String correlators on AdS_3

Andrea Dei
Harvard University

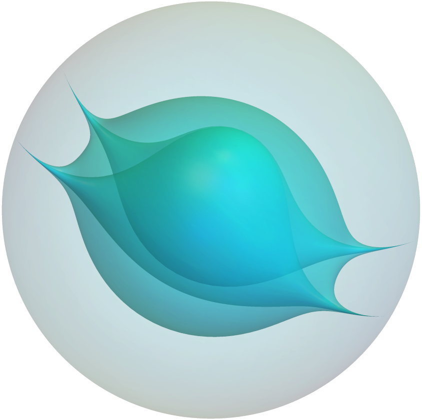
Based on 2105.12130 and work in progress with Lorenz Eberhardt

Motivation

- String theory on AdS is important to understand holography
- The simplest setup is AdS_3 with pure NSNS flux: string theory can be studied via the $\mathrm{SL}(2,\mathbb{R})$ WZW model

  [Giveon, Kutasov, Seiberg '98; Maldacena Ooguri '00-'01, ...]
- $\mathrm{SL}(2,\mathbb{R})$ WZW model is suspected to be completely solvable, but this is still a work in progress...

What's the novelty? Spectral flow!



- The string can have non-trivial winding around the insertion points
- Not many results are known in the literature

[Maldacena, Ooguri '01; Minces, Nunez, Herscovich '05; Cagnacci, Iguri '13]

Setup

- We consider genus zero bosonic string theory

  [Teschner '97-'99; Maldacena, Ooguri, '01]

- Everything will be a consequence of symmetries

- The symmetry algebra on the worldsheet is $\mathfrak{sl}(2,\mathbb{R})_k$

$$
\begin{aligned}
[J^3_m, J^3_n] &= -\frac{k}{2}\, m\, \delta_{m+n,0} \ , \\
[J^3_m, J^\pm_n] &= \pm J^\pm_{m+n} \ , \\
[J^+_m, J^-_n] &= k\, m\, \delta_{m+n,0} - 2J^3_{m+n} \ .
\end{aligned}
$$

Unflowed vertex operators

$$V_j^0(x, z)$$

$\mathfrak{sl}(2,\mathbb{R})$ spin

Boundary coordinate

Worldsheet coordinate

$$
\begin{aligned}
J^+(\zeta)V_j^0(x;z) &\sim \frac{\partial_x V_j^0(x;z)}{\zeta - z} \ , \\
J^3(\zeta)V_j^0(x;z) &\sim \frac{(j + x\partial_x)V_j^0(x;z)}{\zeta - z} \ , \\
J^-(\zeta)V_j^0(x;z) &\sim \frac{(2jx + x^2\partial_x)V_j^0(x;z)}{\zeta - z} \ .
\end{aligned}
$$

Notice that $j = h \equiv$ spacetime conformal dimension.

Flowed vertex operators

Spectral flow, $w \in \mathbb{Z}_{\geq 0}$

$$V^{w}_{j,h}(x, z)$$

Worldsheet coordinate

$\mathfrak{sl}(2, \mathbb{R})$ spin

Boundary coordinate

Spacetime conformal weight

OPEs of the currents with these vertex operators have higher-order poles.

Our strategy: symmetries

We require correlators to obey

- Global symmetries
- Affine symmetry
- Knizhnik–Zamolodchikov equation
- Null vector equation

We solve these constraints case by case for all the w_i's with $\sum w_i \leq 10$ and make a proposal for 3- and 4-point functions

Our strategy: symmetries

We require correlators to obey

- Global symmetries
- Affine symmetry
- Knizhnik–Zamolodchikov equation
- Null vector equation

Unflowed correlator $\left\langle \prod_{i=1}^{n} V^0_{j_i}(x_i, z_i) \right\rangle$ $\Longrightarrow$ Flowed correlator $\left\langle \prod_{i=1}^{n} V^{w_i}_{j_i,h_i}(x_i, z_i) \right\rangle$

Global Ward identities

We can use global Ward identities to set

$$x_1 = z_1 = 0 \ , \qquad x_2 = z_2 = 1 \ , \qquad x_3 = z_3 = \infty \ .$$

The x_i and z_i dependence of 3-point functions is fully fixed while 4-point functions still depend on the cross ratios

$$x \equiv x_4 \ , \qquad z \equiv z_4 \ .$$

Affine symmetry

Affine symmetry implies additional constraints, for example

$$
\begin{aligned}
&\left(h_1 - \frac{k}{2} - 1 + j_1\right) \left\langle V^1_{h_1-1} V^1_{h_2} V^1_{h_3} \prod_{i=4}^{n} V^1_{h_i} \right\rangle \\
&= \left[h_1 - h_2 - h_3 + \sum_{i=4}^{n} \frac{h_i \left(z_i - 2x_i\right) + x_i \left(z_i - x_i\right) \partial_{x_i}}{z_i} \right] \left\langle V^1_{h_1} V^1_{h_2} V^1_{h_3} \prod_{i=4}^{n} V^1_{h_i} \right\rangle \\
&\quad + \sum_{i=2}^{n} \left(h_i - \frac{k}{2} + 1 - j_i\right) \frac{x_i^2}{z_i^2} \left\langle V^1_{h_1} V^1_{h_2} V^1_{h_3} V^1_{h_i+1} \prod_{\ell \neq \{1,2,3,i\}} V^1_{h_\ell} \right\rangle
\end{aligned}
$$

when all the vertex operators have $w_i = 1$.

[Eberhardt, Gaberdiel, Gopakumar '19]

How to solve local Ward identities?

We Fourier transform in order to turn recursion relations into PDEs

$$V^{w}_{j,h,\bar{h}}(x;z) = \int \mathsf{d}^2 y\, y^{\frac{kw}{2}+j-h-1}\,\bar{y}^{\frac{kw}{2}+j-\bar{h}-1}\, V^{w}_{j}(x;y;z)$$

h_i recursion relations are traded for a system of linear PDEs in the y_i. For example,

$$\begin{aligned}0 = &\big((1-y_1)\,(1-z)^2\partial_{y_1} + (1-y_2)\,y_2(1-z)^2\partial_{y_2} + (1-y_3)\,(1-z)^2\partial_{y_3}\\ &+\big(x^2-2zy_4x+2y_4x-2x+z^2y_4-y_4+1\big)\,\partial_{y_4}\\ &+(-xk+k-zj_1+j_1+zj_2-j_2-zj_3+j_3-2xj_4+zj_4+j_4\\ &\quad -2zj_2y_2+2j_2y_2)\,(z-1)-(x-1)(x-z)(z-1)\partial_x\big)\,\langle\ldots\rangle\end{aligned}$$

The general solution

The solution of the global symmetries in the unflowed sector is

$$\left\langle \prod_{i=1}^{4} V_{j_i}^{0}(x_i,z_i) \right\rangle = x_{12}^{-j_1-j_2+j_3-j_4} x_{13}^{-j_1+j_2-j_3+j_4} \times \quad x_{23}^{j_1-j_2-j_3+j_4} x_{34}^{-2j_4} F\left(\frac{x_{23}\, x_{14}}{x_{12}\, x_{34}}, z\right)$$

The solution of affine symmetry constraints in the flowed sector has a similar structure, with x_{ij} traded for the 'generalised differences' $X_{ij}(x,z,y_i,y_j)$. When $\sum_i w_i$ is even,

$$\left\langle \prod_{i=1}^{4} V_{j_i}^{w_i}(x_i,y_i,z_i) \right\rangle = X_{\emptyset}^{j_1+j_2+j_3+j_4-k} X_{12}^{-j_1-j_2+j_3-j_4} \times X_{13}^{-j_1+j_2-j_3+j_4} X_{23}^{j_1-j_2-j_3+j_4} X_{34}^{-2j_4} F\left(\frac{X_{23}X_{14}}{X_{12}X_{34}}, z\right)$$

The general solution

For $\sum_i w_i$ even,

$$\left\langle \prod_{i=1}^4 V_{j_i}^{w_i}(x_i,y_i,z_i)\right\rangle = X_\emptyset^{j_1+j_2+j_3+j_4-k} X_{12}^{-j_1-j_2+j_3-j_4} \times X_{13}^{-j_1+j_2-j_3+j_4} X_{23}^{j_1-j_2-j_3+j_4} X_{34}^{-2j_4} F\left(\frac{X_{23}X_{14}}{X_{12}X_{34}}, z\right)$$

For example

$$X_\emptyset = P_{(w_1,w_2,w_3,w_4)}(x;z) \ ,$$

$$X_{12} = P_{(w_1+1,w_2+1,w_3,w_4)}(x;z) + y_1 P_{(w_1-1,w_2+1,w_3,w_4)}(x;z) + y_2 P_{(w_1+1,w_2-1,w_3,w_4)}(x;z) + y_1 y_2 P_{(w_1-1,w_2-1,w_3,w_4)}(x;z) \ .$$

Additional constraints

For $\sum_i w_i$ even,

$$\left\langle \prod_{i=1}^4 V_{j_i}^{w_i}(x_i, y_i, z_i) \right\rangle = X_\emptyset^{j_1+j_2+j_3+j_4-k} X_{12}^{-j_1-j_2+j_3-j_4} \\ \times X_{13}^{-j_1+j_2-j_3+j_4} X_{23}^{j_1-j_2-j_3+j_4} X_{34}^{-2j_4} F\left(\frac{X_{23}X_{14}}{X_{12}X_{34}}, z\right)$$

Can we constrain $F(c, z)$ further?

Yes! We can impose the Knizhnik–Zamolodchikov equation and for special values of the spins, null vector equations.

Knizhnik–Zamolodchikov equation

The spacetime translation can be written as

$$\partial_z V^w_{j,h}(0;z) = L_{-1} V^w_{j,h}(0;z) = \\ = \frac{1}{k-2}\Big(-(J^3J^3)_{-1} + \tfrac{1}{2}(J^+J^-)_{-1} + \tfrac{1}{2}(J^-J^+)_{-1}\Big) V^w_{j,h}(0;z) \ ,$$

We get a differential equation for the unknown function $F(c,z)$.

The differential equation for $F(c,z)$ coincides with the KZ equation of the unflowed correlator!

KZ equation for flowed correlator	$\Longrightarrow$	Unflowed KZ equation for $F(c,z)$

A conjecture for the full correlator

This motivates the conjecture

$$F(c,z) = \left\langle V^0_{j_1}(0,0) V^0_{j_2}(1,1) V^0_{j_3}(\infty,\infty) V^0_{j_4}(c,z) \right\rangle$$

and for the full correlator with $\sum w_i$ even

$$\left\langle \prod_{i=1}^4 V^{w_i}_{j_i,h_i,\bar{h}_i}(x_i;z_i) \right\rangle = \int \prod_{i=1}^4 \mathrm{d}^2 y_i \, y_i^{\frac{kw_i}{2}+j_i-h_i-1} \, \bar{y}_i^{\frac{kw_i}{2}+j_i-\bar{h}_i-1}$$
$$\times \; X_\emptyset^{j_1+j_2+j_3+j_4-k} X_{12}^{-j_1-j_2+j_3-j_4} X_{13}^{-j_1+j_2-j_3+j_4} X_{23}^{j_1-j_2-j_3+j_4}$$
$$\times \; X_{34}^{-2j_4} \left\langle V^0_{j_1}(0,0) V^0_{j_2}(1,1) V^0_{j_3}(\infty,\infty) V^0_{j_4}\left(\frac{X_{23}X_{14}}{X_{12}X_{34}}, z\right) \right\rangle$$

Unflowed correlators give flowed correlators for free!

Work in progress and open questions

- Worldsheet correlators have intriguing singularity structure
- Near the singularity the string correlator becomes the correlator of $\mathrm{Sym}^N\left(\mathbb{R}_{Q=\frac{k-3}{\sqrt{k-2}}}\times X\right)$
- Proof of our formula via free-field Wakimoto representation?
- Supersymmetric and extremal correlators