{cases}$$

Xu, Zhang, Chang '85

Massive:

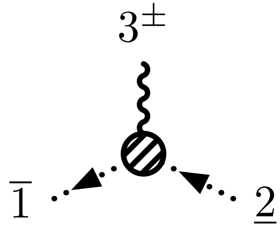
$$\varepsilon^{ab}_{p\mu} = \frac{i\langle p^{(a}|\sigma_{\mu}|p^{b)}]}{\sqrt{2}m} \quad \Rightarrow \quad \begin{cases} p\cdot\varepsilon_p^{ab} = 0 \\ \varepsilon^{ab}_{p\mu}\varepsilon_{p\nu ab} = -\eta_{\mu\nu} + \dfrac{p_{\mu}p_{\nu}}{m^2} \\ \varepsilon_p^{ab}\cdot\varepsilon_{pcd} = -\delta^{(a}_{(c}\delta^{b)}_{d)} \end{cases}$$

Guevara, AO, Vines '18
Chung, Huang, Kim, Lee '18

and (symmetrized) tensor products thereof

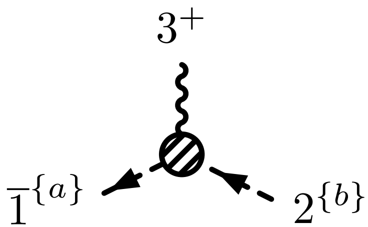
3-pt gravitational amplitudes

Spin 0:

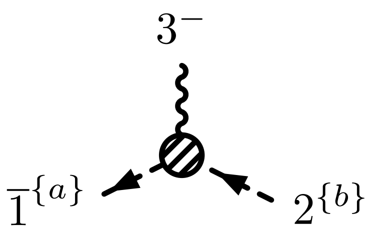


$$= -i\kappa(p_1 \cdot \varepsilon_3)^2 = -\frac{i\kappa}{2} m^2 x_{\pm}^2$$

Spin 1:



$$= -\frac{i\kappa}{2} x_{+}^2 \langle 1^{(a_1} 2^{(b_1} \rangle \langle 1^{a_2)} \overline{2}^{b_2)} \rangle$$



$$= -\frac{i\kappa}{2} x_{-}^2 [1^{(a_1} 2^{(b_1}][1^{a_2)} \overline{2}^{b_2)}]$$

double-copy aspects in Johansson, AO '19

Minimal 3-pt amplitudes

Arkani-Hamed, Huang, Huang '17

$\langle 1^{(a_1} 2^{(b_1} \rangle \ldots \langle 1^{a_{2s})} 2^{b_{2s})} \rangle$

$$\mathcal{M}_3(\overline{1}^{\{a\}}, \underline{2}^{\{b\}}, 3^+) = -\frac{\kappa}{2}\frac{\langle 1^a 2^b \rangle^{\odot 2s}}{m^{2s-2}} x_+^2$$

$$\mathcal{M}_3(\overline{1}^{\{a\}}, \underline{2}^{\{b\}}, 3^-) = -\frac{\kappa}{2}\frac{[1^a 2^b]^{\odot 2s}}{m^{2s-2}} x_-^2$$

$$x = \sqrt{2}\,\frac{p_1 \cdot \varepsilon_3}{m} : \qquad x_+ = \frac{\langle r|1|3]}{m\langle r\,3\rangle}, \qquad x_- = -\frac{[r|1|3\rangle}{m[r\,3]} = -\frac{1}{x_+}$$

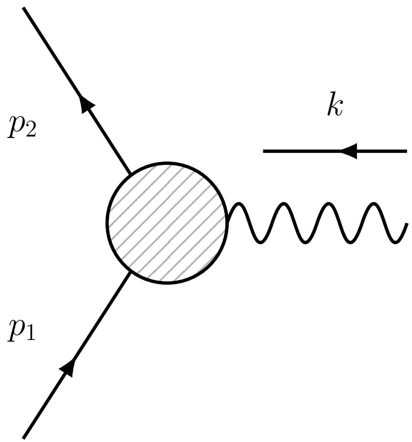
NB! Independent of ref. momentum r

$$p_2^2 - m^2 = 2p_1 \cdot p_3 = \langle 3|1|3] = 0 \qquad \Rightarrow \qquad \exists\, x \in \mathbb{C} : |1|3\rangle = -mx|3]$$

Recognizing spin operators: spin exponentiation

Minimal-coupling 3-pt amplitudes

Arkani-Hamed, Huang, Huang '17



$$\mathcal{M}_3^{(s,+)} = -\frac{\kappa}{2}\frac{\langle 12\rangle^{\odot 2s}}{m^{2s-2}}x^2, \qquad x = -\frac{\sqrt{2}}{m}(p_1\cdot\varepsilon^+)$$

$$\mathcal{M}_3^{(s,-)} = -\frac{\kappa}{2}\frac{[12]^{\odot 2s}}{m^{2s-2}}x^{-2}, \qquad = \left[\frac{\sqrt{2}}{m}(p_1\cdot\varepsilon^-)\right]^{-1}$$

e.g. $\mathcal{M}_3^{(0,\pm)} = -\kappa(p_1\cdot\varepsilon^\pm)^2$

Angular-momentum structure inside:

$$\mathcal{M}_3^{(s,+)} = \mathcal{M}_3^{(0,+)}\frac{\langle 12\rangle^{\odot 2s}}{m^{2s}} = \frac{\mathcal{M}_3^{(0,+)}}{m^{2s}}[2|^{\odot 2s}\exp\left(-i\frac{k_\mu\varepsilon_\nu^+\bar{\sigma}^{\mu\nu}}{p_1\cdot\varepsilon^+}\right)|1]^{\odot 2s}$$

$$\mathcal{M}_3^{(s,-)} = \mathcal{M}_3^{(0,-)}\frac{[12]^{\odot 2s}}{m^{2s}} = \frac{\mathcal{M}_3^{(0,-)}}{m^{2s}}\langle 2|^{\odot 2s}\exp\left(-i\frac{k_\mu\varepsilon_\nu^-\sigma^{\mu\nu}}{p_1\cdot\varepsilon^-}\right)|1\rangle^{\odot 2s}$$

Guevara, AO, Vines '18

inspired by soft theorems, e.g. Cachazo, Strominger '14

Angular-momentum exponential of Kerr

Vines '17

Stress-energy tensor (eff. source) for lin. Kerr BH:[*]

$$T^{\mu\nu}_{\mathsf{BH}}(x) = \frac{1}{m}\int d\tau\, p^{(\mu} \exp(a * \partial)^{\nu)}{}_{\rho} p^{\rho} \delta^{(4)}(x - u\tau), \qquad p^{\mu} = m u^{\mu}$$

$$T^{\mu\nu}_{\mathsf{BH}}(k) = \hat{\delta}(p \cdot k) p^{(\mu} \exp(-ia * k)^{\nu)}{}_{\rho}\, p^{\rho}, \qquad S^{\mu} = m a^{\mu}$$

Couple to on-shell graviton $h_{\mu\nu}(k) \to \hat{\delta}(k^2)\varepsilon_\mu \varepsilon_\nu$:

$$h_{\mu\nu}(k) T^{\mu\nu}_{\mathsf{BH}}(-k) = \hat{\delta}(k^2)\hat{\delta}(p \cdot k)(p \cdot \varepsilon)^2 \exp\!\left(-i \frac{k_\mu \varepsilon_\nu S^{\mu\nu}}{p \cdot \varepsilon}\right),$$

where $\qquad S^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} p_\rho a_\sigma$

[*] Hat notation absorbs straightforward powers of 2π.

Kerr $\Leftarrow$ minimal coupling to gravity

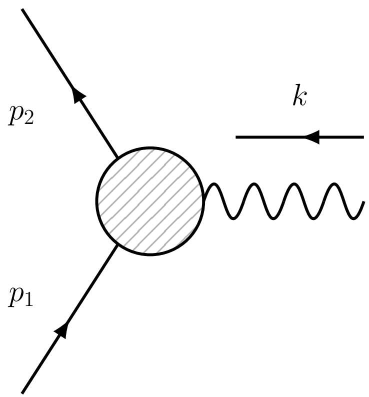
Guevara, AO, Vines '18

$$h_{\mu\nu}(k)T^{\mu\nu}_{\mathsf{BH}}(-k) = \hat{\delta}(k^2)\hat{\delta}(p\cdot k)(p\cdot\varepsilon)^2 \exp\left(-i\frac{k_\mu \varepsilon_\nu S^{\mu\nu}}{p\cdot\varepsilon}\right)$$

Compare to

$$\mathcal{M}_3^{(s,+)} = \frac{\mathcal{M}_3^{(0,+)}}{m^{2s}}[2|^{\odot 2s}\exp\left(-i\frac{k_\mu \varepsilon^+_\nu \bar{\sigma}^{\mu\nu}}{p_1\cdot\varepsilon^+}\right)|1]^{\odot 2s}$$

$$\mathcal{M}_3^{(s,-)} = \frac{\mathcal{M}_3^{(0,-)}}{m^{2s}}\langle 2|^{\odot 2s}\exp\left(-i\frac{k_\mu \varepsilon^-_\nu \sigma^{\mu\nu}}{p_1\cdot\varepsilon^-}\right)|1\rangle^{\odot 2s}$$



Matching spin-induced multipole structure!

complementary picture: 1-body EFT of Kerr by Levi, Steinhoff '15

match to Wilson coeffs by Chung, Huang, Kim, Lee '18

Spin exponentiation in covariant form

Covariant formulation:

Bautista, Guevara '19

$$\mathcal{M}_3^{(s)} = \mathcal{M}_3^{(0)} \varepsilon_2 \cdot \exp\left(-i\frac{k_\mu \varepsilon_\nu \Sigma^{\mu\nu}}{p_1 \cdot \varepsilon}\right) \cdot \varepsilon_1$$

Lorentz generators:

$$(\Sigma^{\mu\nu})^{\sigma_1 \dots \sigma_s}{}_{\tau_1 \dots \tau_s} = \Sigma^{\mu\nu,\sigma_1}{}_{\tau_1} \delta^{\sigma_2}_{\tau_2} \dots \delta^{\sigma_s}_{\tau_s} + \dots + \delta^{\sigma_1}_{\tau_1} \dots \delta^{\sigma_{s-1}}_{\tau_{s-1}} \Sigma^{\mu\nu,\sigma_s}{}_{\tau_s},$$

$$\Sigma^{\mu\nu,\sigma}{}_\tau = i[\eta^{\mu\sigma}\delta^\nu_\tau - \eta^{\nu\sigma}\delta^\mu_\tau]$$

Polarization tensors:

Guevara, AO, Vines '18, Chung, Huang, Kim, Lee '18

$$\varepsilon^{a_1 \dots a_{2s}}_{p\mu_1 \dots \mu_s} = \varepsilon^{(a_1 a_2}_{p\mu_1} \dots \varepsilon^{a_{2s-1} a_{2s})}_{p\mu_s}, \qquad \varepsilon^{ab}_{p\mu} = \frac{i\langle p^{(a}|\sigma_\mu|p^{b)}]}{\sqrt{2}m}$$

Spinor-helicity formulation:

Guevara, AO, Vines '19

$$\mathcal{M}_3^{(s,+)} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}}[2|^{\odot 2s} \exp\left(-i\frac{k_\mu \varepsilon^+_\nu \bar{\sigma}^{\mu\nu}}{p_1 \cdot \varepsilon^+}\right)|1]^{\odot 2s} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}}[2|^{\odot 2s} \exp(-2k \cdot a)|1]^{\odot 2s}$$

$$\mathcal{M}_3^{(s,-)} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}}\langle 2|^{\odot 2s} \exp\left(-i\frac{k_\mu \varepsilon^-_\nu \sigma^{\mu\nu}}{p_1 \cdot \varepsilon^-}\right)|1\rangle^{\odot 2s} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}}\langle 2|^{\odot 2s} \exp(2k \cdot a)|1\rangle^{\odot 2s}$$

$$a^{\mu,}{}_\alpha{}^\beta = \frac{1}{2m^2}\epsilon^{\mu\nu\rho\sigma} p_{a\nu} \sigma_{\rho\sigma,\alpha}{}^\beta, \qquad a^{\mu,\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{2m^2}\epsilon^{\mu\nu\rho\sigma} p_{a\nu} \bar{\sigma}_{\rho\sigma,}{}^{\dot{\alpha}}{}_{\dot{\beta}}$$

$$\sigma^{\mu\nu} = \tfrac{i}{2}\sigma^{[\mu}\bar{\sigma}^{\nu]}, \qquad \bar{\sigma}^{\mu\nu} = \tfrac{i}{2}\bar{\sigma}^{[\mu}\sigma^{\nu]}$$

(and tensor generalizations)

Spin quantization

Define Pauli-Lubanski vector operator $\Sigma_\lambda = \frac{1}{2m}\epsilon_{\lambda\mu\nu\rho}\Sigma^{\mu\nu}p^\rho$

Its 1-particle matrix elements are

$$
\begin{aligned}
S_{p\mu}^{\{a\}\{b\}} &= (-1)^s \varepsilon_p^{\{a\}} \cdot \Sigma_\mu \cdot \varepsilon_p^{\{b\}} \\
&= -\frac{s}{2m}\big\{\langle p^{(a_1}|\sigma_\mu|p^{(b_1}] + [p^{(a_1}|\bar{\sigma}_\mu|p^{(b_1}\rangle\big\}\epsilon^{a_2b_2}\ldots\epsilon^{a_{2s})b_{2s})}
\end{aligned}
$$

Spin quantized explicitly:

$$
\frac{\varepsilon_{p\{a\}} \cdot \Sigma^\mu \cdot \varepsilon_p^{\{a\}}}{\varepsilon_{p\{a\}} \cdot \varepsilon_p^{\{a\}}} = \begin{cases} s s_p^\mu, & a_1 = \ldots = a_{2s} = 1, \\ (s-1)s_p^\mu, & \sum_{j=1}^{2s} a_j = 2s+1, \\ (s-2)s_p^\mu, & \sum_{j=1}^{2s} a_j = 2s+2, \\ \ldots & \\ -s s_p^\mu, & a_1 = \ldots = a_{2s} = 2, \end{cases}
$$

in terms of unit spin vector

$$
\begin{aligned}
s_p^\mu &= -\frac{1}{2m}\big\{\langle p_1|\sigma^\mu|p^1] + [p_1|\bar{\sigma}^\mu|p^1\rangle\big\} \\
&= \frac{1}{2m}\bar{u}_{p1}\gamma^\mu\gamma^5 u_p^1 = -\frac{1}{2m}\bar{u}_{p2}\gamma^\mu\gamma^5 u_p^2
\end{aligned}
$$

$$
p \cdot s_p = 0 \qquad s_p^2 = -1
$$

Spin asymmetry of chiral reps

Puzzle:

two reps of $\mathcal{M}_3^{(s,+)} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}} \langle 21 \rangle^{\odot 2s} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}} [2|^{\odot 2s} e^{-2k\cdot a} |1]^{\odot 2s}$
seem to depend differently on a^μ

Fix:

Guevara, AO, Vines '18

"divide" by

$$\lim_{s\to\infty} \varepsilon_2 \cdot \varepsilon_1 = \lim_{s\to\infty} \frac{1}{m^{2s}} \langle 2|^{\odot 2s} e^{k\cdot a} |1\rangle^{\odot 2s} = \lim_{s\to\infty} \frac{1}{m^{2s}} [2|^{\odot 2s} e^{-k\cdot a} |1]^{\odot 2s}$$

Hint:

Levi, Steinhoff '15

"spin-induced higher multipoles should naturally be considered in the body-fixed frame"

Solution:

Bautista, Guevara '19
Guevara, AO, Vines '19
also in Arkani-Hamed, Huang, O'Connell '19
Aoude's talk on Aoude, Haddad, Helset '20

must only compare states of same momentum!

Lorentz-boost exponentials

Bautista, Guevara '19
Guevara, AO, Vines '19
also in Arkani-Hamed, Huang, O'Connell '19

Consider $p_1 \to p_2$ boost:

$$p_2^\rho = \exp\big(\tfrac{i}{m^2} p_1^\mu k^\nu \Sigma_{\mu\nu}\big)^\rho{}_\sigma p_1^\sigma$$
$$|2^b\rangle = U_{12}{}^b{}_a \exp\big(\tfrac{i}{m^2} p_1^\mu k^\nu \sigma_{\mu\nu}\big)|1^a\rangle$$
$$|2^b] = U_{12}{}^b{}_a \exp\big(\tfrac{i}{m^2} p_1^\mu k^\nu \bar\sigma_{\mu\nu}\big)|1^a]$$

$$k^2 = (p_2 - p_1)^2 = 0$$
$$U_{12} \in \mathrm{SU}(2)$$

Self-duality of $\sigma^{\mu\nu}$, $\bar\sigma^{\mu\nu}$ implies

$$\frac{i}{m^2} p_1^\mu k^\nu \sigma_{\mu\nu,\alpha}{}^\beta = k \cdot a_\alpha{}^\beta, \qquad \frac{i}{m^2} p_1^\mu k^\nu \bar\sigma_{\mu\nu,}{}^{\dot\alpha}{}_{\dot\beta} = -k \cdot a^{\dot\alpha}{}_{\dot\beta}$$

in terms of left- and right-handed reps of Pauli-Lubanski vector

$$a^{\mu,}{}_\alpha{}^\beta = \frac{1}{2m^2} \epsilon^{\mu\nu\rho\sigma} p_{\mathrm{a}\nu} \sigma_{\rho\sigma,\alpha}{}^\beta, \qquad a^{\mu,\dot\alpha}{}_{\dot\beta} = \frac{1}{2m^2} \epsilon^{\mu\nu\rho\sigma} p_{\mathrm{a}\nu} \bar\sigma_{\rho\sigma,}{}^{\dot\alpha}{}_{\dot\beta}$$

Spin exponentials from Lorentz boosts

Arbirary-spin reps boost as

$$|2\rangle^{\odot 2s} = e^{k\cdot a}\{U_{12}|1\rangle\}^{\odot 2s}, \qquad |2]^{\odot 2s} = e^{-k\cdot a}\{U_{12}|1]\}^{\odot 2s}$$
$$\langle 2|^{\odot 2s} = \{U_{12}\langle 1|\}^{\odot 2s} e^{-k\cdot a}, \qquad [2|^{\odot 2s} = \{U_{12}[1|\}^{\odot 2s} e^{k\cdot a}$$

Back to spin dependence of 3-pt amplitude:*

$$\begin{aligned}\mathcal{M}_3^{(s,+)} &= \frac{\mathcal{M}_3^{(0)}}{m^{2s}}\langle 21\rangle^{\odot 2s} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}}\{U_{12}\langle 1|\}^{\odot 2s} e^{-k\cdot a}|1\rangle^{\odot 2s} \\ &= \frac{\mathcal{M}_3^{(0)}}{m^{2s}}[2|^{\odot 2s} e^{-2k\cdot a}|1]^{\odot 2s} = \frac{\mathcal{M}_3^{(0)}}{m^{2s}}\{U_{12}[1|\}^{\odot 2s} e^{-k\cdot a}|1]^{\odot 2s} \\ &\xrightarrow[s\to\infty]{} \mathcal{M}_3^{(0)} e^{-k\cdot a}\lim_{s\to\infty}(U_{12})^{\odot 2s} \quad \text{unambigiously!}\end{aligned}$$

a^μ is now classical (C-number) spin of Kerr BH

*m^{2s} cancels due to $\langle p^a p^b\rangle = -[p^a p^b] = -m\epsilon^{ab}$.

Classical limit: spin coherent states

Want: extract classical spin dependence ($S^\mu \in \mathbb{R}^4$)
from quantum spin amplitudes ($s \in \mathbb{Z}_+$)

Kosower, Maybee, O'Connell '18
Maybee, O'Connell, Vines '19
Bern, Luna, Roiban, Shen, Zeng '20
de la Cruz, Maybee, O'Connell, Ross '20

Impulse formulae

LO impulses:

Kosower, Maybee, O'Connell '18

Maybee, O'Connell, Vines '19

imp. par.

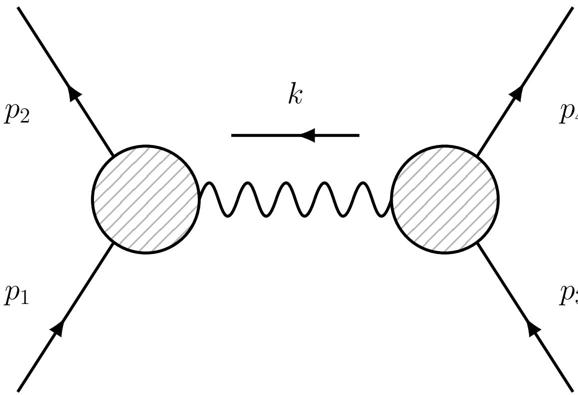
$$\Delta p_{\rm a}^{\mu} = \left\langle\!\!\left\langle \int \hat{d}^4 k\, \hat{\delta}(2p_{\rm a}\cdot k)\hat{\delta}(2p_{\rm b}\cdot k) k^{\mu} e^{-ik\cdot b/\hbar} i\mathcal{M}_4(k) \right\rangle\!\!\right\rangle$$

$$\Delta S_{\rm a}^{\mu} = \left\langle\!\!\left\langle \int \hat{d}^4 k\, \hat{\delta}(2p_{\rm a}\cdot k)\hat{\delta}(2p_{\rm b}\cdot k) e^{-ik\cdot b/\hbar} \times \left(-\frac{i}{m_{\rm a}^2} p_{\rm a}^{\mu} S_{\rm a}^{\nu} k_{\nu} \mathcal{M}_4(k) + \left[S_{\rm a}^{\mu}, i\mathcal{M}_4(k)\right] \right) \right\rangle\!\!\right\rangle$$

S^μ acts on LG indices

$S^{\mu}_{b_1\ldots b_{2s}}{}^{a_1\ldots a_{2s}} \propto (\varepsilon^{b_1\ldots b_{2s}})^* \cdot \Sigma^{\mu} \cdot \varepsilon_1^{a_1\ldots a_{2s}}$

$$p_{\rm a} = \frac{1}{2}(p_1 + p_2)$$



$$p_{\rm b} = \frac{1}{2}(p_3 + p_4)$$

Net effect of $\langle\!\langle \ldots \rangle\!\rangle$:

$$k^{\mu} = \hbar \bar{k}^{\mu} \to 0, \qquad p_1^{\mu}, p_2^{\mu} \to m_{\rm a} u_{\rm a}^{\mu}, \qquad p_3^{\mu}, p_4^{\mu} \to m_{\rm b} u_{\rm b}^{\mu}$$

$$S_1^{\mu}, S_2^{\mu} \to m_{\rm a} a_{\rm a}^{\mu}, \qquad S_3^{\mu}, S_4^{\mu} \to m_{\rm b} a_{\rm b}^{\mu}$$

Wavepackets

Momentum wavepacket in

$$\exp\left(\frac{-p^2}{2m^2 \ell_c^2/\ell_\omega^2}\right)$$

Kosower, Maybee, O'Connell '18

$$\phi(p) = \mathcal{N} \exp\left(-\frac{p \cdot u}{m\xi}\right) \quad \Rightarrow \quad \frac{\langle (\Delta p)^2 \rangle}{\langle p \rangle^2} = \mathcal{O}(\xi), \qquad \xi \equiv \frac{\ell_{\mathsf{c}}^2}{\ell_\omega^2} \to 0$$

Classical limit for momentum vs spin:

- amplitudes are functions of continously changing momenta
- vs little-group index dependence on spin

To model classical spin $S^\mu = m a^\mu$,
need spinning wavefunctions such that

$$\langle p^\mu \rangle = m u^\mu, \qquad \langle p^\mu p^\nu \rangle = m^2 u^\mu u^\nu + \text{negligible}, \qquad \text{etc.}$$

$$\langle S^\mu \rangle = m a^\mu, \qquad \langle S^\mu S^\nu \rangle = m^2 a^\mu a^\nu + \text{negligible}, \qquad \text{etc.}$$

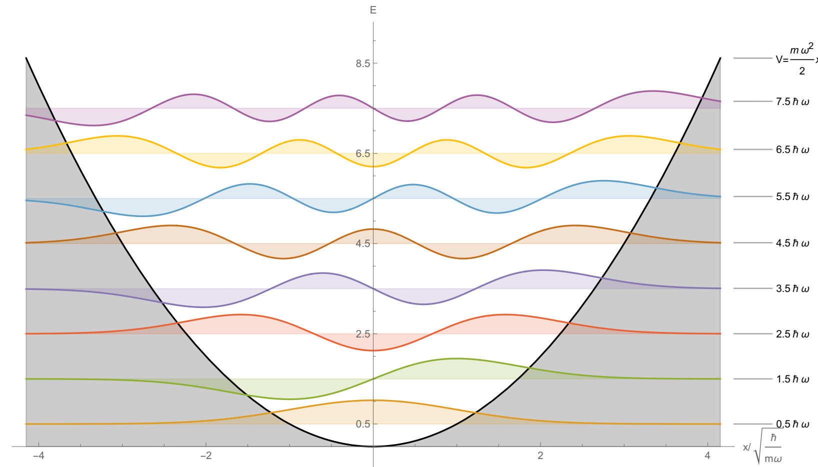
Canonical coherent states

Harmonic oscillator:

$$E_n = \hbar\omega\left(n + {}^1\!/_2\right), \qquad \Delta_n x = \sqrt{\frac{\hbar}{m\omega}\left(n + {}^1\!/_2\right)}$$

$$\Delta_n p = \sqrt{m\omega\hbar\left(n + {}^1\!/_2\right)}$$

$\uparrow 0$ $\uparrow \infty$



$[a, a^\dagger] = 1$

Def $|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} |0\rangle$ (vacuum) $\Rightarrow$ $a|\alpha\rangle = \alpha|\alpha\rangle \; ; \; \alpha \in \mathbb{C}$

$= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$

$O = f(a, a^\dagger) \Rightarrow \langle\alpha| N[O] |\alpha\rangle = f(\alpha, \alpha^*)$

(order imp.)

$\langle x\rangle_\alpha = \sqrt{\frac{\hbar}{2m\omega}}(\alpha + \alpha^*)$

$\langle n\rangle_\alpha = |\alpha|^2$

$\langle p\rangle_\alpha = \ldots (\alpha - \alpha^*)$

$\langle E\rangle_\alpha = \hbar\omega\left(|\alpha|^2 + \frac{1}{2}\right)$

$\Delta_\alpha x = \sqrt{\frac{\hbar}{2m\omega}} \; ; \; \Delta_\alpha p = \sqrt{\frac{m\omega\hbar}{2}} \quad \forall \alpha$

Classical: "$\hbar \to 0$", $|\alpha| = O(\hbar^{-\frac{1}{2}}) \to \infty$

$\langle O\rangle_\alpha \to f(\alpha, \alpha^*) \Rightarrow \langle O_1 O_2\rangle_\alpha \to f_1(\alpha, \alpha^*) f_2(\alpha, \alpha^*) = \langle O_1\rangle_\alpha \times \langle O_2\rangle_\alpha$

Spin coherent states [Schwinger '52 → modern sp.-hel. language]

$[a^a, a^\dagger_b] = \delta^a_b$; $(a^a)^\dagger = a^\dagger_a$ $\quad$ $a,b = 1,2$ – SU(2) spinor indices

Def. $\vec{J} = \frac{\hbar}{2} a^\dagger_a \vec{\sigma}^a{}_b a^b$ $\quad$ $[\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$ fund. rep. of SU(2)

$\Rightarrow [J^i, J^j] = i\hbar\epsilon^{ijk} J^k$ $\quad \Rightarrow$ arb.-spin rep of SU(2)

Def SU(2)-covariant j-spin state

$$|j, \{a_1, \ldots, a_{2j}\}\rangle = \frac{1}{\sqrt{(2j)!}} a^\dagger_{a_1} \ldots a^\dagger_{a_{2j}} |0\rangle$$

($\{a_1, \ldots, a_{2j}\}$: symm.; $|0\rangle$: scalar state)

$$|j, m\rangle = \sqrt{\frac{(2j)!}{(j+m)!(j-m)!}} \, |j, \{\underbrace{1, \ldots, 1}_{j+m}, \underbrace{2, \ldots, 2}_{j-m}\}\rangle$$

"Fock" space for spin

Def. $|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha^a a_a^\dagger}|0\rangle \qquad \alpha \in \mathbb{C}^2$

$\Rightarrow \quad a^a|\alpha\rangle = \alpha^a|\alpha\rangle$

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{j=0,\frac{1}{2}}^{\infty} \sum_{a_1\ldots a_{2j}} \frac{\alpha^{a_1}\ldots\alpha^{a_{2j}} = \alpha^{\odot(2j)}}{\sqrt{(2j)!}} |j,\{a_1\ldots a_{2j}\}\rangle$$

$$= e^{-\frac{1}{2}|\alpha|^2} \sum_{j=0,\frac{1}{2}}^{\infty} \sum_{m=-j}^{j} \frac{(\alpha^1)^{j+m}(\alpha^2)^{j-m}}{\sqrt{(j+m)!(j-m)!}} |j,m\rangle$$

$$\langle\alpha|\vec{J}|\alpha\rangle = \frac{\hbar}{2}\vec{\sigma}^a{}_b\langle\alpha|a_a^\dagger a^b|\alpha\rangle = \frac{\hbar}{2}(\alpha_a^*\vec{\sigma}^a{}_b\alpha^b)$$

Cl. limit: "$\hbar \to 0$" $\quad \sqrt{\alpha_a^*\alpha^a \equiv |\alpha|^2} = O(\hbar^{-\frac{1}{2}}) \to \infty \quad \Rightarrow \langle J\rangle_\alpha = \text{finite}$

$$\langle J^i J^j\rangle_\alpha = \langle J^i\rangle_\alpha\langle J^j\rangle_\alpha + \underbrace{\frac{\hbar^2}{4}\left[\delta^{ij}\alpha_a^*\alpha^a + i\epsilon^{ijk}(\alpha^*\sigma^k\alpha)\right]}_{\text{negligible } O(\hbar)}$$

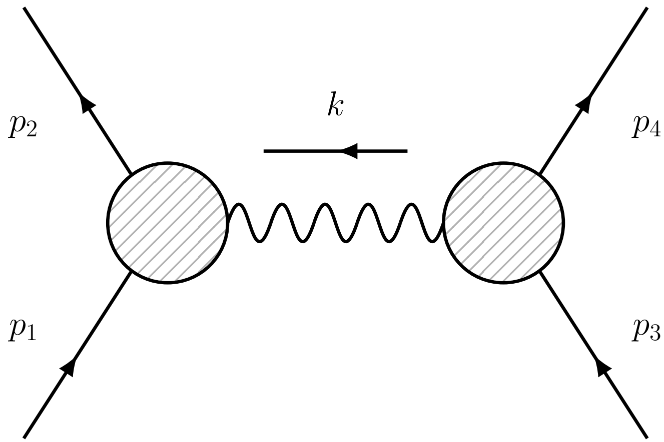
$J^i = L^i{}_\mu S^\mu \; ; \quad S^\mu$ – Pauli-Lub. spin op.

$L^i{}_\mu p^\mu = 0$

1PM for general spins

Holomorphic Classical Limit (HCL)

Cachazo, Guevara '17
Guevara '17

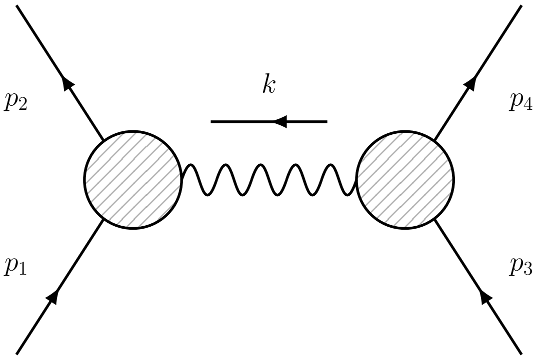


Idea: Replace $k^\mu = \hbar \bar{k}^\mu \to 0$ by non-zero on-shell $t = k^2 \to 0$
Indeed, $k^2 = 0 \quad \Rightarrow \quad p_i \cdot k = \mathcal{O}(t) = 0$

$$\mathcal{M}_4^{(s_\mathrm{a}, s_\mathrm{b})}(p_1, -p_2, p_3, -p_4)$$
$$= \frac{-1}{t} \sum_{\pm} \mathcal{M}_3^{(s_\mathrm{a})}(p_1, -p_2, k^\pm) \mathcal{M}_3^{(s_\mathrm{b})}(p_3, -p_4, -k^\mp) + \mathcal{O}(t^0)$$

4-pt “classical amplitude” from HCL

Guevara, AO, Vines ’19



$$\gamma = \frac{1}{\sqrt{1-v^2}} = \frac{p_{\rm a}\cdot p_{\rm b}}{m_{\rm a} m_{\rm b}} \to u_{\rm a}\cdot u_{\rm b}$$

$$\mathcal{M}_4 = \frac{-(\kappa/2)^2\gamma^2}{m_{\rm a}^{2s_{\rm a}-2} m_{\rm b}^{2s_{\rm b}-2} t}\Big((1-v)^2 \big\{U_{12}\langle 1|\big\}^{\odot 2s_{\rm a}} e^{-k\cdot a_{\rm a}} |1\rangle^{\odot 2s_{\rm a}} \big\{U_{34}[3|\big\}^{\odot 2s_{\rm b}} e^{-k\cdot a_{\rm b}} |3]^{\odot 2s_{\rm b}}$$
$$+(1+v)^2 \big\{U_{12}[1|\big\}^{\odot 2s_{\rm a}} e^{k\cdot a_{\rm a}} |1]^{\odot 2s_{\rm a}} \big\{U_{34}\langle 3|\big\}^{\odot 2s_{\rm b}} e^{k\cdot a_{\rm b}} |3\rangle^{\odot 2s_{\rm b}}\Big)$$

Remove parity-oddness using

$$k\cdot a_{\rm a,b} = ik\cdot w * a_{\rm a,b}, \qquad [w * a_{\rm a,b}]_\mu = \frac{\epsilon_{\mu\nu\rho\sigma} a_{\rm a,b}^\nu p_{\rm a}^\rho p_{\rm b}^\sigma}{m_{\rm a} m_{\rm b} \gamma v}$$

$$\langle \mathcal{M}_4(k)\rangle = -\left(\frac{\kappa}{2}\right)^2 \frac{m_{\rm a}^2 m_{\rm b}^2}{k^2}\gamma^2 \sum_\pm (1\pm v)^2 \exp[\pm i(k\cdot w * a_0)], \quad a_0^\mu = a_{\rm a}^\mu + a_{\rm b}^\mu$$

4-pt scattering function

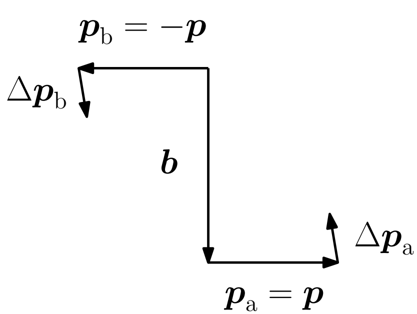
Guevara, AO, Vines '19

from momentum transfer/mismatch k^μ

$$\langle \mathcal{M}_4(k)\rangle = -\Big(\frac{\kappa}{2}\Big)^2 \frac{m_\mathrm{a}^2 m_\mathrm{b}^2}{k^2}\gamma^2 \sum_{\pm}(1\pm v)^2 \exp[\pm i(k\cdot w * a_0)]$$

to impact parameter b^μ

$$\begin{aligned}\langle \mathcal{M}_4(b)\rangle &= \int \hat{d}^4k\,\hat{\delta}(2p_\mathrm{a}\cdot k)\hat{\delta}(2p_\mathrm{b}\cdot k)e^{-ik\cdot b}\langle \mathcal{M}_4(k)\rangle \\ &= -Gm_\mathrm{a}m_\mathrm{b}\frac{\gamma}{v}\sum_{\pm}(1\pm v)^2 \log\sqrt{-(b\mp w * a_0)^2}\end{aligned}$$



$$b^2 < 0$$
$$b\cdot p_\mathrm{a} = 0$$
$$b\cdot p_\mathrm{b} = 0$$

eikonal Fourier transform e.g. in Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove '18

Linear and angular impulses from scattering function

Guevara, AO, Vines '19

$$\langle \mathcal{M}_4(b) \rangle = -G m_{\rm a} m_{\rm b} \frac{\gamma}{v} \sum_{\pm} (1 \pm v)^2 \log \sqrt{-(b \mp w * a_0)^2}$$

$$\Delta p_{\rm a}^{\mu} = \left\langle\!\!\left\langle \int \hat{d}^4 k \, \hat{\delta}(2p_{\rm a} \cdot k) \hat{\delta}(2p_{\rm b} \cdot k) k^{\mu} e^{-ik\cdot b/\hbar} i\mathcal{M}_4(k) \right\rangle\!\!\right\rangle = -\frac{\partial}{\partial b_{\mu}} \langle \mathcal{M}_4(b) \rangle$$

$$\Delta a_{\rm a}^{\mu} = \frac{1}{m_{\rm a}} \left\langle\!\!\left\langle \int \hat{d}^4 k \, \hat{\delta}(2p_{\rm a} \cdot k) \hat{\delta}(2p_{\rm b} \cdot k) e^{-ik\cdot b/\hbar} \right.\right.$$

$$\left.\left. \times \left(-\frac{i}{m_{\rm a}^2} p_{\rm a}^{\mu} S_{\rm a}^{\nu} k_{\nu} \mathcal{M}_4(k) + \left[S_{\rm a}^{\mu}, i\mathcal{M}_4(k) \right] \right) \right\rangle\!\!\right\rangle$$

$$\stackrel{*}{=} \frac{1}{m_{\rm a}^2} \left[p_{\rm a}^{\mu} a_{\rm a}^{\nu} \frac{\partial}{\partial b^{\nu}} - \epsilon^{\mu\nu\rho\sigma} p_{{\rm a}\nu} a_{{\rm a}\rho} \frac{\partial}{\partial a_{\rm a}^{\sigma}} \right] \langle \mathcal{M}_4(b) \rangle$$

Complete match to 1PM classical solution!

Vines '17

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*Relied on little-group $\mathrm{so}(3)$ algebra of $S_{\rm a}^{\mu}$ in rest frame of $p_{\rm a}$, i.e.

$$[S_{\rm a}^{\mu}, S_{\rm a}^{\nu}] = \frac{i}{m_{\rm a}} \epsilon^{\mu\nu\rho\sigma} p_{{\rm a}\rho} S_{{\rm a}\sigma} \quad \Rightarrow \quad [S_{\rm a}^{\mu}, \mathcal{M}_4] = \frac{i}{m_{\rm a}} \epsilon^{\mu\nu\rho\sigma} p_{{\rm a}\nu} S_{{\rm a}\rho} \frac{\partial \mathcal{M}_4}{\partial S_{\rm a}^{\sigma}}.$$

1PM classical solution for general spin orientations

Vines '17

Linear and angular impulses

$$\Delta p_{\rm a}^{\mu} = G m_{\rm a} m_{\rm b} \Re Z^{\mu}$$

$$\Delta a_{\rm a}^{\mu} = -\frac{G m_{\rm b}}{m_{\rm a}} \left[p_{\rm a}^{\mu} (a_{\rm a} \cdot \Re Z) + \epsilon^{\mu\nu\rho\sigma} (\Im Z_{\nu}) p_{{\rm a}\rho} a_{{\rm a}\sigma} \right]$$

in terms of an auxiliary complex vector

$$Z^{\mu} = \frac{\gamma}{v} \sum_{\pm} \left(1 \pm v\right)^{2} [\eta^{\mu\nu} \mp i(*w)^{\mu\nu}] \frac{(b \mp w * a_0)_{\nu}}{(b \mp w * a_0)^2}$$

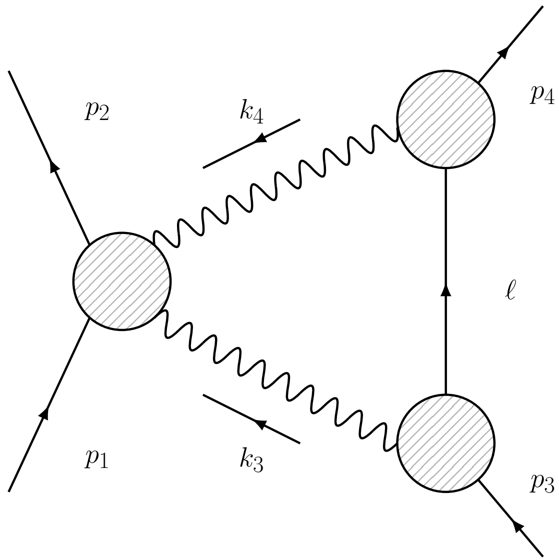
Z^{μ} automatic from scattering function $\langle \mathcal{M}_4(b) \rangle$

$$\frac{\partial}{\partial b^{\mu}} \langle \mathcal{M}_4(b) \rangle = -G m_{\rm a} m_{\rm b} \Re Z_{\mu}, \qquad \frac{\partial}{\partial a^{\mu}} \langle \mathcal{M}_4(b) \rangle = G m_{\rm a} m_{\rm b} \Im Z_{\mu}$$

2PM for aligned spins

Classical contributions from loops

- 1 loop triangles with massive propagators



- 1 loop boxes contribute to $\Delta p^{\mu}_{\mathrm{a,b}}$ Kosower, Maybee, O'Connell '18

  but not to scattering angle θ

  Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove '18

  Bern, Luna, Roiban, Shen, Zeng '20

- higher loops discussed in talks by Bern, Cheung, Roiban, Shen, Solon, Zeng '19, '20

  Bern, Parra-Martinez, Roiban, Ruf, Shen, Solon, Zeng '21

2PM aligned-spin scattering angle from 1 loop

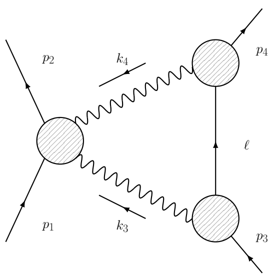
Guevara, AO, Vines '18

- Incoming spins $\perp$ to scattering plane
  $\Rightarrow$ outgoing spins stay aligned, $\Delta a_{\mathrm{a,b}} = 0$, scattering within plane
  $\Rightarrow$ scattering angle θ implies $\Delta p_{\mathrm{a,b}}$
- Use known non-spinning formula from eikonal

$$2 \sin \frac{\theta}{2} = \frac{-E}{(2 m_a m_b \gamma v)^2} \frac{\partial}{\partial b} \int \frac{d^2 \boldsymbol{k}}{(2\pi)^2} e^{-i \boldsymbol{k} \cdot \boldsymbol{b}} \lim_{s_a, s_b \to \infty} \langle \mathcal{M}_4^{(s_a, s_b)} \rangle + \mathcal{O}(G^3)$$

Kabat, Ortiz '92; Akhoury, Saotome '13
Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove '18

- Triangle contributions encode θ



- Compute triangle coeffs in HCL

Cachazo, Guevara '17; Guevara '17

- Extract angular-momentum dependence from spin exponentials

2PM aligned-spin scattering angle result

Guevara, AO, Vines '18

$$\theta_{\triangleleft} = \pi G^2 E \frac{m_b}{2v^4} \frac{\partial}{\partial b} \oint_{R>1/v} \frac{dz}{2\pi i} \frac{(1-vz)^4}{(z^2-1)^{3/2}} \left| b - z a_{\mathrm{b}} - \frac{z-v}{1-vz} a_{\mathrm{a}} \right|^{-1}$$

$$\theta^{\text{1-loop}} = \theta_{\triangleleft} + \theta_{\triangleright} = -\pi G^2 E \, \frac{\partial}{\partial b} \Big[m_{\mathrm{b}} f(a_{\mathrm{a}}, a_{\mathrm{b}}) + m_a f(a_{\mathrm{b}}, a_{\mathrm{a}}) \Big],$$

where

$$E = \sqrt{m_{\mathrm{a}}^2 + m_{\mathrm{b}}^2 + 2 m_{\mathrm{a}} m_{\mathrm{b}} \sqrt{1-v^2}},$$

$$f(\sigma, a) = \frac{1}{2a^2} \left[-b + \frac{(\jmath + \varkappa - 2a)^5}{4v\varkappa \big[(\jmath + \varkappa)^2 - (2va)^2 \big]^{3/2}} \right] + \mathcal{O}(\sigma^5),$$

$$\jmath = vb + \sigma + a, \qquad \varkappa = \sqrt{\jmath^2 - 4va(b + v\sigma)}$$

true at least through $\mathcal{O}(a^2)$, possibly wrong beyond $\mathcal{O}(a^4)$

Bini, Damour '18
Vines, Steinhoff, Buonanno '18

2PM aligned-spin scattering angle vs PN theory

$$\theta = \frac{GE}{v^2}\sum_{\pm}\frac{(1\pm v)^2}{b\pm(a_{\rm a}+a_{\rm b})} - \pi G^2 E\,\frac{\partial}{\partial b}\Big[m_{\rm b} f(a_{\rm a},a_{\rm b}) + m_{a} f(a_{\rm b},a_{\rm a})\Big] + \mathcal{O}(G^3)$$

(taken through $\mathcal{O}(G^2S^4)$)

- agrees with older PN results through NNLO S^2

  Porto, Rothstein '06, '08; Levi '08, '10; Perrodin '10; Porto '10
  Levi '11; Levi, Steinhoff '14 '15 '16

- agrees with newer PM results through S^2

  Bern, Luna, Roiban, Shen, Zeng '20
  Liu, Porto, Yang '21
  Kosmopoulos, Luna '21

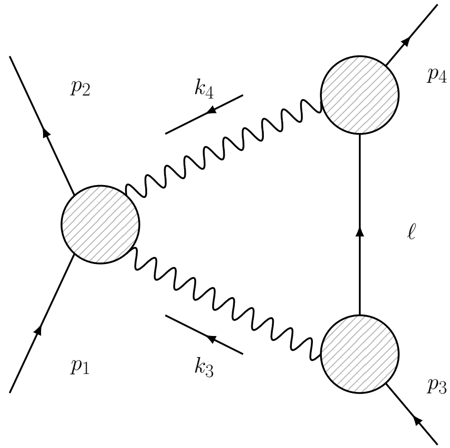
- conjectures new results at 4.5PN (NLO S^3) and 5PN (NLO S^4)

PN order	0	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5
	N	1PN		2PN		3PN		4PN		5PN			
			LO SO		NLO SO		NNLO SO		NNNLO SO				
				LO S^2		NLO S^2		NNLO S^2		NNNLO S^2			
							LO S^3		NLO S^3				
								LO S^4		NLO S^4			
											LO S^5		NLO S^5
												LO S^6	

need up to
1PM / tree
2PM / 1-loop
3PM / 2-loop
4PM / 3-loop
5PM / 4-loop
6PM / 5-loop

table by Vines as of '19

Gravitational Compton amplitude



$$\mathcal{M}_4^{(s)}(p_1, -p_2, k_3^+, k_4^-) = -\left(\frac{\kappa}{2}\right)^2 \langle 2|^{\odot 2s} \exp\left(-i\frac{k_{4\mu}\varepsilon_{4\nu}^- \sigma^{\mu\nu}}{p_1\cdot\varepsilon_4^-}\right)|1\rangle^{\odot 2s}$$

$$= -\left(\frac{\kappa}{2}\right)^2 [2|^{\odot 2s} \exp\left(-i\frac{k_{3\mu}\varepsilon_{3\nu}^+ \bar{\sigma}^{\mu\nu}}{p_1\cdot\varepsilon_3^+}\right)|1]^{\odot 2s}$$

$$= \left(\frac{\kappa}{2}\right)^2 \frac{\langle 4|1|3]^{4-2s}\big([13]\langle 42\rangle + \langle 14\rangle[32]\big)^{\odot 2s}}{(2p_1\cdot k_4)(2p_2\cdot k_4)(2k_3\cdot k_4)}$$

- first appeared in — Arkani-Hamed, Huang, Huang '17
- problematic at $s \geq 2$ — double-copy aspects in Johansson, AO '19
- alternatives proposed in — Chung, Huang, Kim, Lee '18; Falkowski, Machado '20

Summary & outlook

- Spin exponentiation pattern inherent to Kerr BH

  Guevara, AO, Vines '18
  Bautista, Guevara '19
  Guevara, AO, Vines '19
  Arkani-Hamed, Huang, O'Connell '19

- 1PM match for general spin (to all orders in spin) — Vines '17
- 2PM results for aligned spins:
  - match at presently known orders in spins
  - conjectured results for higher orders in spins
- Used HCL and KMOC formalism,
  interplay between classical-limit approaches

  Guevara '17
  Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove '18
  Cheung, Rothstein, Solon '18
  Kosower, Maybee, O'Connell '18
  Koemans Collado, Di Vecchia, Russo '19
  Bjerrum-Bohr, Cristofoli, Damgaard, Vanhove '19
  Maybee, O'Connell, Vines '19
  Damgaard, Haddad, Helset '19
  Kälin, Porto '19
  Aoude, Hadded, Helset '20
  Jakobsen, Mogull, Plefka, Steinhoff '21
  Di Vecchia, Heissenberg, Russo, Veneziano '21
  Bjerrum-Bohr, Damgaard, Planté, Vanhove '21

- Open questions at higher orders in G and spin