Double Copy Relations in Massive Theories

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Double Copy

- Double copy relations between Yang Mills and GR scattering amplitudes proved to be a useful tool for GW calculations, because they simplify the scattering amplitude calculations a lot.
- The most studied double copy is between YM and GR, however there are more examples of this relation between different theories, for example, NLSM and special Galileon, NLSM, YM and DBI, supersymmetric theories, YM and GR with higher derivative operators and many more.
  [Bern, Carrasco, Chiodaroli, Johansson, Roiban 2019]
- What makes these theories special and how general is double copy?

Massive Double Copy

- Are there examples of double copy relations between massive spin-1 and massive spin-2 theories?
- Maybe they could help to simplify the calculations in massive gravity? (for example GW, BH, Vainshtein mechanism)
- Related work on double copy with massive matter: [Plefka, Shi, Wang, Johansson, Ochirov, Carrasco, Vazquez-Holm, Haddad, Helset, Brandhuber, Chen, Travaglini, Wen]

Massive Yang Mills and Massive Gravity

It is natural to begin with massive Yang Mills theory:

[Johnson, Jones, Paranjape, 2020, Momeni, Rumbutis, Tolley, 2020]

$$\mathcal{L}_{mYM} = -\frac{1}{4}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) - \frac{1}{2}m^2\mathrm{tr}(A_\mu A^\mu),$$

and check if it is related to dRGT massive gravity theory:

[de Rham, Gabadadze, Tolley 2010]

$$\mathcal{L}_{mGR} = \frac{M_{\mathrm{Pl}}^2}{2}\sqrt{-g}R + \frac{m^2 M_{\mathrm{Pl}}^2}{4}\sqrt{-g}\sum_{n=0}^{4}\kappa_n \mathcal{U}_n[\mathcal{K}]$$

$$\mathcal{K}^\mu_\nu(f,g) = \delta^\mu_\nu - \left(\sqrt{g^{-1}f}\right)^\mu_\nu .$$

Massive Yang Mills and Massive Gravity: EFT Cutoffs

Both of these theories are effective field theories (EFTs). They are only valid up to a certain scale called the cutoff. For mYM the cutoff, Λ is given by:

$$\Lambda = \frac{m}{g},$$

while for mGR it is Λ_3 equal to:

$$\Lambda_3 = (m^2 M_{pl})^{1/3}.$$

These EFTs greatly simplify if we only look at the energies close to cutoff, which is known as decoupling limit.

Massive Yang Mills and Massive Gravity: Decoupling Limits

One of the reasons to expect a relation between these two theories is the double copy relation between their decoupling limits:

$$\lim_{m\to 0,\Lambda \text{ fixed}} \mathcal{L}_{mYM} = \text{NLSM},$$

$$\lim_{m\to 0,\Lambda_3 \text{fixed}} \mathcal{L}_{mGR} = \text{Gal},$$

$$\text{sGal} = \text{NLSM} \otimes \text{NLSM}.$$

Degrees of Freedom

To find the spectrum of double copy theory we need to decompose the product of two massive A_μ into irreps of $SO(d-1)$:

$$A_\mu \otimes A_\nu = h_{\mu\nu} \oplus B_{\mu\nu} \oplus \phi,$$

$h_{\mu\nu}$ is a massive spin-2 field, $B_{\mu\nu}$ is a massive 2-form field which is dual to a massive spin-1 field in four dimensions and ϕ is a massive scalar field.

3pt Amplitudes

In terms of polarization and momentum vectors the three-point on-shell vertex for massive Yang-Mills is exactly same as that of massless Yang-Mills:

$$A_3(1^a, 2^b, 3^c) = \sqrt{2} g f_{abc}(-\epsilon_1 \cdot \epsilon_2 \; \epsilon_3 \cdot p_1 + \epsilon_1 \cdot \epsilon_3 \; \epsilon_2 \cdot p_1 - \epsilon_1 \cdot p_2 \; \epsilon_2 \cdot \epsilon_3).$$

while 3pt dRGT MG vertex is:

$$\begin{aligned} M_3 = & i\kappa\Big((\epsilon_1^{\mu\nu} \epsilon_{3\mu\nu} \epsilon_{2\alpha\beta} p_1^\alpha p_1^\beta + 2\epsilon_{1\mu\nu} \epsilon_2^{\mu\alpha} \epsilon_{3\beta}^{\nu} p_{1\alpha} p_{2\beta} + \text{permutations}) \\ & + \frac{3}{2}(1+\kappa_3) \epsilon_1^{\mu\nu} \epsilon_{2\nu\alpha} \epsilon_{3\mu}^{\alpha} m^2 \Big), \end{aligned}$$

4pt Amplitudes

4pt massive YM amplitude can be written as:

$$A_4(1^a, 2^b, 3^c, 4^d) = g^2 \left(\frac{c_s n_s}{s - m^2} + \frac{c_t n_t}{t - m^2} + \frac{c_u n_u}{u - m^2} \right) ,$$

where colour factors satisfy Jacobi identity $c_s + c_t + c_u = 0$. The same identity is satisfied by the kinematic factors directly obtained from Feynman rules:

$$n_s + n_t + n_u \propto p_4 \cdot \epsilon_4 = 0.$$

4pt Amplitudes

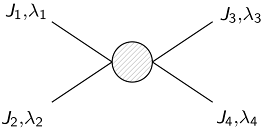
Since the colour kinematics duality is satisfied we can just square the numerators to obtain external spin-2 scattering:

$$M_4 = i\left(\frac{\kappa}{2}\right)^2\left(\frac{n_s^2}{s-m^2}+\frac{n_t^2}{t-m^2}+\frac{n_u^2}{u-m^2}\right),$$

and compare it with 4pt massive gravity amplitude:

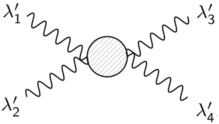
$$\begin{aligned} M_4 &= M_4^{\mathrm{mGr}}\big|_{\kappa_3=-1,\kappa_4=\frac{7}{24}} \\ &\quad - i\frac{3}{16}\kappa^2 m^4\left(\frac{\epsilon_{1\mu\nu}\epsilon_2^{\mu\nu}\epsilon_{3\alpha\beta}\epsilon_4^{\alpha\beta}}{s-m^2}+\frac{\epsilon_{1\mu\nu}\epsilon_3^{\mu\nu}\epsilon_{2\alpha\beta}\epsilon_4^{\alpha\beta}}{t-m^2}+\frac{\epsilon_{1\mu\nu}\epsilon_4^{\mu\nu}\epsilon_{3\alpha\beta}\epsilon_2^{\alpha\beta}}{u-m^2}\right).\end{aligned}$$

4pt Amplitudes

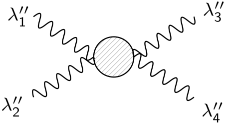


$$=$$

$$= \sum_{\{\lambda'\},\{\lambda''\}} C^{J_1,\lambda_1}_{\lambda'_1\lambda''_1} \dots C^{J_4,\lambda_4}_{\lambda'_4\lambda''_4}$$



$$\otimes$$



4pt Amplitudes

Considering other combinations if n's we constructed 4pt amplitudes with external ϕ and $B_{\mu\nu}$. Then from them we constructed an action with Λ_3 cutoff matching all of these 4pt double copy amplitudes:

$$
\begin{aligned}
S = \int d^4x\sqrt{-g}\bigg(& \frac{2}{\kappa^2}R[g] + \frac{m^2}{\kappa^2}\sum_{n=2}^{4}\kappa_n\,\mathcal{U}_n\,[\mathcal{K}] \\
& - \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}m^2A_\mu A^\mu \\
& -\frac{1}{2}\mathcal{K}^{\mu\nu}F_{\nu\alpha}F_\mu{}^\alpha + \frac{1}{8}\mathcal{K}^\mu_\mu F_{\nu\alpha}F^{\nu\alpha} - \frac{1}{4}\nabla_\mu\phi\nabla_\nu\phi\left(\mathcal{K}^{\mu\nu} - g^{\mu\nu}\mathcal{K}^\alpha_\alpha\right) - \frac{\sqrt{3}}{2}\frac{m^2}{\kappa}\phi\left(\mathcal{K}^{\mu\nu}\mathcal{K}_{\mu\nu} - \mathcal{K}^\mu_\mu\mathcal{K}^\nu_\nu\right) \\
& + \frac{1}{24\sqrt{3}}\frac{\kappa}{m^2}\phi\left([\Phi]^2 - [\Phi^2]\right) + \frac{-3}{8\sqrt{3}}\kappa m^2\phi^3 - \frac{\kappa}{\sqrt{3}}m^2A^\mu A_\mu\phi - \frac{\kappa}{16\sqrt{3}}F^{\mu\nu}F_{\mu\nu}\phi + \text{quartic contact terms}\bigg).
\end{aligned}
$$

Decoupling Limit

However taking the decoupling of this action does not give a special Galileon action as expected, but rather a bi-Galileon theory coupled to spin-2:

$$\mathcal{L}_{DL} = \frac{1}{2}\tilde{h}^{\mu\nu}\mathcal{E}\tilde{h}_{\mu\nu} - \frac{3}{4}(\partial\pi)^2 - \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}(\partial\chi)^2 + \mathcal{L}^{\text{int}}_{\text{bi-Galileon}}(\phi,\pi) + \frac{1}{24\Lambda_3^6}\varepsilon^{abcd}\varepsilon_{ABCD}\tilde{h}_a^A\tilde{\Pi}_b^B\tilde{\Pi}_c^C\tilde{\Pi}_d^D + \mathcal{L}_{A,V},$$

showing that double copy does not commute with taking DL!

Decoupling Limit

The reason for this is CK duality:

$$
\begin{aligned}
n_s &= \frac{s-m^2}{m^3}\Sigma(s,t,u) + \frac{1}{m^2}\hat{n}_s, \\
n_t &= \frac{t-m^2}{m^3}\Sigma(s,t,u) + \frac{1}{m^2}\hat{n}_t, \\
n_u &= \frac{u-m^2}{m^3}\Sigma(s,t,u) + \frac{1}{m^2}\hat{n}_u\,,
\end{aligned}
$$

where $\hat{n}$'s are finite in DL.

Decoupling Limit

$$
\begin{aligned}
A_4^{\text{mYM}} &= g^2 \left(\frac{c_s n_s}{s - m^2} + \frac{c_t n_t}{t - m^2} + \frac{c_u n_u}{u - m^2} \right) \\
&+ \frac{1}{m\Lambda^2} \Sigma(s, t, u) \left(c_s + c_t + c_u \right) \\
&= \frac{1}{\Lambda^2} \left(\frac{c_s \hat{n}_s}{s - m^2} + \frac{c_t \hat{n}_t}{t - m^2} + \frac{c_u \hat{n}_u}{u - m^2} \right) ,
\end{aligned}
$$

$$
\begin{aligned}
M_4 =& \frac{1}{M_{\rm Pl}^2} \left(\frac{n_s n'_s}{s - m^2} + \frac{n_t n'_t}{t - m^2} + \frac{n_u n'_u}{u - m^2} \right) \\
\sim& \frac{\Sigma \Sigma'}{M_{\rm Pl}^2 m^6} \left((s - m^2) + (t - m^2) + (u - m^2) \right) + \dots \\
&\sim \frac{\Sigma \Sigma'}{\Lambda_3^6} + \dots
\end{aligned}
$$

Decoupling Limit

$\Sigma(s,t,u)$ term is responsible for this discontinuity in DL:

- mYM $\xrightarrow{\text{DL}}$ $\Sigma(s,t,u)$ cancels, we get NLSM $\xrightarrow{\text{DC}}$ sGal
- mYM $\xrightarrow{\text{DC}}$ $\Sigma(s,t,u)$ remains for CK, we get MG $\xrightarrow{\text{DL}}$ bi-Gal ($\Sigma(s,t,u)$ still survives)

5pt Amplitudes

However at 5pt there is a problem: squaring massive YM numerators obeying CK duality gives spurious poles...

[Johnson, Jones, Paranjape, 2020]

Massive Double Copy for n-pt Amplitudes

The n-point tree level gauge theory amplitude, A_n, can be written in matrix form as:

$$A_n = g^{n-2} c^T D^{-1} n.$$

The Jacobi identities and CK duality in matrix form are

$$Mc = 0 \rightarrow Mn = 0,$$

then the double copy is

$$M_n = i \left(\frac{\kappa}{2}\right)^{n-2} n^T D^{-1} n.$$

Shifting the Numerators

The kinematics factors directly calculated from Feynman diagrams may not satisfy Jacobi relations, therefore they must be shifted as

$$n \to n + \Delta n, \tag{1}$$

such that the amplitude is unchanged, which can be achieved by setting

$$D^{-1}\Delta n = M^T v, \tag{2}$$

where v is some vector. CK duality :

$$M(n + \Delta n) = 0, \tag{3}$$

which combined with Eq. (2) gives

$$MDM^T v = -Mn. \tag{4}$$

Shifting the Numerators

We need to invert the non-zero block matrix in MDM^T, we call it A:

$$v = -(A^{-1}U, 0, .., 0), \tag{5}$$

where

$$Mn = (U, 0, ..., 0). \tag{6}$$

Final Expression for Double Copy

Substituting the solution for Δn:

$$
\begin{aligned}
-i\left(\frac{\kappa}{2}\right)^{-(n-2)} M_n &= (n+\Delta n)^T D^{-1}(n+\Delta n) \\
&= (n+\Delta n)^T D^{-1} n + (M(n+\Delta n))^T v \\
&= n^T D^{-1} n + \Delta n^T D^{-1} n \\
&= n^T D^{-1} n + v^T M n \\
&= n^T D^{-1} n - U^T A^{-1} U,
\end{aligned} \tag{7}
$$

Spurious poles

In addition to physical poles, D_{ij}, the double copy amplitude has poles coming from $\det(A)$, which in general is some complicated polynomial of Mandelstam variables.
For example for 5pt massive YM:

$$\det(A) = m^8 (\prod_{i<j} D_{ij}) P(s_{kl}, m), \tag{8}$$

$$\begin{aligned}
P(s_{kl}, m) = {} & 320m^8 + 36m^6(9s_{12} + 4(s_{13} + s_{14} + s_{23} + s_{24})) \\
& + m^4 \Big(117s_{12}^2 + 108s_{12}(s_{13} + s_{14} + s_{23} + s_{24}) + 4\,(s_{13}(13s_{14} + 4s_{23} + 17s_{24}) \\
& \qquad +4s_{13}^2 + 4s_{14}^2 + 17s_{14}s_{23} + 4s_{14}s_{24} + 4s_{23}^2 + 13s_{23}s_{24} + 4s_{24}^2)\Big) \\
& + 2m^2 \Big(9s_{12}^3 + 13s_{12}^2(s_{13} + s_{14} + s_{23} + s_{24}) + s_{12}\,(s_{13}(10s_{14} + 6s_{23} + 17s_{24}) \\
& \qquad +4s_{13}^2 + 4s_{14}^2 + s_{14}(17s_{23} + 6s_{24}) + 2(2s_{23} + s_{24})(s_{23} + 2s_{24})) \\
& \qquad +2\,\Big(s_{13}^2(s_{14} + 2s_{24}) + s_{13}\,\Big(s_{14}^2 + s_{14}(s_{23} + s_{24}) + s_{24}(s_{23} + 2s_{24})\Big) \\
& \qquad +s_{23}\,\Big(s_{24}(s_{14} + s_{23}) + 2s_{14}(s_{14} + s_{23}) + s_{24}^2\Big)\Big)\Big) \\
& + 2s_{24}\,\Big(s_{23}\,\Big(s_{12}^2 + s_{12}(s_{13} + s_{14}) - s_{13}s_{14}\Big) + s_{12}(s_{12} + s_{13})(s_{12} + s_{13} + s_{14})\Big) \\
& + (s_{12}(s_{12} + s_{13} + s_{14}) + s_{23}(s_{12} + s_{14}))^2 + s_{24}^2(s_{12} + s_{13})^2,
\end{aligned} \tag{9}$$

Singular Matrix A

For more general masses in the spectrum we can have $\det A = 0$ (for example when $m = 0$). Then we can find Δn satisfying CK ONLY if U lives in the space orthogonal to null vectors of A:

$$U.\mathrm{null}(A) = 0.$$

These conditions in massless Yang-Mills are equivalent to BCJ relations.
For 5pt massless YM amplitudes A has rank 5 and in the 5 dimensional subspace its inverse does not have any spurious poles.

Spectral Conditions

If the propagators in a massive theory obey they same algebraic relations as the massless ones, for example at 4pt,

$$s - m_{12}^2 + t - m_{13}^2 + u - m_{14}^2 = 0,$$

A is guaranteed to have the same rank as in the massless case. Such constraints on masses are known as spectral conditions [Johnson, Jones, Paranjape, 2020].

However, the interactions still need to satisfy the BCJ relations.

Kaluza-Klein Theories

Kaluza-Klein (KK) theories satisfy these spectral conditions.
We tried to start with KK spectrum (conservation of mass) and operators the same as in compactifications of [Broedel, Dixon, 2012]

$$-\frac{1}{4}\mathrm{tr}(F^2) + \frac{G_{5d}}{\Lambda^2}\mathrm{tr}(F^3) - \frac{9G_{5d}^2}{16\Lambda^4}\mathrm{tr}([F,F]^2)$$

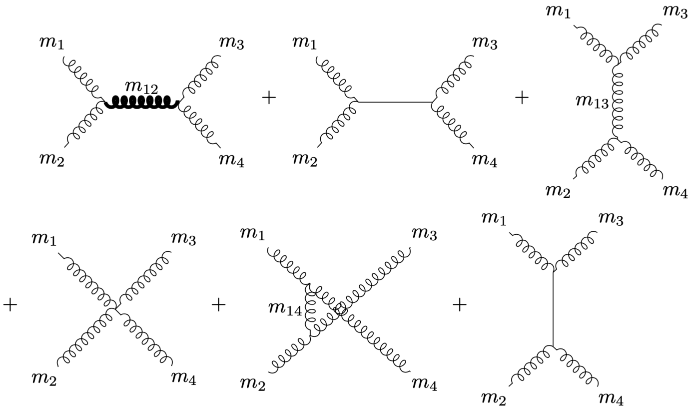
and construct a general EFT for mass spin-1 fields, then impose BCJ relations to check if we can get something other than KK compatible with CK duality.[Momeni, Rumbutis, Tolley, 2020]

Kaluza-Klein Theories

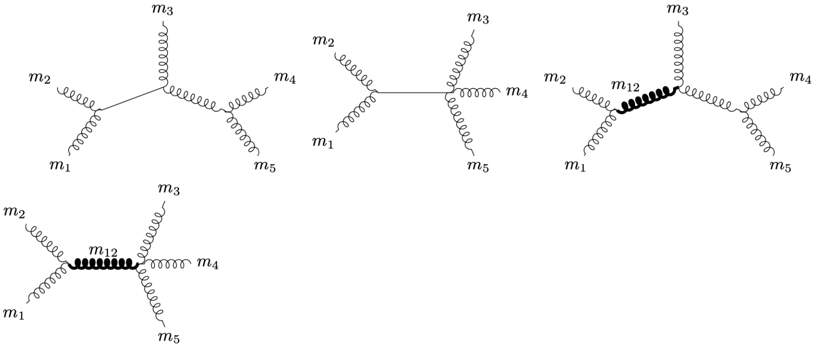
$$\mathcal{L} = \mathcal{L}^{F^2} + \mathcal{L}^{F^3} + \mathcal{L}^{F^4},$$

$$\begin{aligned}
\mathcal{L}^{F^2} &= \mathrm{tr}\left(-\frac{1}{2} D_\mu \phi D^\mu \phi - \frac{1}{4} F^0_{\mu\nu} F^{0\mu\nu} - \frac{1}{2} \sum_{i \in \mathbb{Z}_{\neq 0}} \frac{1}{2} |D_\mu A^i_\nu - D_\nu A^i_\mu|^2 - 2 g_i A^{i\mu} A^{-i\nu} F^0_{\mu\nu} + m_i^2 |A^i_\mu|^2 \right) \\
&\quad + \mathcal{L}_{AAA} + \mathcal{L}_{AA\phi} + \mathcal{L}_{AA\phi\phi} + \mathcal{L}_{AAAA}, \\
\mathcal{L}^{F^3} &= \frac{1}{\Lambda^2} \mathrm{tr}\left(G F^0_{\mu\nu} F^{0\nu\rho} F^{0\mu}_\rho + \sum_{i \in \mathbb{Z}_{\neq 0}} \left(3 G_i D_\mu A^{i\nu} D_\nu A^{-i\rho} F^\mu_{0\rho} \right) \right) \\
&\quad \mathcal{L}^{F^3}_{AAA1} + \mathcal{L}^{F^3}_{AAA2} + \mathcal{L}^{F^3}_{AA\phi} + \mathcal{L}^{F^3}_{A\phi\phi} + \mathcal{L}^{F^3}_{AAAA1} + \mathcal{L}^{F^3}_{AAAA2} + \mathcal{L}^{F^3}_{AA\phi\phi1} + \mathcal{L}^{F^3}_{AA\phi\phi2} + \mathcal{L}^{F^3}_{AAA\phi1} + \mathcal{L}^{F^3}_{AAA\phi2} \\
&\quad + \mathcal{L}^{F^3}_{AAAAA} + \mathcal{L}^{F^3}_{\phi AAAA} + \mathcal{L}^{F^3}_{\phi\phi AAA1} + \mathcal{L}^{F^3}_{\phi\phi AAA2} \\
\mathcal{L}^{F^4} &= \mathcal{L}^{F^4}_{AAAA1} + \mathcal{L}^{F^4}_{AAAA2} + \mathcal{L}^{F^4}_{AAA\phi1} + \mathcal{L}^{F^4}_{AAA\phi2} + \mathcal{L}^{F^4}_{AA\phi\phi1} + \mathcal{L}^{F^4}_{AA\phi\phi2} + \mathcal{L}^{F^4}_{AA\phi\phi3} + \mathcal{L}^{F^4}_{\phi\phi\phi\phi} \\
&\quad + \mathcal{L}^{F^4}_{AAAAA1} + \mathcal{L}^{F^4}_{AAAAA2} + \mathcal{L}^{F^4}_{\phi AAAA1} + \mathcal{L}^{F^4}_{\phi AAAA2} + \mathcal{L}^{F^4}_{\phi AAAA3} + \mathcal{L}^{F^4}_{\phi\phi AAA1} + \mathcal{L}^{F^4}_{\phi\phi AAA2} + \mathcal{L}^{F^4}_{\phi\phi AAA3} \\
&\quad + \mathcal{L}^{F^4}_{\phi\phi\phi AA1} + \mathcal{L}^{F^4}_{\phi\phi\phi AA2},
\end{aligned}$$

Kaluza-Klein Theories: 4pt Diagrams



Kaluza-Klein Theories: 5pt Diagrams



Kaluza-Klein Theories

In every individual scattering process there seems to be some freedom to satisfy the 4pt BCJ relation ($n_s + n_t + n_u = 0$):

$$
\begin{aligned}
i\mathcal{A}_4 \propto &\left(V_{g_{12}}^{AAA} + V_{G_{12}}^{AAA1} + V_{\hat{G}_{12}}^{AAA2}\right) \frac{i}{s - m_{12}^2} \left(V_{g_{34}}^{AAA} + V_{G_{34}}^{AAA1} + V_{\hat{G}_{34}}^{AAA2}\right) \\
&+ \left(V_{g_{13}}^{AAA} + V_{G_{13}}^{AAA1} + V_{\hat{G}_{13}}^{AAA2}\right) \frac{i}{t - m_{13}^2} \left(V_{g_{24}}^{AAA} + V_{G_{24}}^{AAA1} + V_{\hat{G}_{24}}^{AAA2}\right) \\
&+ \left(V_{g_{14}}^{AAA} + V_{G_{14}}^{AAA1} + V_{\hat{G}_{14}}^{AAA2}\right) \frac{i}{u - m_{14}^2} \left(V_{g_{23}}^{AAA} + V_{G_{23}}^{AAA1} + V_{\hat{G}_{23}}^{AAA2}\right) \\
&+ \left(V_{g_{1234}}^{AAAA} + V_{G_{1234}}^{AAAA1} + V_{\hat{G}_{1234}}^{AAAA2} + V_{c_{1234}}^{AAAA1} + V_{C_{1234}}^{AAAA2}\right),
\end{aligned}
$$

$$
\begin{aligned}
&g_{1234} - \frac{18 m_1 m_2 m_3 m_4}{\Lambda^4} c_{1234} = g_{12} g_{34} = g_{13} g_{24} = g_{14} g_{23} = \frac{G_{1234}^2}{c_{1234}}, \\
&G_{1234} = G_{12} g_{34} = G_{13} g_{24} = G_{14} g_{23} = g_{12} G_{34} = g_{13} G_{24} = g_{14} G_{23}, \\
&G_{ij} = \hat{G}_{ij}, \quad G_{1234} = \hat{G}_{1234}, \quad c_{1234} = C_{1234}, \\
&c_{1234} = G_{12} G_{34} = G_{13} G_{24} = G_{14} G_{23}.
\end{aligned}
$$

Kaluza-Klein Theories: 4pt BCJ Relation Constraints

	coefficient	CK constrained value		coefficient	CK constrained value
$\mathcal{L}_{AAA}$	g_{ijk}	g	$\mathcal{L}^{F^3}_{AA\phi\phi 1}$	G_{ijss}	$g(2G_{0ss} - G) - \frac{3\sqrt{2}m_i^2 G^2}{\Lambda^2}$
$\mathcal{L}_{AA\phi}$	g'_{ijs}	$m_i g$	$\mathcal{L}^{F^3}_{AA\phi\phi 2}$	$\hat{G}_{ijss}$	gG
$\mathcal{L}_{AAAA}$	g_{ijkl}	$g^2 + \frac{m_i m_j m_k m_l}{\Lambda^4} G^2$	$\mathcal{L}^{F^3}_{AAA\phi 1}$	$\hat{G}_{ijks}$	gG
$\mathcal{L}_{AA\phi\phi}$	g_{ijss}	g^2	$\mathcal{L}^{F^3}_{AAA\phi 2}$	G_{ijks}	gG
$\mathcal{L}^{F^4}_{AAF0}$	g_i	$g_i = g - G\frac{3\sqrt{2}m_i^2}{\Lambda^2}$	$\mathcal{L}^{F^4}_{AAAA1}$	c_{ijkl}	G^2
$\mathcal{L}^{F^3}_{AAF0}$	G_i	$G_i = G$	$\mathcal{L}^{F^4}_{AAAA2}$	C_{ijkl}	G^2
$\mathcal{L}^{F^3}_{AAA1}$	G_{ijk}	G	$\mathcal{L}^{F^4}_{AAA\phi 1}$	c_{ijks}	G^2
$\mathcal{L}^{F^3}_{AAA2}$	$\hat{G}_{ijk}$	G	$\mathcal{L}^{F^4}_{AAA\phi 2}$	C_{ijks}	G^2
$\mathcal{L}^{F^3}_{AA\phi}$	G'_{ijs}	$m_i G$	$\mathcal{L}^{F^4}_{AA\phi\phi 1}$	c_{ijss}	$G_{0ss} G$
$\mathcal{L}^{F^3}_{A\phi\phi}$	G_{0ss}	not constrained	$\mathcal{L}^{F^4}_{AA\phi\phi 2}$	$c^{(2)}_{ijss}$	G^2
$\mathcal{L}^{F^3}_{AAAA1}$	G_{ijkl}	gG	$\mathcal{L}^{F^4}_{AA\phi\phi 3}$	$c^{(3)}_{ijss}$	$G_{0ss} G + \frac{\sqrt{2}\Lambda^2}{6m_i^2} g\,(G_{0ss} - G)$
$\mathcal{L}^{F^3}_{AAAA2}$	$\hat{G}_{ijkl}$	gG	$\mathcal{L}^{F^4}_{\phi\phi\phi\phi}$	$c_{\phi 4}$	G^2_{0ss}

Table: Coefficients of the interactions constrained the 4pt BCJ relation.

Kaluza-Klein Theories: 4 and 5 pt BCJ Relation Constraints

- Imposing 4pt BCJ relation on all 4pt amplitudes leaves us with a single free coefficient which is fixed by 5pt BCJ relations.
- All of the interactions must be the same as in the KK theory.
- This suggests that these constraints are very strong.

Massive Double Copy in 3d

Under closer inspection of the polynomial $P(s_{kl}, m)$ in

$$\det(A) = m^8 (\prod_{i<j} D_{ij}) P(s_{kl}, m),$$

we find that it can be expressed as:

$$P(s_{kl}, m) = 16 \det(p_i \cdot p_j). \quad i, j < 5, \tag{10}$$

which is zero in $d < 4$.

Further Work

- Are there any other massive spin-1 theories compatible with CK?
- Does the double copy formalism have to be changed for massive theories? (different rank of A and different number of BCJ relations?)
- Are there any new examples of massive double copy in lower spacetime dimensions?