Black-Hole Scattering, Eikonal Resummation & Gravitational Waves

GGI CONFERENCE

CORTONA YOUNG 2021

Carlo Heissenberg

[NORDITA & Uppsala University]

BASED ON: 2008.12743, 2101.05772, 2104.03256

IN COLLABORATION WITH:

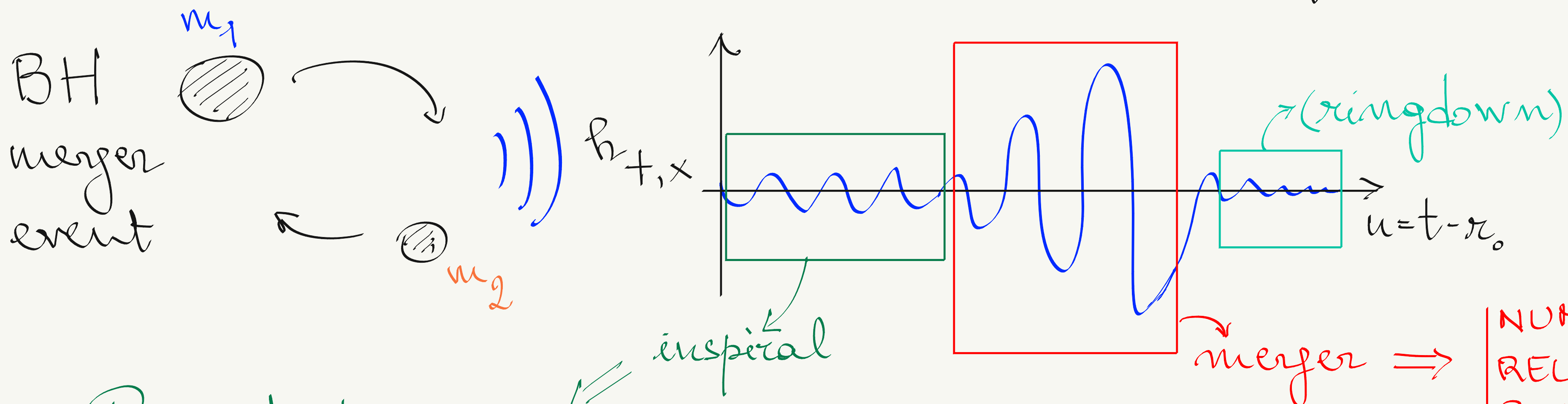
P. Di Vecchia, R. Russo, G. Veneziano

AND ON: 2105.04594.

MOTIVATION: Gravitational Merger Events

- Ligo / Virgo / Lisa: Gravitational wave "spectroscopy"

  $\Longrightarrow$ Need for precision waveform templates



$\Longrightarrow$ NUMERICAL RELATIVITY SIMULATIONS

- Perturbative

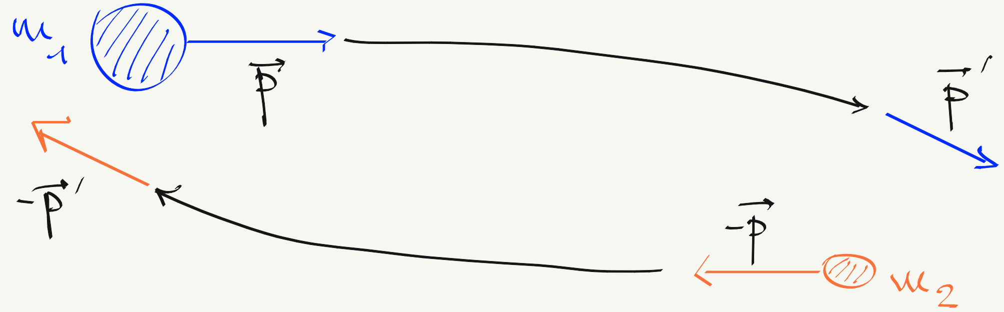
  - Post-Minkowskian (PM) expansion $\boxed{\dfrac{Gm}{r} \ll 1}$ $\quad m \sim m_1 \sim m_2$

    "in powers of G" $\quad r =$ relative separation

  - Post-Newtonian (PN) expansion $\boxed{\dfrac{Gm}{r} \sim \dfrac{1}{2} v^2 \ll 1}$ $\quad v =$ relative speed

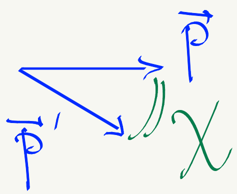
    "in powers of $G \sim v^2$"

Gravitational Scattering

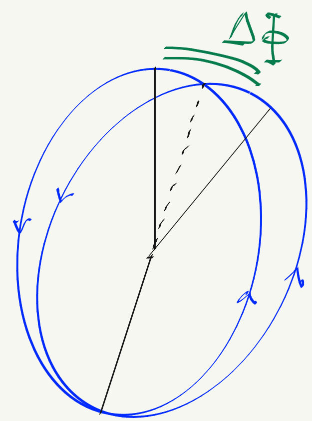


- What can be learned from BH scattering? E.g.
  - Effective potential V_{eff} [C. Cheung, I. Rothstein, M. Solon - 1808.02489]

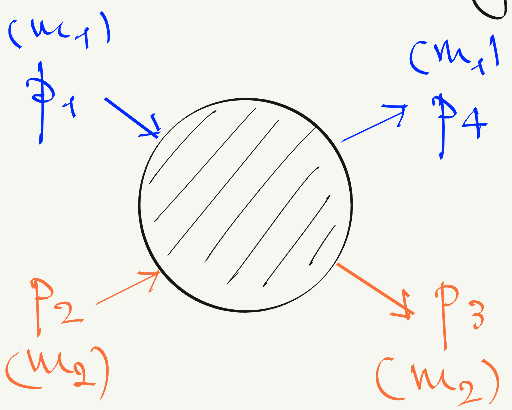
    valid for bound or unbound problem!
  - Deflection angle $\Longrightarrow$ Precession angle



ANALYTIC CONTINUATION [E.g. Kälin, R. Porto - 1910.03008, 1911.09130]



- Scattering Amplitudes probe the PM regime:



$\mathcal{A} =$	$\mathcal{A}_0$ +	$\mathcal{A}_1$ +	$\boxed{\mathcal{A}_2}$ +	$\mathcal{A}_3$ + ...
	tree	1 loop	2 loops	3 loops
	$\mathcal{O}(G)$	$\mathcal{O}(G^2)$	$\mathcal{O}(G^3)$	$\mathcal{O}(G^4)$
	1PM	2PM	3PM	4PM

[STATE OF THE ART]

Generic v (Lorentz invariance)

A 3PM Puzzle

- 3PM effective potential and deflection angle derived from amplitude techniques

$$A_2 \longrightarrow V_{eff}^{(3PM)} \longrightarrow \chi_{3PM}$$

[Z. Bern, C. Cheung, R. Roiban, C.-H. Shen, M. Solon, M. Zeng – 1901.04424, 1908.01493]

- $\chi_{3PM}(E, b, m_1, m_2)$: high-energy limit $\boxed{E \gg m_1, m_2}$

  E: C.O.M. energy; b: impact parameter

$$\boxed{\chi_{3PM} \sim \left(\frac{GE}{b}\right)^3 \log\left(\frac{E^2}{m_1 m_2}\right)}$$ log-enhanced

- $\chi_{3PM}^{ACV}(E, b)$ $\boxed{\text{MASSLESS}}$:

  (shockwave scattering)

$$\boxed{\chi_{3PM}^{ACV} \sim \left(\frac{GE}{b}\right)^3}$$ no log!?

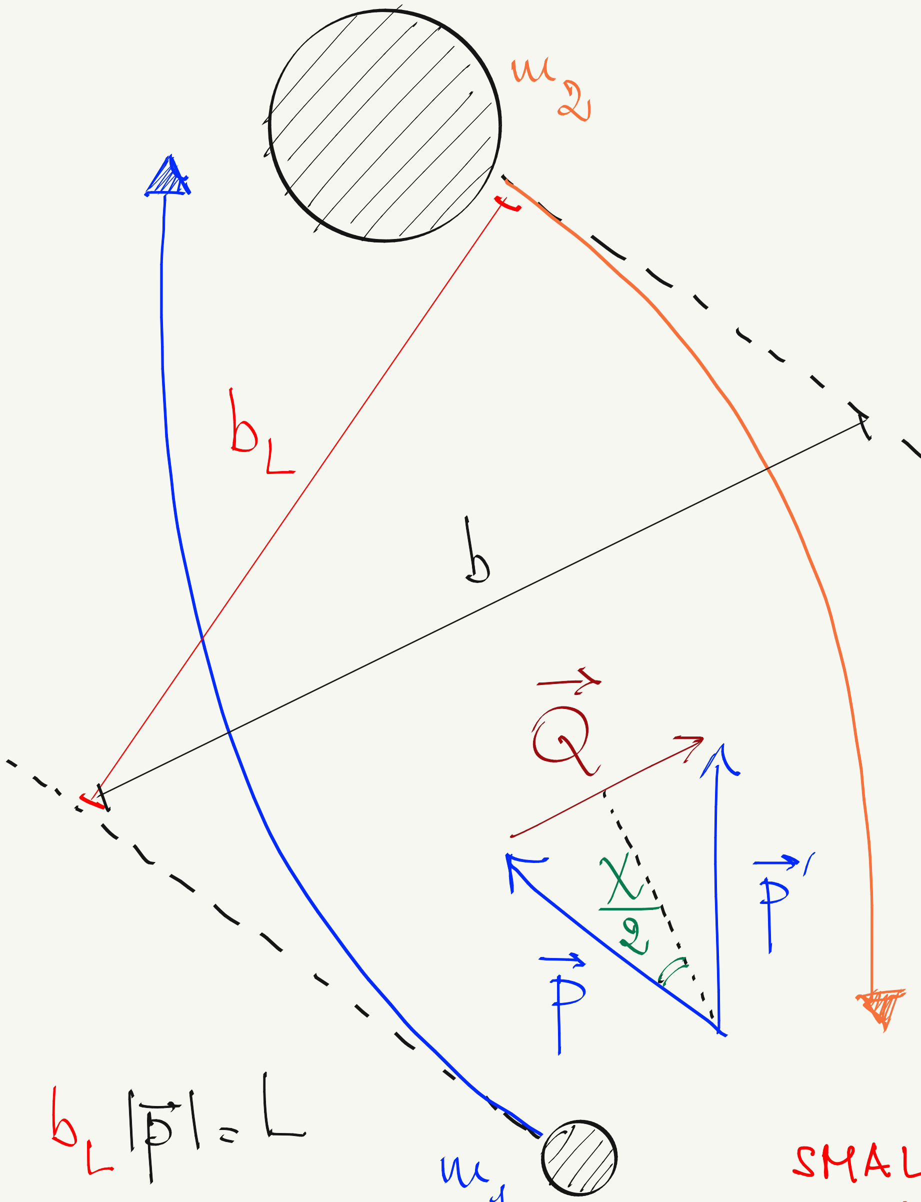
UNIVERSAL: GR, $\mathcal{N}=8$ ...

[D. Amati, M. Ciafaloni, G. Veneziano – 1990]

Outline

- **Classical Limit of Scattering Amplitudes**
  - EIKONAL EXPONENTIATION $\quad 1 + i\mathcal{A} \sim e^{2i\delta}$
  - EIKONAL PHASE $\longrightarrow$ DEFLECTION ANGLE χ
- **3PM Eikonal & Deflection Angle**
  - χ_{3PM} IN MASSIVE $\mathcal{N}=8$ SUGRA
  - FROM $\mathcal{N}=8$ TO GENERAL RELATIVITY

Classical Limit & Gravitational Scattering



$b_L |\vec{p}| = L$

- Length scales:

$$\boxed{\frac{\hbar}{m} \ll Gm \ll b}$$

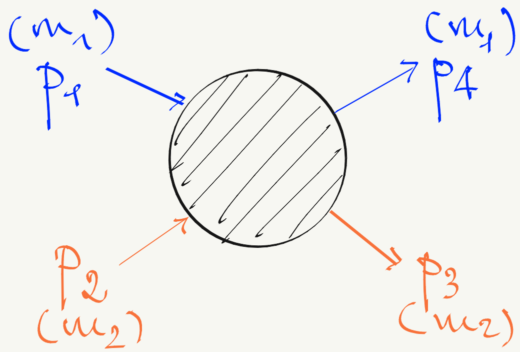
CLASSICAL LIMIT

PM regime

Compton wavelength λ

Schwarzschild radius R

impact param.

- 

$$s = -(p_1+p_2)^2 = E^2$$

$$\boxed{q = p_4 - p_1}$$

▶ From q-space...

$$\boxed{\mathcal{A}(s,q)} = \mathcal{A}_0(s,q) + \mathcal{A}_1(s,q) + \mathcal{A}_2(s,q) + \dots$$

▶ ... to b-space

SMALL q ⇕ LARGE b

$$\boxed{\tilde{\mathcal{A}}(s,b)} = \int e^{ibq} \frac{\mathcal{A}(s,q)}{4E|\vec{p}|} \frac{d^{2-2\epsilon}q}{(2\pi)^{2-2\epsilon}}$$

$D = 4 - 2\epsilon$

Amplitude in b-space

- Long-range terms in the $\frac{1}{b}$-expansion of $\tilde{A}(s,b)$
  $\Leftarrow$ non-analytic terms in the q^2-expansion of $A(s,q)$:

- Schematically:

$$\boxed{\begin{aligned} &\lambda \equiv \frac{\hbar}{m}, \quad R \equiv Gm(b^{2\epsilon}) \quad D=4-2\epsilon \\ &\frac{R}{\lambda} \gg 1 \ \text{classical limit} \qquad \frac{R}{b} \ll 1 \ \text{PM regime} \end{aligned}}$$

$(m \sim m_1 \sim m_2)$

- $\tilde{A}_0(s,b) = c_0\left(\frac{R}{\lambda}\right)$

- $\tilde{A}_1(s,b) = c_{1,0}\left(\frac{R}{\lambda}\right)^2 + c_{1,1}\left(\frac{R}{\lambda}\right)\left(\frac{R}{b}\right) + c_{1,2}\left(\frac{R}{b}\right)^2\left[1+\mathcal{O}\left(\frac{\lambda}{b}\right)\right]$

- $\tilde{A}_2(s,b) = c_{2,0}\left(\frac{R}{\lambda}\right)^3 + c_{2,1}\left(\frac{R}{\lambda}\right)^2\left(\frac{R}{b}\right) + c_{2,2}\left(\frac{R}{\lambda}\right)\left(\frac{R}{b}\right)^2 + c_{2,3}\left(\frac{R}{b}\right)^3\left[1+\mathcal{O}\left(\frac{\lambda}{b}\right)\right]$

$\vdots$

$$c_{L,n-1}(\sigma) \qquad \sigma \equiv \frac{s-m_1^2-m_2^2}{2m_1m_2} = \frac{1}{\sqrt{1-v^2}}$$

(relative velocity)

Eikonal Exponentiation

- $$1 + i\tilde{A}(s,b) = e^{2i\delta(s,b)}\left[1 + 2i\Delta(s,b)\right]$$

(classical) eikonal $\left[e^{\frac{i}{\hbar}S_{\text{classical}}}\right]$

quantum remainder

$$\boxed{\delta = \delta_0 + \delta_1 + \delta_2 + \dots}$$

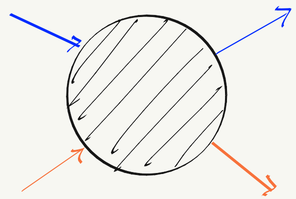
$$\delta_{n-1} \sim \frac{R}{\lambda}\left(\frac{R}{b}\right)^{n-1}$$

$$\boxed{\Delta = \Delta_1 + \Delta_2 + \dots}$$

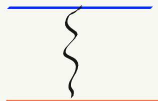
$$\Delta_{n-1} \sim \left(\frac{R}{b}\right)^n \left[1 + O\left(\frac{\lambda}{b}\right)\right]$$

$$\boxed{\frac{R}{\lambda} \gg 1 \quad \text{classical} \qquad \frac{R}{b} \ll 1 \quad \text{PM}}$$

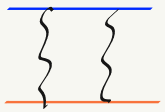
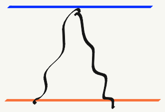
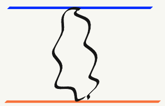
- For "small G":

$A =$ 

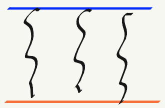
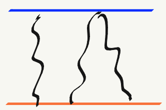
1PM

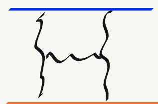
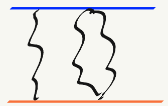
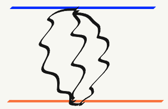
- $i\tilde{A}_0(s,b) = 2i\delta_0$ 

2PM

- $i\tilde{A}_1(s,b) = \frac{(2i\delta_0)^2}{2!}$  $+ \; 2i\delta_1$  $+ \; 2i\Delta_1$ 

3PM

- $i\tilde{A}_2(s,b) = \frac{(2i\delta_0)^3}{3!}$  $+ \; 2i\delta_0 \, 2i\delta_1$  $+ \left[2i\delta_2 + 2i\delta_0 \, 2i\Delta_1\right] + 2i\Delta_2$

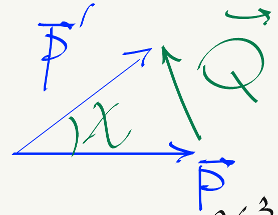
⋮

[D. Kabat, M. Ortiz - 9203082;
R. Akhoury, R. Saotome, G. Sterman - 1308.5204;
N. Bjerrum-Bohr, P.H. Damgaard, G. Festuccia, L. Planté, P. Vanhove - 1806.04920]

Eikonal & Deflection Angle

- Stationary phase approximation: after exponentiation

$$\mathcal{A}(s,\vec{Q}) \sim \int e^{-i\vec{b}\cdot\vec{Q}+2i\delta}\, d^{2-2\epsilon}b \quad \underset{\delta \gg 1}{\Longrightarrow} \quad \boxed{\vec{Q} = \frac{\partial}{\partial \vec{b}} \mathrm{Re}(2\delta)}$$



$$|\vec{Q}| \simeq 2|\vec{p}|\sin\frac{\chi}{2} \simeq |\vec{p}|\left(\chi - \frac{\chi^3}{24}\right)$$

- $\delta = \delta_0 + \delta_1 + \delta_2 + \dots$

$$2\delta_0 \sim G m_1 m_2 \frac{b^{2\epsilon}}{2\epsilon} \quad \Rightarrow \quad \chi_{1PM} \sim \frac{G m_1 m_2}{|\vec{p}|\, b} \sim \left(\frac{Gm}{b}\right)^1$$

$$2\delta_1 \sim G m_1 m_2 \frac{G(m_1+m_2)}{b^{1-4\epsilon}} \quad \Rightarrow \quad \chi_{2PM} \sim \frac{G^2 m_1 m_2 (m_1+m_2)}{|\vec{p}|\, b^2} \sim \left(\frac{Gm}{b}\right)^2$$

$$2\delta_2 = G m_1 m_2 \frac{G^2 m_1 m_2}{b^{2-6\epsilon}}\left[(\dots) + \frac{i}{\epsilon}(\dots)\right] \quad \Rightarrow \quad \chi_{3PM} \sim \frac{G^3 (m_1 m_2)^2}{|\vec{p}|\, b^3} \sim \left(\frac{Gm}{b}\right)^3$$

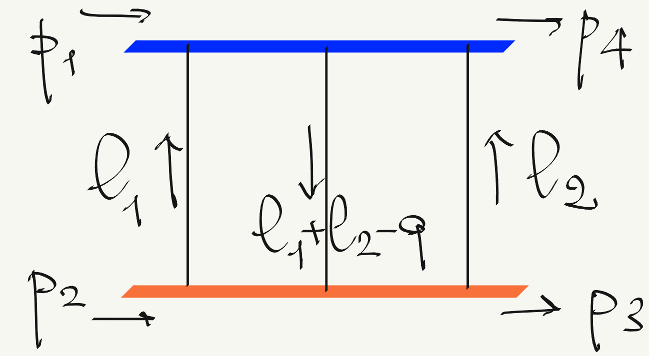
IR finite real part

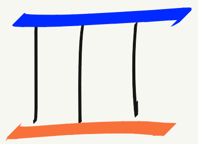
IR divergent imaginary part due to

[S. Weinberg – 1965]

Does the coefficient scale as $\log\left(\frac{s}{m_1 m_2}\right)$ for $s \gg m_{1,2}$?

Evaluation of the Loop Integrals



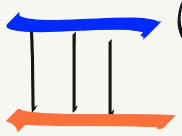


$$= \iint \frac{d^D\ell_1 \, d^D\ell_2}{\left[2\bar{p}_1\cdot\ell_1 - i0 + (\ell_1^2 - \ell_1 q)\right]\left[-2\bar{p}_2\cdot\ell_1 - i0 + (\ell_1^2 - \ell_1 q)\right]\left[-2\bar{p}_1\cdot\ell_2 - i0 + (\ell_2^2 - \ell_2 q)\right]\left[2\bar{p}_2\cdot\ell_2 - i0 + (\ell_2^2 - \ell_2 q)\right]}$$

$$\times \frac{1}{(\ell_1^2 - i0)(\ell_2^2 - i0)((\ell_1+\ell_2-q)^2 - i0)}$$

$$p_1 = \bar{p}_1 - q/2$$
$$p_2 = \bar{p}_2 + q/2$$
$$p_3 = \bar{p}_2 - q/2$$
$$p_4 = \bar{p}_1 + q/2$$

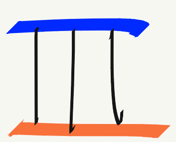
- Hard region: $q^2 \to 0$, $\boxed{\ell_1, \ell_2 \sim \mathcal{O}(m_{1,2})}$



$$^{(h)} = c_0 (q^2)^0 + c_1 (q^2)^1 + \cdots$$

irrelevant for the long range behavior of $\delta(s,b)$

- Soft region: $q^2 \to 0$, $\boxed{\ell_1, \ell_2 \sim \mathcal{O}(q)}$



$$^{(s)} = \frac{C_{\text{III}}^{(SSCL)}}{(q^2)^{1+2\epsilon}} + \frac{C_{\text{III}}^{(SCL)}}{(q^2)^{\frac{1}{2}+2\epsilon}} + \frac{C_{\text{III}}^{(CL)}}{(q^2)^{2\epsilon}} + \cdots$$

relevant region!

Soft vs Potential Region

- The soft region further splits into two regions if we introduce the near static limit

$\Longrightarrow$ SMALL-VELOCITY expansion $v \ll 1$

↳ Potential Region: $q^2 \to 0$, $\boxed{\begin{array}{l} (l_1)^0, (l_2)^0 \sim O(vq) \\ \vec{l}_1 , \vec{l}_2 \sim O(q) \end{array}}$

* off-shell intermediate gravitons

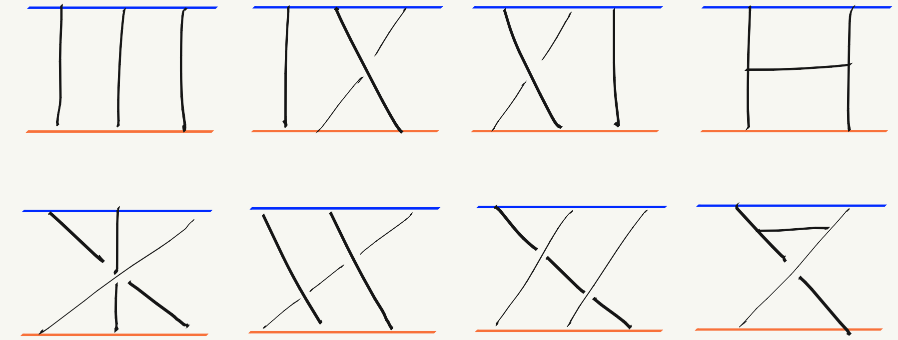
* contributes to the V_{eff} of BCRSSZ

↳ Radiation Region: $q^2 \to 0$, $\boxed{l_1 , l_2 \sim O(vq)}$

* contains on-shell gravitons

* not included in the V_{eff} of BCRSSZ

$\mathrm{Re}\,\delta_2$ and χ_{3PM} in $\mathcal{N}=8$ SUGRA



- Exponentiation works!

3PM eikonal:

$$\mathrm{Re}\, 2\delta_2 = -\frac{16 m_1^2 m_2^2 G_N^3 \sigma^4}{b^2(\sigma^2-1)} \cosh^{-1}(\sigma)\left[\boxed{1} - \frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{3}{2}}}\right] + \frac{16 m_1^2 m_2^2 G_N^3 \sigma^6}{b^2(\sigma^2-1)^2}$$

Also appearing in the potential region

[J. Parra-Martinez, M. Ruf, M. Zeng - 2104.03256]

New terms appearing in the full soft region

[P. Di Vecchia, CH, R. Russo, G. Veneziano - 2008.12743]

IR finite, ϵ has been sent to zero.

- 3PM scattering angle: (BYPASSING $V_{eff}^{(3PM)}$!)

$$\chi_{3PM} = -\frac{m_1^3 m_2^3 \sigma^6 G_N^3}{L^3(\sigma^2-1)}\left[\frac{16}{3}\frac{1}{\sqrt{\sigma^2-1}} - \frac{32 m_1 m_2}{s}\right] - \frac{32 m_1^4 m_2^4 \sigma^4 G_N^3}{L^3 s}\left[\cosh^{-1}(\sigma) - \frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{3}{2}}}\cosh^{-1}(\sigma)\right]$$

- High-energy limit: $E^2 = s = m_1^2 + m_2^2 + 2m_1 m_2 \sigma \gg m_{1,2}^2$

$$\chi_{3PM} = -\frac{m_1^3 m_2^3 \sigma^6 G^3}{L^3(\sigma^2-1)}\left[\frac{16}{3}\frac{1}{\sqrt{\sigma^2-1}} - \frac{32 m_1 m_2}{s}\right] \to \frac{4}{3}\left(\frac{Gs}{L}\right)^3 \sim \left(\frac{GE}{b}\right)^3$$

ACV90 ✓✓

$$-\underbrace{\frac{32\, m_1^4 m_2^4 \sigma^4 G^3}{L^3 s}}_{\sim\left(\frac{GE}{b}\right)^3} \underbrace{\cosh^{-1}(\sigma)}_{\sim \log\left(\frac{E^2}{m_1 m_2}\right)} \underbrace{\left[1 - \frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{3}{2}}}\right]}_{\sim \frac{m_1 m_2}{E^2}}$$

- Near-static limit: $\sqrt{\sigma^2-1} = \frac{E_{CM}|\vec{P}|}{m_1 m_2}$ $\qquad \sigma - 1 = O(v^2) \ll 1$

$$\chi_{3PM} = -\frac{32 m_1^3 m_2^3 \sigma^6 G^3}{L^3}\left[\underbrace{\boxed{\frac{1}{6}\frac{1}{(\sigma^2-1)^{\frac{3}{2}}}}}_{\text{0PN}} - \boxed{\frac{m_1 m_2}{s(\sigma^2-1)}}\right]$$

$$-\frac{32\, m_1^4 m_2^4 \sigma^4 G^3}{L^3 s}\left[\underbrace{\boxed{\cosh^{-1}(\sigma)}}_{\text{2PN}} - \boxed{\frac{\sigma(\sigma^2-2)}{(\sigma^2-1)^{\frac{3}{2}}}\cosh^{-1}(\sigma)}\right]$$

$\sim \left(\frac{Gm}{b}\right)^3 \frac{1}{v^3}$

1.5 PN

RADIATION REACTION

$G_N^m (v^2)^n \Leftrightarrow (m+n)\text{PN}$

$$\left(\cosh^{-1}(\sigma) \sim \sqrt{\sigma^2-1}\right)$$

Towards GR

- Radiation-Reaction (RR) effects can be added to the conservative result:

$$\boxed{\chi_{3PM}^{cons.} + \chi_{3PM}^{RR}} \to \text{no } \log\left(\frac{s}{m_1 m_2}\right) \to \text{agreement with ACV90}$$

[T. Damour - 2010.01641]

- Amplitude approach? In the full $\mathcal{N}=8$ result:

$$\mathrm{Re}\, 2\delta_2^{RR} = \frac{G^3 m_1^2 m_2^2}{b^2} \frac{16\sigma^4}{(\sigma^2-1)^2}\left[\sigma^2 + \frac{\sigma(\sigma^2-2)}{\sqrt{\sigma^2-1}}\cosh^{-1}(\sigma)\right]$$

$$i\,\mathrm{Im}\, 2\delta_2 = -\frac{i}{\pi\epsilon}\frac{G^3 m_1^2 m_2^2}{b^2} \frac{16\sigma^4}{(\sigma^2-1)^2}\left[\sigma^2 + \frac{\sigma(\sigma^2-2)}{\sqrt{\sigma^2-1}}\cosh^{-1}(\sigma)\right]\left[1-\epsilon\log(\sigma^2-1)\right]$$

$+$ other $O(\epsilon^0)$ terms [PDV, CH, RR, GV - 2101.05772, 2104.03256]

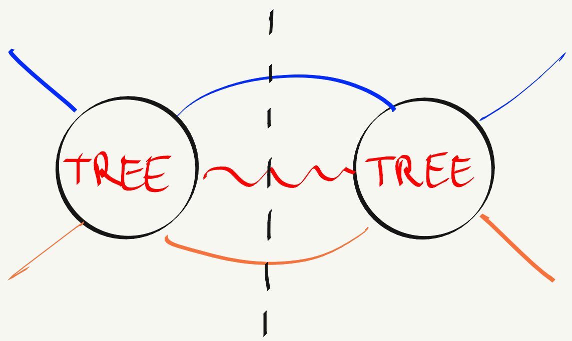
$$\boxed{(\mathrm{Re}\, 2\delta_2)^{RR} = \lim_{\epsilon\to 0}\left(-\pi\epsilon\, \mathrm{Im}\, 2\delta_2\right)}$$

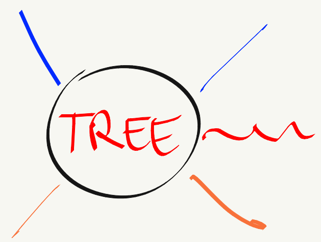
in fact

$$\boxed{\lim_{\epsilon\to 0}\left(-4\epsilon\, \mathrm{Im}\, 2\delta_2\right) = \left.\frac{dE_{GW}}{d\omega}\right|_{\omega=0}}$$

Radiation Reaction in $\mathcal{N}=8$ and GR

- $\mathcal{I}m\, 2\delta_2$ from UNITARITY:



$= \int d\Gamma_{int} \Bigg|$  $\Bigg|^2$

$$\longrightarrow \boxed{2\, \mathcal{I}m\, 2\delta_2 = \int \frac{\left|\tilde{\mathcal{A}}_{5pt}(s,b,\vec{K})\right|^2}{2|\vec{K}|} \frac{d^{3-2\epsilon}\vec{K}}{(2\pi)^{3-2\epsilon}}}$$

- Weinberg soft limit $|\vec{K}| \ll q \ll m \overset{q\sim\frac{1}{b}}{\Longrightarrow} \tilde{\mathcal{A}}_{5pt}(s,b,\vec{K}) \sim \frac{1}{|\vec{K}|} f(s,b,\hat{K})$

$$\Longrightarrow \boxed{-\frac{i}{\pi\epsilon}\left[1-\epsilon \log v^2\right]} \quad \text{for } v \ll 1$$

[PDV, CH, RR, GV - 2101.05772]

- Full $\tilde{\mathcal{A}}_{5pt}(s,b,K) \Longrightarrow \boxed{-\frac{i}{\pi\epsilon}\left[1-\epsilon \log(\sigma^2-1)\right]}$

[PDV, CH, RR, GV - 2104.03256]

- $(\mathrm{Re}\, 2\delta_2)^{RR}$ from ANALYTICITY:

$$\boxed{\frac{i}{\pi}\log(\sigma^2-1)} = i\, \mathcal{I}m\left[\frac{i}{\pi}\log\left(-(\sigma^2-1)-i0\right)\right] \longrightarrow \mathrm{Re}\left[\frac{i}{\pi}\log\left(-(\sigma^2-1)-i0\right)\right] = \boxed{+1}$$

Radiation Reaction, IR divergences and Gravitational Waves

$$\boxed{\left(\mathrm{Re}\, 2\delta_2\right)^{RR} = \lim_{\epsilon \to 0}\left(-\pi\epsilon\, \mathrm{Im}\, 2\delta_2\right)} \Longrightarrow \chi_2^{RR} \begin{cases} \mathcal{N}=8 & \checkmark\checkmark \\ GR & \checkmark\checkmark \end{cases}$$

- The IR-divergent part of $2\delta_2$ can be also obtained from the EXPONENTIATION of IR divergences in q-space

$$\boxed{A_2 = A_2^0\, e^{W}}$$

[S. Weinberg - 1965]

$\Longrightarrow$ ✓ $\mathrm{Re}\, 2\delta_2$ finite

✓ $\mathrm{Im}\, 2\delta_2$ div. part obtained from tree-level & 1-loop

IR-finite

[CH - 2105.04594]

- $\tilde{A}_{5pt}(s, b, \omega\hat{x})$ IS (up to a factor) the WAVEFORM!

[G.U. Jakobsen, G. Mogull, J. Plefka, J. Steinhoff - 2101.12688; S. Mougiakakos, M. Riva, S. Vernizzi - 2102.08339]

[PDV, CH, RR, GV - 2104.03256, in progress]

Conclusions

- Eikonal exponentiation for $\boxed{\mathcal{N}=8 \text{ SUGRA}}$ ✓✓
- Deflection angle χ from $\mathrm{Re}\,\delta_2$:
  - smooth @ high-energy & agrees with ACV90
  - RADIATION-REACTION EFFECTS
  - the latter are captured by $\mathcal{O}(\frac{1}{\epsilon})$ terms in $\mathrm{Im}\,\delta_2$ in $\boxed{\mathcal{N}=8 \text{ and } GR}$

Outlook:
- Real radiation in the eikonal (operator)
- Comparison w/ EOB, effective field theory, direct evaluation of $\langle \vec{Q} \rangle$
- $\mathcal{O}(G^4)$ RR and tail effect.