Large-N limit and θ-dependence from the lattice: $2d$ CP^{N-1} models vs $4d$ $SU(N)$ gauge theories

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Based on: CB, Bonati, D'Elia, 2018 [arXiv:1807.11357];
Berni, CB, D'Elia, 2019 [arXiv:1911.03384];
Berni, CB, D'Elia, 2020 [arXiv:2009.14056]; CB, Bonati, D'Elia, 2020 [arXiv:2012.14000]

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The topological charge in $4d$ gauge theories

The topological charge of the gluon field $A_\mu(x)$

$$Q = \frac{g^2}{16\pi^2}\int d^4x \ \mathrm{Tr}\left\{\widetilde{G}^{\mu\nu}(x)G_{\mu\nu}(x)\right\} \in \mathbb{Z}, \qquad \widetilde{G}^{\mu\nu}(x) \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}G_{\rho\sigma}(x)$$

can be coupled to the QCD action via the dimensionless parameter θ:

$$S_{QCD} \to S_{QCD}(\theta) = S_{QCD} + \theta Q,$$

introducing a **non-trivial dependence on** θ in the theory.

The θ-dependence of the free energy (density), defined in Euclidean time as

$$f(\theta) = -\frac{1}{V}\log\int[d\overline{\psi}d\psi dA]e^{-S_{QCD}+i\theta Q}, \qquad f(\theta) = \frac{1}{2}\chi\theta^2\left(1+\sum_{n=1}^{\infty}b_{2n}\theta^{2n}\right),$$

$$\chi = \left.\frac{\langle Q^2\rangle}{V}\right|_{\theta=0}, \qquad b_2 = -\frac{1}{12}\left.\frac{\langle Q^4\rangle - 3\langle Q^2\rangle^2}{\langle Q^2\rangle}\right|_{\theta=0}, \qquad b_{2n} \propto \left.\frac{\langle Q^{2n+2}\rangle_c}{\langle Q^2\rangle}\right|_{\theta=0}$$

has been extensively investigated in several different physical contexts.

Motivations to compute large-N $f(\theta)$ from the lattice

Physics of η' related to θ-dependence of large-N $SU(N)$ gauge theories:

$$N \to \infty : \qquad V_{eff}(\eta') \sim f_{YM}(\theta), \quad \chi \sim m^2_{\eta'}, \quad b_2 \sim \lambda_{4\eta'}.$$

$SU(N)$ gauge theories $\to$ large-N arguments + Witten–Veneziano eq. give

Large-N: $\quad \chi = \bar{\chi} + O(1/N^2), \quad b_{2n} = \bar{b}_{2n}/N^{2n}\,[1 + O(1/N^2)],$

Witten–Veneziano: $\quad \bar{\chi} = m^2_{\eta'} f^2_\pi/6 \simeq (180 \text{ MeV})^4.$

Lattice data confirm within $\sim 5\%$ accuracy LO large-N behavior for χ and within $\sim 15\%$ accuracy LO large-N behavior for b_2 (Bonati et al., 2016).

2d CP^{N-1} models $\to$ θ-dependence known analytically in the large-N limit:

$\xi^2\chi = \frac{1}{2\pi}\frac{1}{N} - 0.0605\frac{1}{N^2} + O\left(\frac{1}{N^3}\right)$ (D'Adda et al., 1978; Rossi et al., 1991),

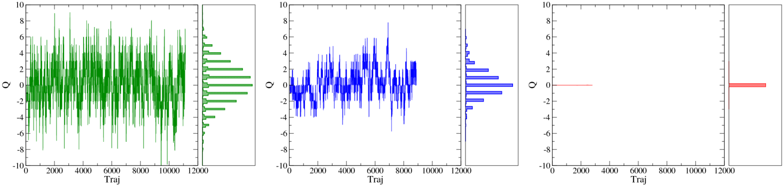
$b_2 = -\frac{27}{5}\frac{1}{N^2} + O\left(\frac{1}{N^3}\right)$ (Del Debbio et al., 2006).

Lattice data disagree with large-N predictions of NLO coeff. of χ and LO coeff. of b_2. **Larger values of N are needed to clarify this issue.**

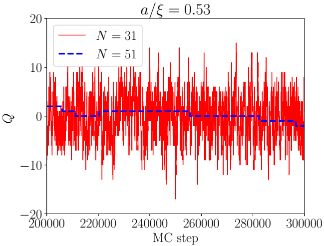
Improvements limited by the severe critical slowing down of standard algorithms approaching the large-N and the continuum limit.

Critical Slowing Down (CSD) of topological modes

Approaching the continuum limit, fluctuations of Q during the simulation become extremely rare. In the continuum theory topological sectors are separated by infinite free-energy barriers.



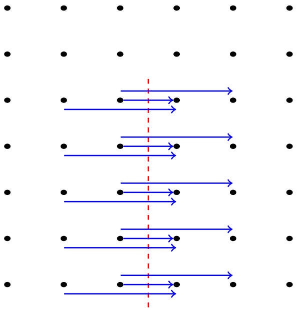
Left to right: $a = 0.082$ fm, 0.057 fm and 0.040 fm (figs. Bonati et al., 2016)

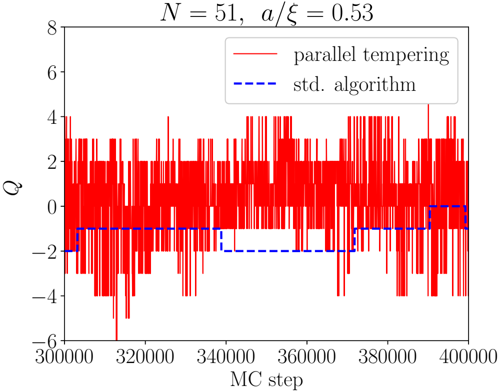


Moreover, CSD worsens increasing N $\Longrightarrow$ approaching the continuum limit for N large is extremely challenging. **Goal: improve large-N validation of $2d$ CP^{N-1} models using the recently-proposed Hasenbusch algorithm.**

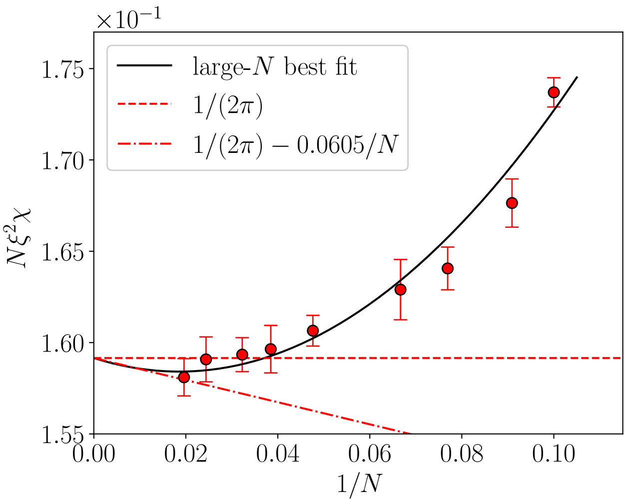
The Hasenbusch algorithm: parallel tempering of defect

Simulate collection of lattice copies with different **boundary conditions**, interpolating periodic and open ones. Each replica has an independent evolution and different copies are swapped from time to time. Charge is quickly changed in the open replica, then the configuration is transferred to the periodic replica through the swaps (Hasenbusch, 2018).





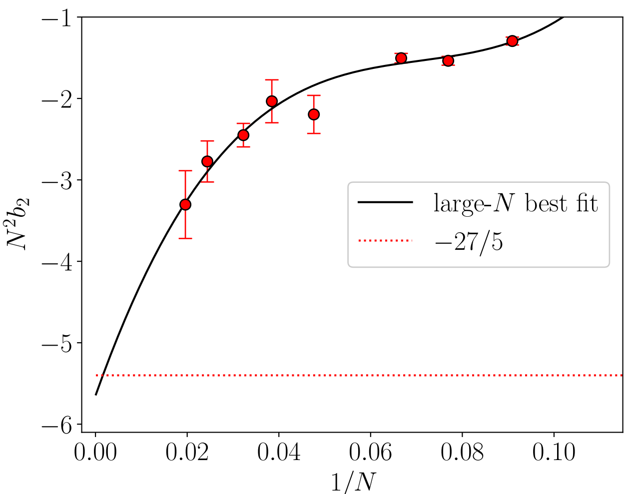
Large-N behavior of $N\xi^2\chi$ in $2d$ CP^{N-1} models



Fit results up to $O(1/N^2)$ terms, $N \in [10, 51]$

$$N\xi^2\chi = 1/(2\pi) - 0.08(2)(1/N) + 2.2(3)(1/N^2)$$

Large-N behavior of $N^2 b_2$ in $2d$ CP^{N-1} models



Fit results up to $O(1/N^3)$ terms, $N \in [11, 51]$

$$\bar{b}_2^{theo} \equiv \lim_{N\to\infty} N^2 b_2 = -27/5 = -5.4,$$

$$(N^2 b_2)_{fit} = -5.7(1.1) + 160(60)(1/N) + \ldots$$

Pathological behavior of CP^{N-1} models when $N \to 2$

The $1/N$ expansion converges very slowly at large-N for $2d$ CP^{N-1} models, unlike $SU(N)$ Yang–Mills theories. **Why?**

A possible explanation could reside in the pathological behavior of CP^{N-1} models in the opposite limit $N \to 2$.

- Semiclassical methods predict $d_I \sim \rho^{N-3}$. For $N = 2$: UV divergence of instanton density $\implies \chi \sim \int d_I(\rho)\, d\rho \to \infty$
- Situation unclear also for $N = 3$: disagreeing claims in the literature about finiteness of χ (Petcher et al., 1983; Lian et al., 2007).

What about b_{2n}? If small instantons with $\rho \to 0$ dominate dynamics, simplest **guess** is (by virtue of asymptotic freedom) that $f(\theta)$ is close to the behavior predicted by the Dilute Instanton Gas Approximation (**DIGA**):

$$f(\theta)|_{N=2} \sim f_{DIGA}(\theta) = \chi(1 - \cos\theta),$$
$$\implies b_2 = -1/12, \quad b_4 = 1/360, \quad \dots$$

Modified continuum scaling at small N

Usual continuum scaling

$$\langle O \rangle_{lat}(a) = \langle O \rangle_{cont} + k_2 a^2 + o(a^2)$$

could be **modified** for small N. Naive argument assuming non-interacting instantons on a $L \times L$ lattice with spacing a:

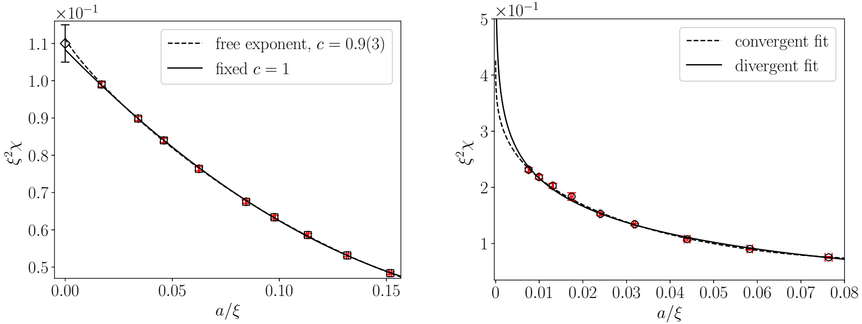
$$\chi \sim \int_a^L \rho^{N-3} d\rho \propto \begin{cases} a^{N-2} & \text{, if } N > 2, \\ \log(a) & \text{, if } N = 2. \end{cases}$$

$\implies$ Modifications expected for $N = 3$ and 2:

$$N = 3 : \chi(a) = \chi + k_1 a + k_2 a^2 + o(a^2)$$
$$N = 2 : \chi(a) = k_0 \log(a/a_0) + k_2 a^2 + o(a^2)$$

These equations must be taken *with a grain of salt*, however, they still constitute an useful guide to study the continuum scaling at small N.

Small-N results for $\xi^2\chi$



Left: $N = 3$. Right: $N = 2$.

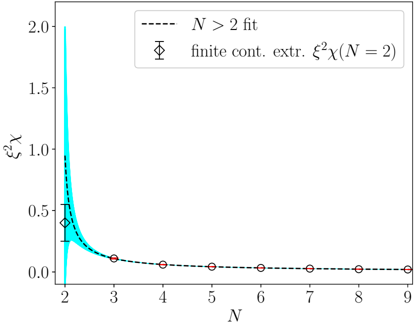
$N = 3$: **good agreement** with dominant $\sim O(a)$ corrections + $O(a^2)$ terms $\implies$ **convergent** continuum limit.

$N = 2$: **no clear conclusion**. Data are both compatible with a dominant logarithmic divergent behavior and with a dominant slowly convergent power-law behavior, with dominant $\sim O(a^c)$ corrections, $c \sim 0.1$ (plus $O(a^2)$ corrections).

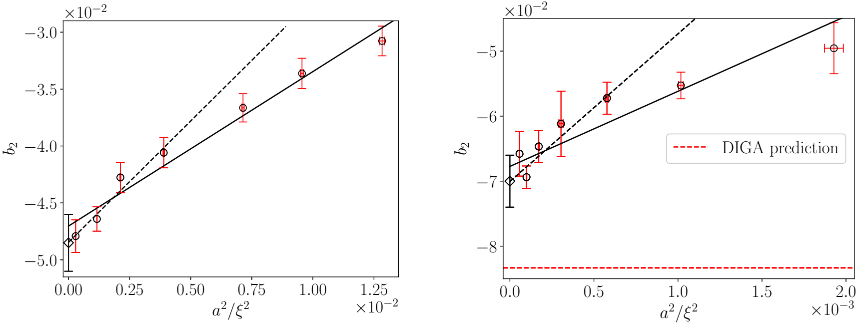
Alternative approach: $N \to 2$ extrapolation of $\xi^2\chi$

Ansatz: since $\chi(N) \sim \int \rho^{N-3} d\rho \sim 1/(N-2) \implies \xi^2\chi(N)$ is extrapolated towards $N = 2$ from $N > 2$ using critical function

$$\xi^2\chi = \frac{A}{(N - N^*)^\gamma}$$



Result: $N^* = 1.90(14) \implies$ No **clear evidence** for a divergent behavior in $N = 2$: again both scenarios are supported.



Left: $N = 3$. Right: $N = 2$.

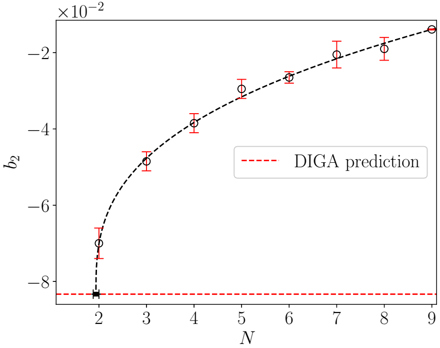
The quartic coefficient b_2, instead, shows the usual scaling down to $N = 2$. Approaching $N \to 2$, b_2 approaches $b_2^{DIGA} = -1/12 \simeq -0.083$. However, we find $b_2(N = 2) = -0.070(4)$, $\sim 3\sigma$ **distant** from b_2^{DIGA}.

Critical small-N extrapolation of b_2

Ansatz: assuming a critical behavior of the type

$$b_2 = b_2^{DIGA} + B(N - N^*)^{\gamma'},$$

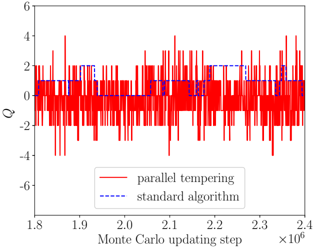
we look for the critical value N^* such that $b_2(N = N^*) = b_2^{DIGA}$.



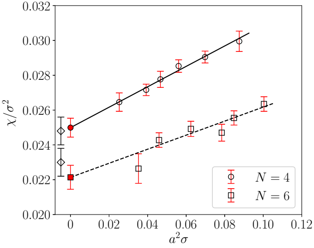
We find $N^* = 1.94(6)$ (**black point**). This value is compatible with $N = 2$ so further refinements are needed in the future to better clarify this issue. Note that it is also well compatible with $N^* = 1.90(14)$, the critical value for which $\xi^2\chi$ **diverges**.

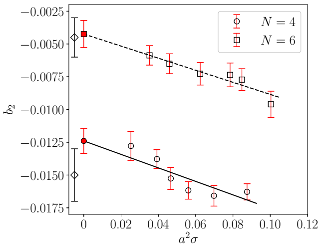
Adapting Hasenbusch Algorithm to $4d$ gauge theories

Standard simulations of $SU(N)$ gauge theories suffer from severe CSD in the large-N limit $\implies$ Hasenbusch algorithm can be adopted to mitigate topological freezing. Difference: now the defect is a **cubic volume**.



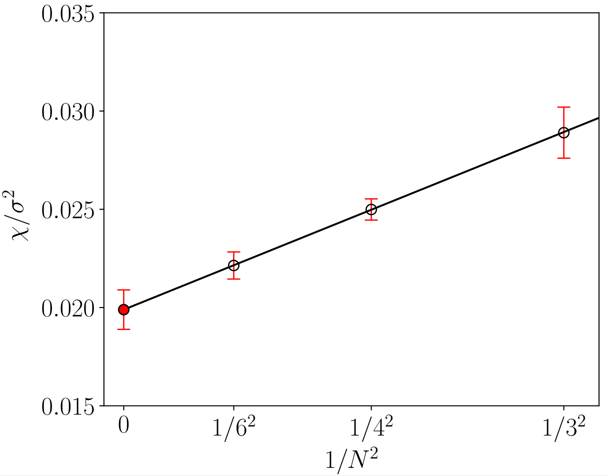
Parallel tempering dramatically improves simulations at large N (fig. on the left: $N = 6$). Performances are exceedingly better without much tuning of the algorithm free parameters (defect volume and swap acceptance).





Continuum limits: diamond pnts (Bonati et al., 2016), full pnts (CB, Bonati, D'Elia, 2020).

Large-N limit of χ in $SU(N)$ pure-gauge theories

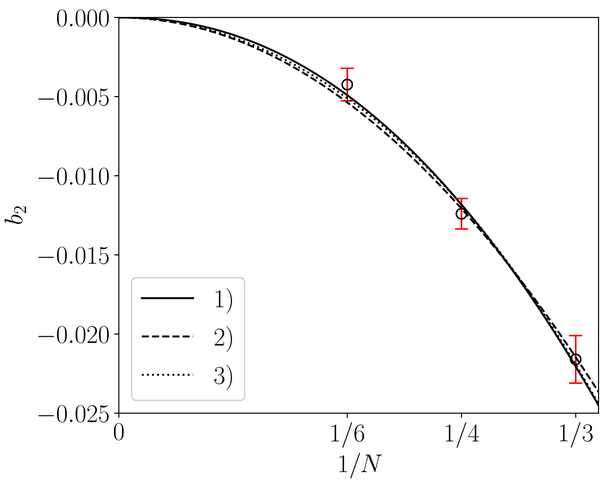


Witten–Veneziano: $\bar{\chi}^{1/4} \simeq 180$ MeV $+ O(1/N^2)$. Fit results:

$$\chi/\sigma^2 = 0.0199(10) + 0.08(2)(1/N^2)$$

$$\bar{\chi}/\sigma^2 = 0.0199(10) \implies \bar{\chi}^{1/4} = 173(8)\ \text{MeV}$$

Large-N limit of b_2 in $SU(N)$ pure-gauge theories



Large-N prediction: $b_2 = \bar{b}_2/N^2 + O(1/N^4)$. Fit results:

1) $b_2 = \bar{b}_2/N^\gamma \to \gamma = 2.17(26)$, cf. $\gamma = 2.0(4)$ (Bonati et al., 2016)

2) $b_2 = \bar{b}_2/N^2 \to \bar{b}_2 = -0.19(1)$, cf. $\bar{b}_2 = -0.23(3)$ (Bonati et al., 2016)

3) $b_2 = \bar{b}_2/N^2 + \bar{b}_2^{(1)}/N^4 \to \bar{b}_2^{(1)} = -0.17(35)$

Conclusions and take-home results

Large-N limit and $1/N$ expansion: $2d$ CP^{N-1} vs $4d$ $SU(N)$ YM

- The Hasenbusch algorithm dramatically mitigates severe topological CSD, both in $2d$ CP^{N-1} models and in $4d$ $SU(N)$ gauge theories.
- Large-N data show slow convergence of $1/N$ series of CP^{N-1} models, explaining discrepancies between early lattice results and analytic predictions.
- Large-N predicted scaling of $4d$ $SU(N)$ θ-dependence holds for $N \geq 3$.

Small-N behavior of $2d$ CP^{N-1} models

- These models at small N are dominated by small instantons, this modifies the continuum scaling of χ for $N = 3$ and $N = 2$.
- For CP^2 linear $O(a)$ corrections appear $\implies$ convergent continuum limit, for CP^1 data are compatible both with slow convergent power and slow divergent log, b_2 instead shows usual scaling.
- Critical fit $\chi \sim 1/(N - N^*)^\gamma$ gives $N^* = 1.90(14)$, while critical fit $b_2 - b_2^{DIGA} \sim (N - N^*)^{\gamma'}$ gives $N^* = 1.94(6)$.

Future outlooks

- Refinement of present study of critical small-N behavior of $2d$ CP^{N-1} models.
- Extend present topology studies to higher momenta of the topological charge density correlator (e.g., χ').
- Some non-topological observables of large-N $SU(N)$ gauge theories are affected by the topological freezing (e.g., glueball masses, k-strings tension) $\implies$ possible improvements of state of the art adopting the Hasenbusch algorithm.

Thank you for your attention!

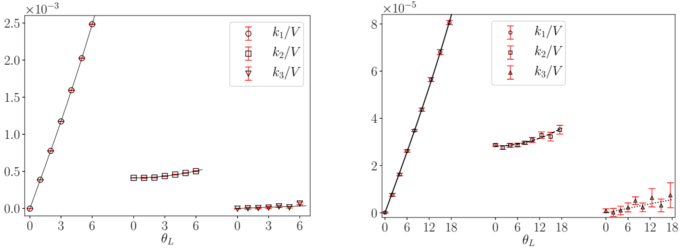
Back-up slides

Higher-order cumulants and imaginary-θ simulations

Signal-to-Noise Ratio (SNR) of b_{2n} (higher-order cumulants) degrades rapidly as the volume grows due to the Central Limit Theorem.
$\Longrightarrow$ large statistics required to keep finite-size effects of b_{2n} under control.
Idea 1: add imaginary-θ term to Euclidean action, so that it acts as a source term for Q, enhancing SNR of higher-order cumulants:

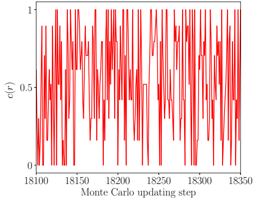
$$S \to S + \theta_I Q, \quad \theta_I \equiv i\theta \qquad \Longrightarrow \qquad k_n \to k_n(\theta_I) \propto \frac{d^n f(\theta_I)}{d\theta_I^n}$$

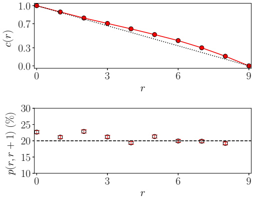
Idea 2: information on χ and b_{2n} now encoded in θ_I-dependence of lower-order cumulants $\Longrightarrow$ extract χ and b_{2n} from combined fit of θ_I-dependence of cumulants k_n. (N. B. odd cumulants non-zero for $\theta_I \neq 0$)



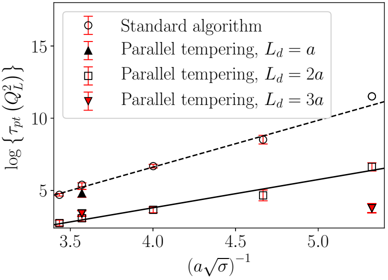
Imaginary-θ fit. Left: $2d$ CP^{N-1} models. Right: $4d$ $SU(N)$ pure-gauge theories.

Links crossing the defect get their coupling multiplied by a factor $c(r)$: $0 \leq c(r) \leq 1$ (r = replica index). In our $SU(N)$ implementation we chose $c(r)$ so that swap acceptance p is $\sim$ const. for couples $(r, r+1)$.

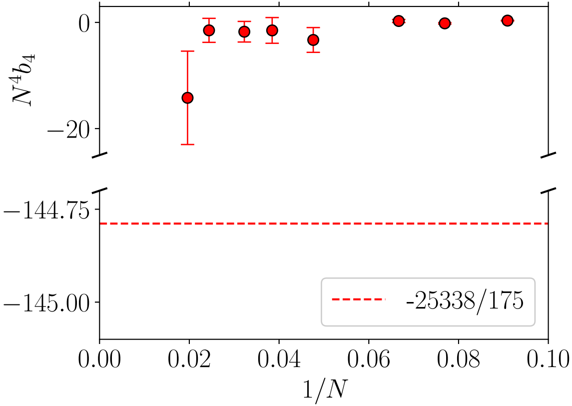




When acc. $\sim$ const. $\implies$ conf. moves freely among different replicas (left fig.) and $c(r)$ deviates from linear interpolation (center fig.). Examples: $N = 4$.



Auto-correlation time of Q^2 scales as $\exp(1/a)$ if defect size L_d is fixed in lattice units as $a \to 0$, however with a **much smaller** slope compared to the standard algorithm. If instead L_d is kept fixed in physical units, scaling with a is **largely improved**. (Fig. on the left: $N = 6$)

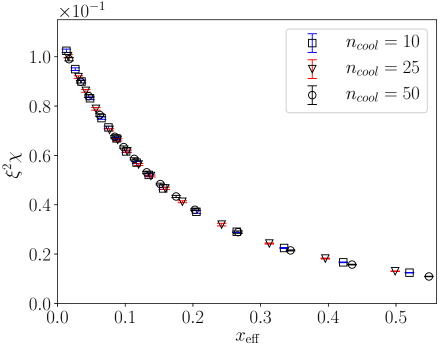


With our statistics b_4 is always compatible with zero. However, we find $|\bar{b}_4| \sim |N^4 b_4| \lesssim 20$, but large-$N$ analytic computations yield $\bar{b}_4 = -25338/175 \simeq -144.79\ldots \implies b_4$ data compatible with slow convergence of $1/N$ series too.

Small-N cont. limit and smoothing, $2d$ CP^{N-1} models

n_{cool} steps $\rightarrow$ smoothing radius r_s with $r_s/a \propto \sqrt{n_{cool}}$.
We kept n_{cool} = const. as $a \rightarrow 0 \implies r_s \sim a$ in the continuum limit $\implies r_s \rightarrow 0$ in the continuum limit, and no relevant signal at the scale of a is smoothed away.

If this hypothesis is correct: $r_s = c(n_{cool})\,a, \implies$
$\chi(a, n_{cool}) = \chi(a', n'_{cool})$ if (a, n_{cool}) and (a', n'_{cool}) have same r_s.



We plot $\xi^2\chi$ vs $x_{eff} \propto ac(n_{cool})/c(n_{cool} = 50) \implies$ all data obtained for different values of n_{cool} collapse on each other $\implies$ choosing a different value of n_{cool} is just a different choice of $a \implies$ continuum limit must be the same. (Fig. on the left: $N = 3$.)