Local Subtraction of Infrared Singularities in QCD

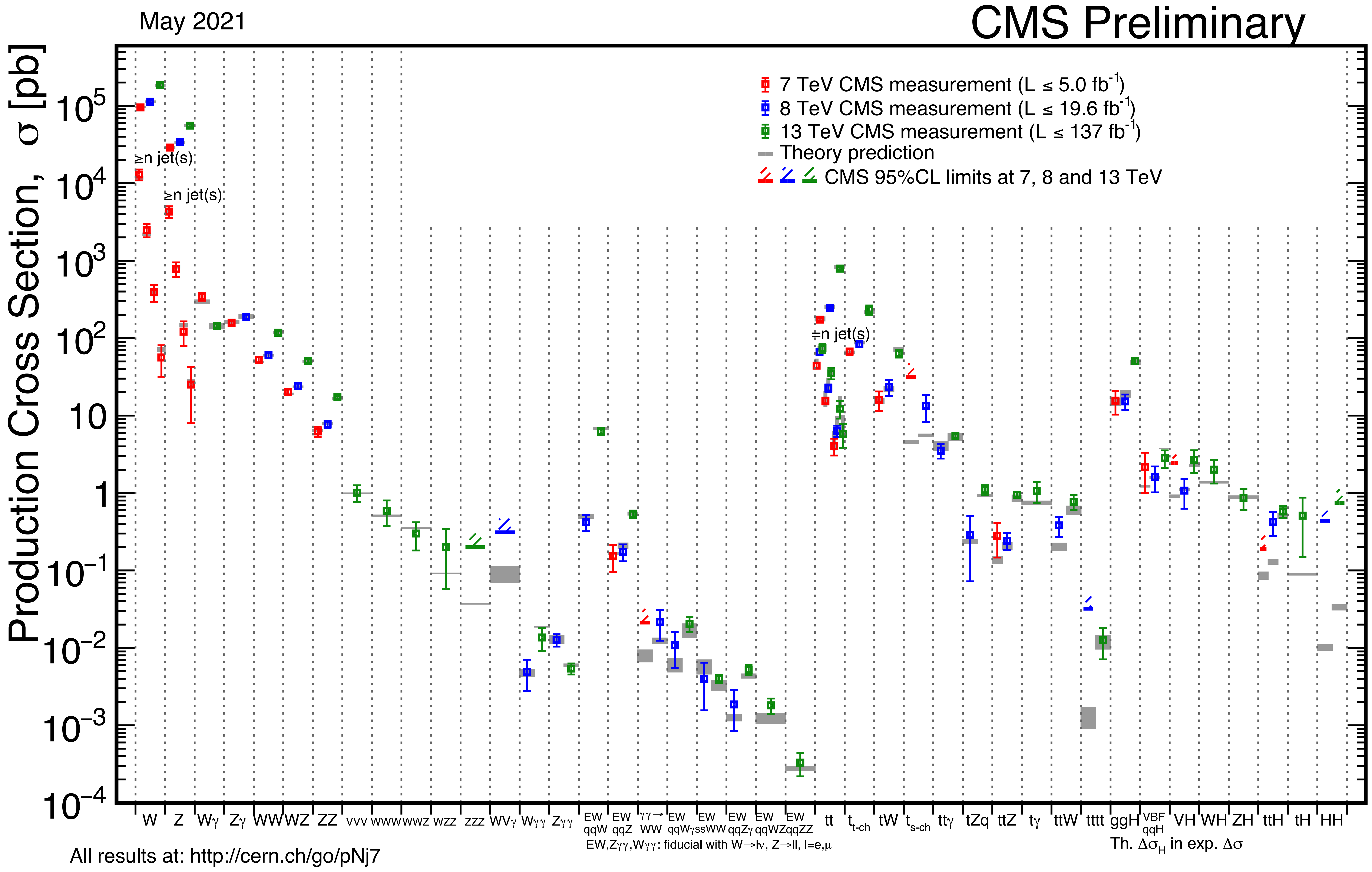
Cortona Young 2021

Chiara Signorile-Signorile

Based on: *Magnea, Maina, Pelliccioli, C.S., Torrielli, Uccirati, JHEP12(2018)107, JHEP02(2021)037*

Motivations

LHC continues to confirm the Standard Model

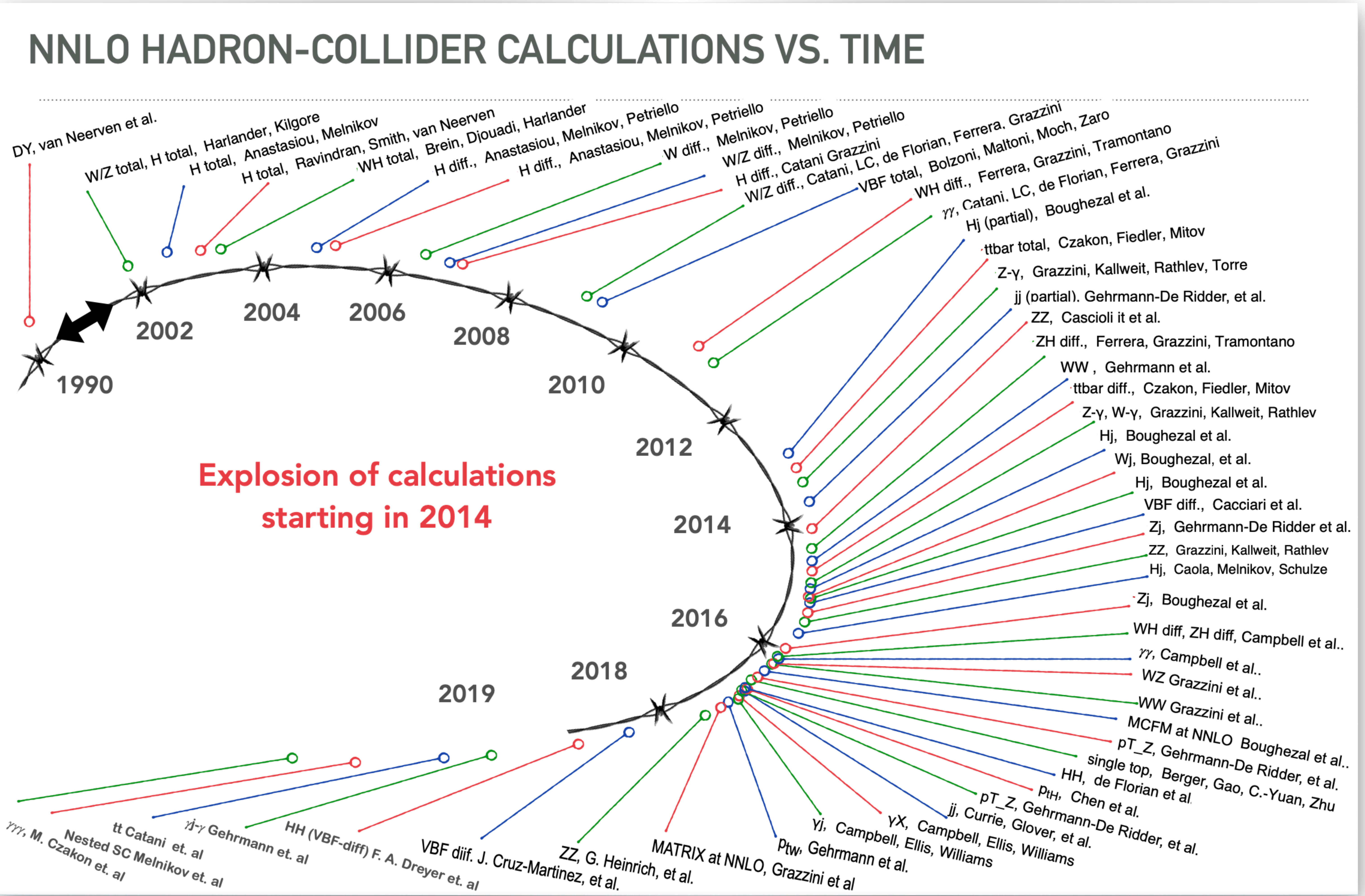


- Direct Search for BSM:
  many proposal,
  no obvious candidate

- Indirect Search for BSM:
  small corrections to SM

High precision
Theoretical Prediction

Motivations



Talk by L. Cieri, WG1 Meeting, 2019

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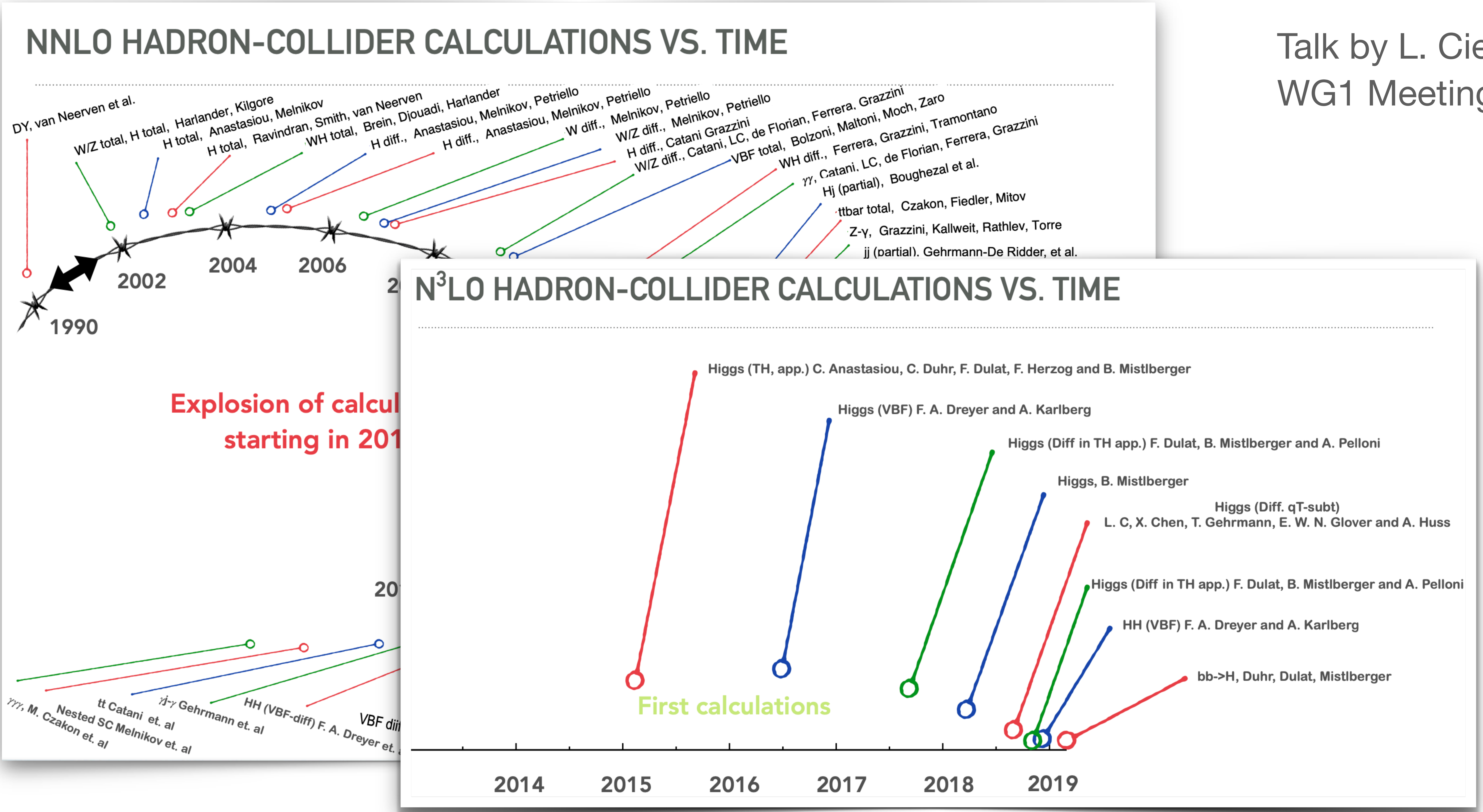

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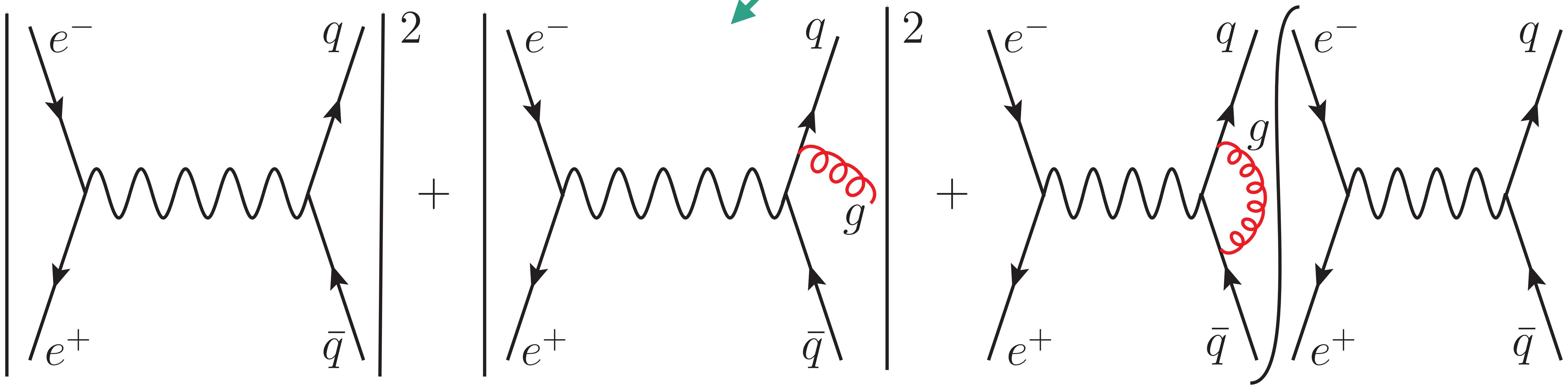
Motivations

Computation of higher orders in perturbative QCD non-trivial due to **IR singularities**

$$d\sigma^{\text{NLO}}_{e^+e^-\to q\bar{q}} = d\sigma^{\text{B}} + \alpha_s\, d\sigma^{\text{R}} + \alpha_s\, d\sigma^{\text{V}}$$



$d = 4 - 2\epsilon$

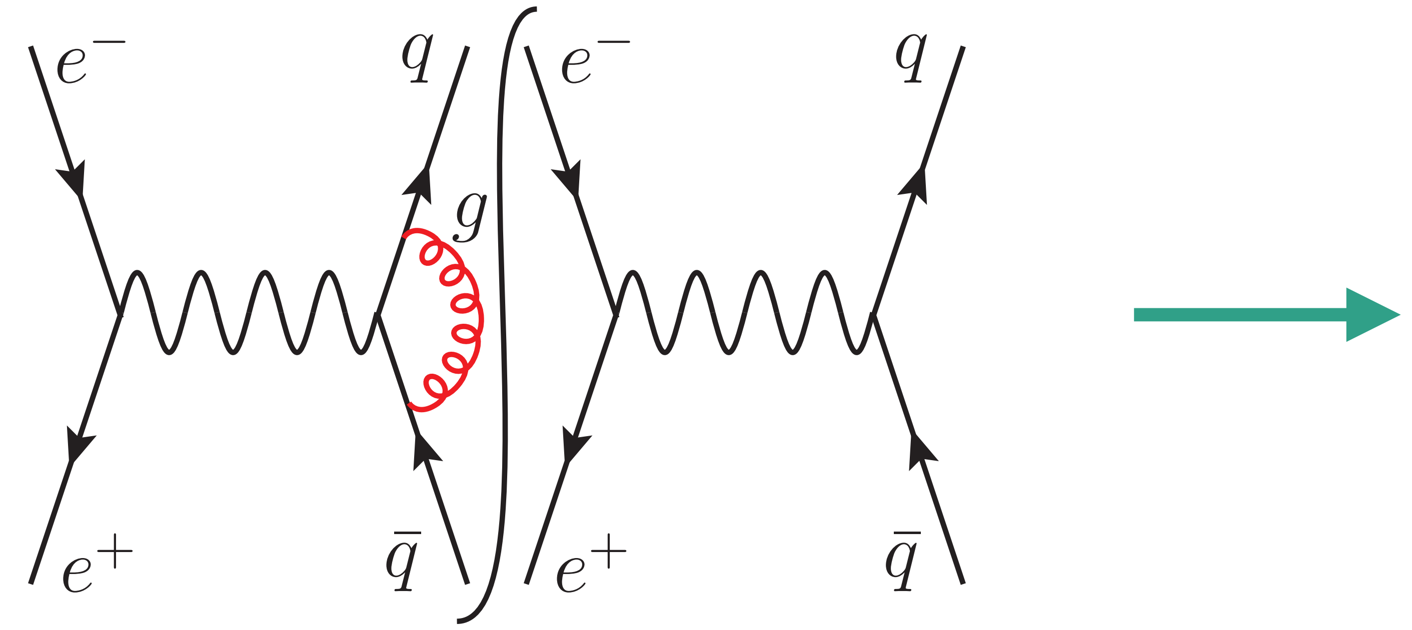
Implicit **phase space singularities**

Explicit **loop singularities**

KLN Theorem

IR singularities vanish if all degenerated states with arbitrary multiplicities in the final state are combined *[Kinoshita ’62; Lee, Nauenberg ’64]*

Real and virtual radiation



In $d = 4 - 2\epsilon$ the explicit poles of **1-loop** and **2-loop** amplitudes are known **independent** of the hard subprocess

$$\mathscr{A}_{1\text{loop}} = \left[\frac{e^{\epsilon\gamma_E}}{\Gamma(1-\epsilon)} \sum_i \left(\frac{1}{\epsilon^2} + \frac{\gamma_i}{\mathbf{T}_i^2}\frac{1}{\epsilon}\right) \sum_{i \neq j} \frac{\mathbf{T}_i \cdot \mathbf{T}_j}{2} \left(\frac{\mu^2}{-s_{ij}}\right)^{\epsilon}\right] \mathscr{A}_{\text{tree}} + \mathscr{A}^{\text{fin}}_{1\text{loop}}$$

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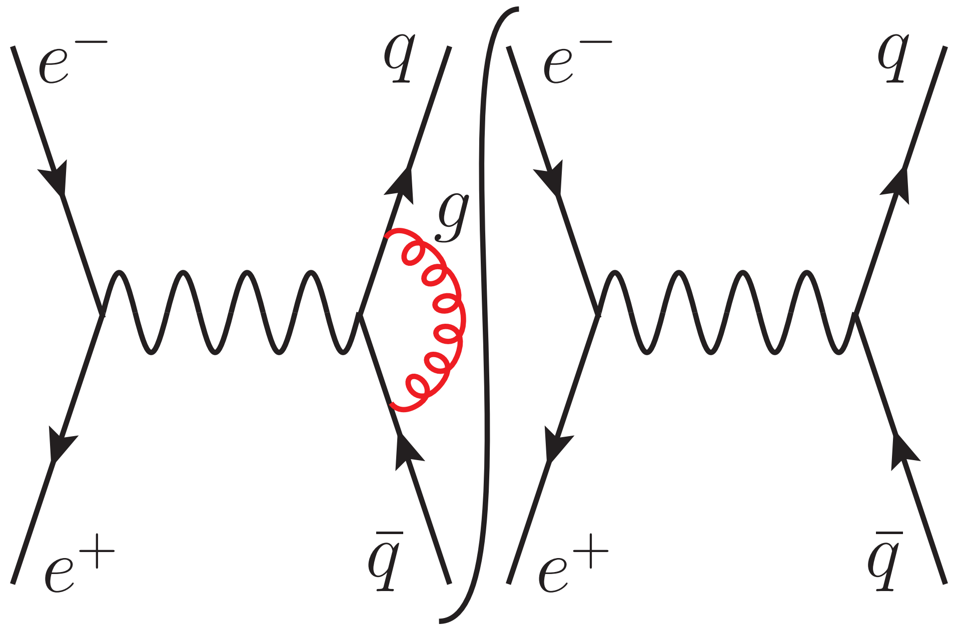

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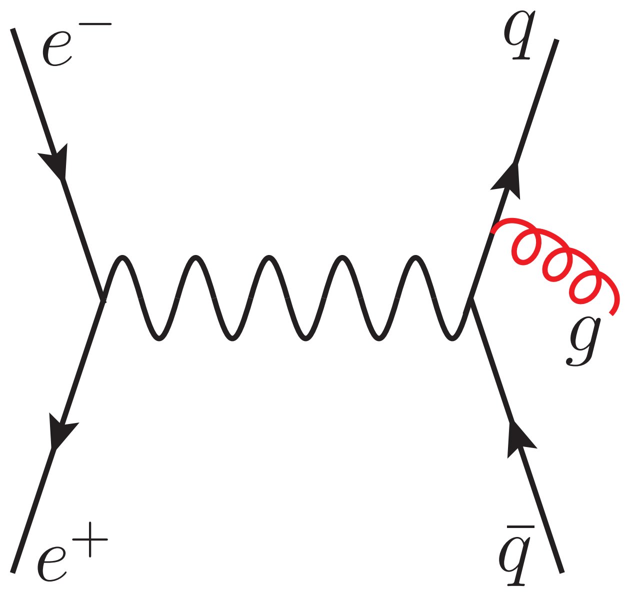

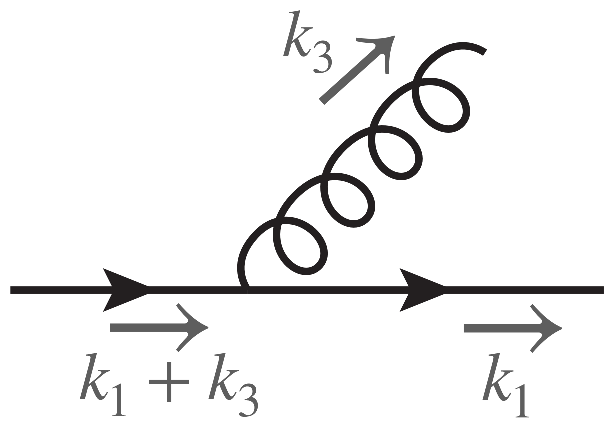

$$\sim \frac{1}{(k_1 + k_3)^2} \sim \frac{1}{E_1 E_3 (1 - \vec{n}_1 \cdot \vec{n}_3)} \to \infty \quad \begin{cases} E_3 \to 0 & \textit{soft} \\ \vec{n}_1 \parallel \vec{n}_3 & \textit{collinear} \end{cases}$$

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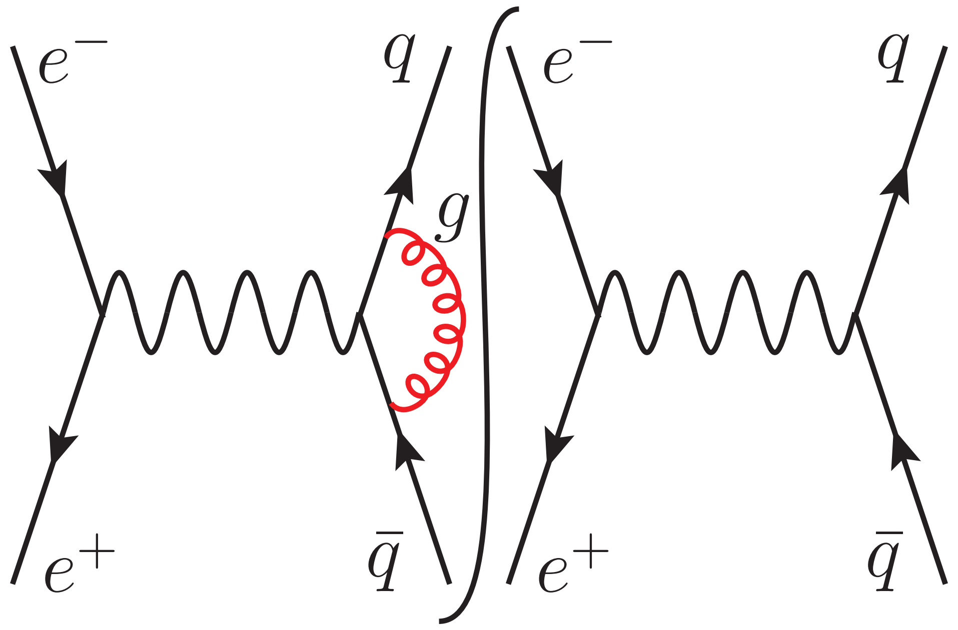

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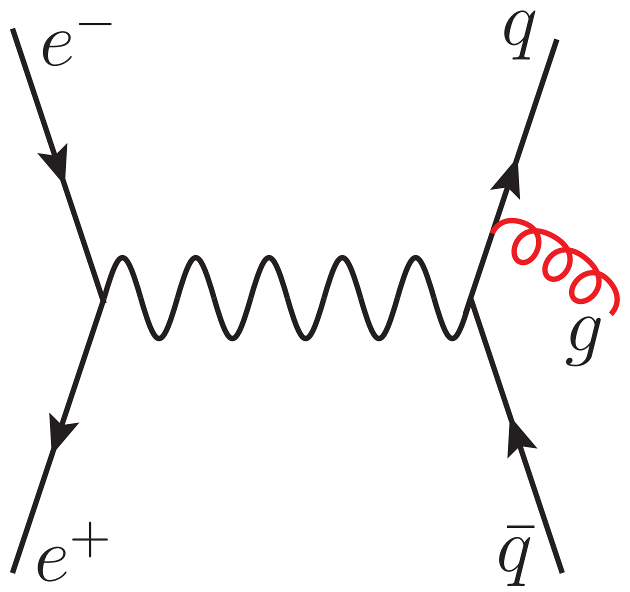

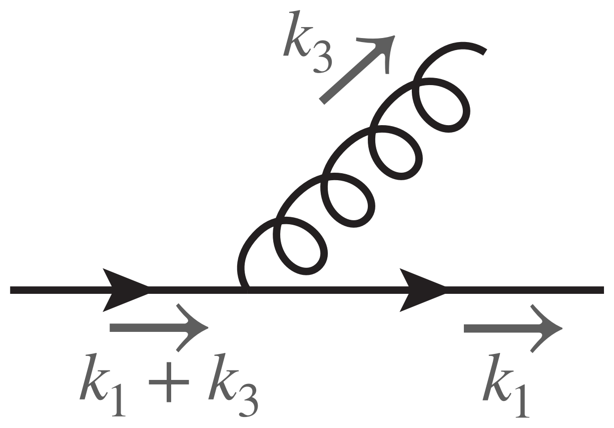

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Singular limits have universal form, independent of the resolved subprocess *[Altarelli, Parisi ’77]*

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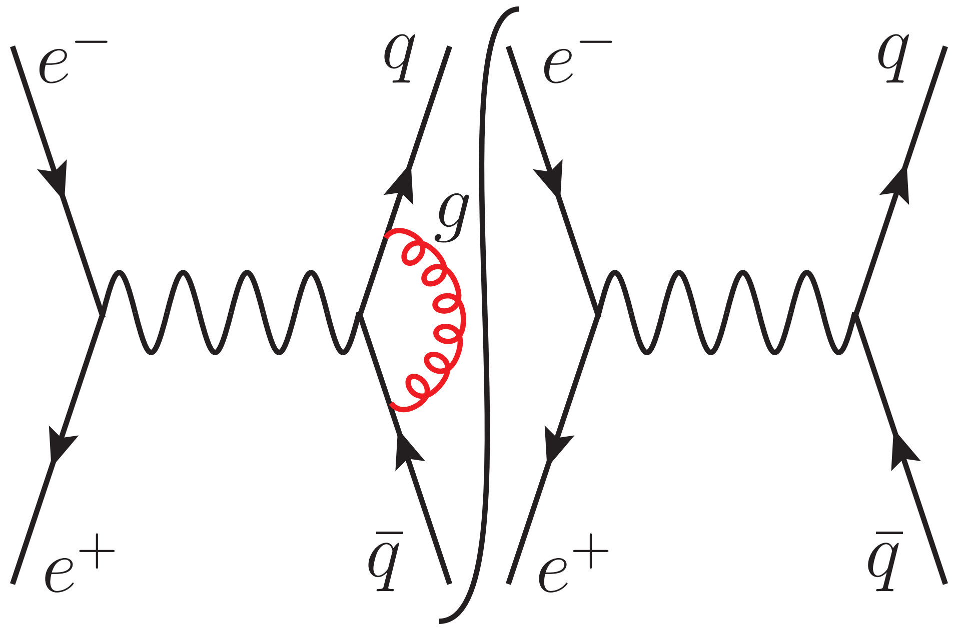

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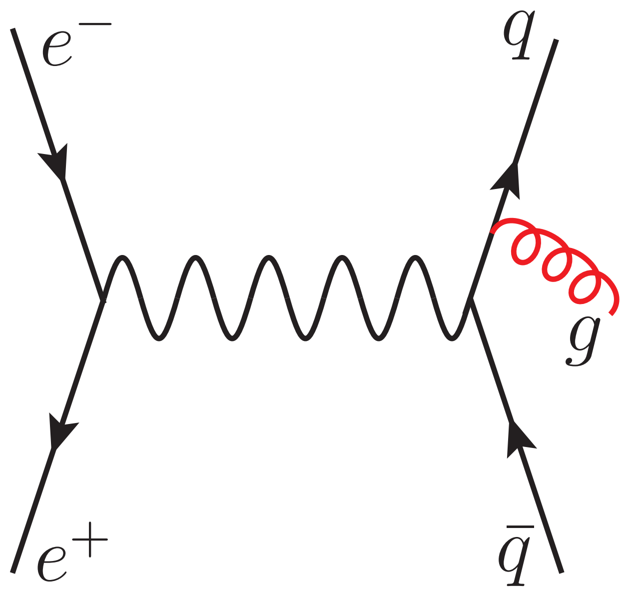

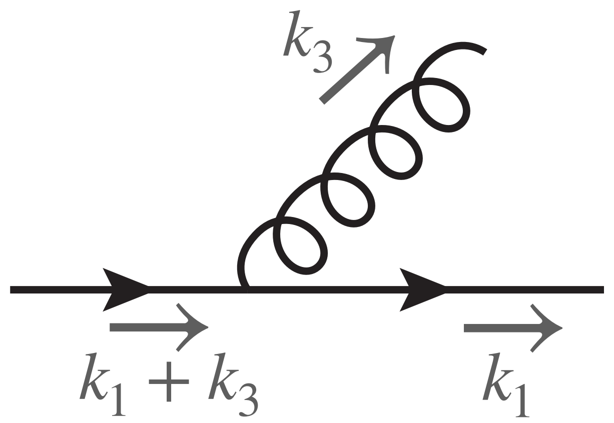

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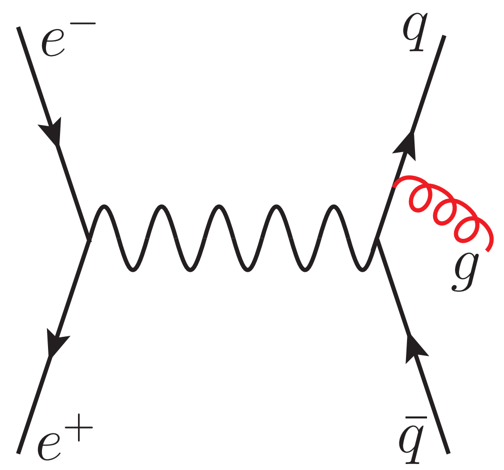
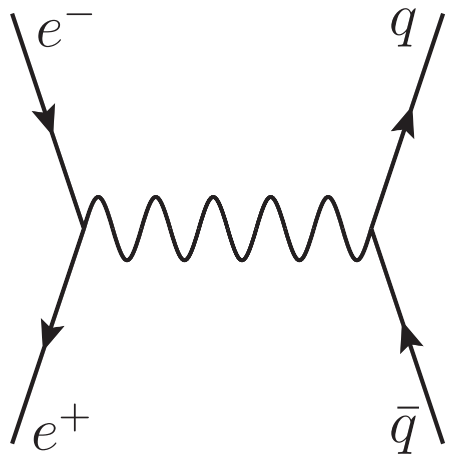
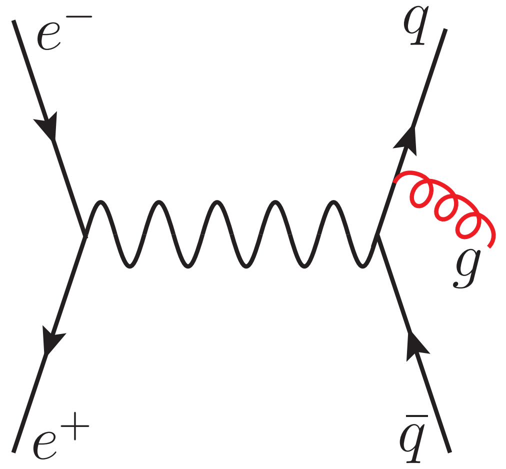
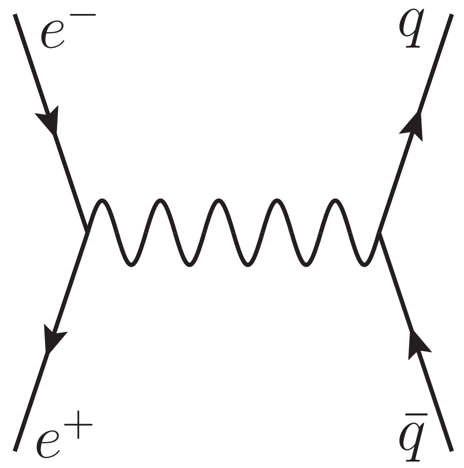
$\left|\right.$  $\left.\right|^2 \underset{k_1\|k_3}{\sim} C_F\, g_s^2 \dfrac{1}{k_1\cdot k_3} \boxed{P_{qg}(z)} \left|\right.$  $\left.\right|^2$

$\left|\right.$  $\left.\right|^2 \underset{E_3\to 0}{\sim} 2C_F\, g_s^2 \boxed{\dfrac{k_1\cdot k_2}{(k_1\cdot k_3)(k_2\cdot k_3)}} \left|\right.$  $\left.\right|^2$

eikonal kernel

Splitting kernel $\quad P_{qg}(z) = \dfrac{1+z^2}{1-z} - \epsilon(1-z)\,, \qquad z = \dfrac{s_{12}}{s_{12}+s_{23}}$

The problem

1. **Regulate** infrared singularities of the **real emission**
2. **Extract** infrared $\frac{1}{\epsilon}$ poles in d-dimension **without integrating over the resolved phase space**

   $\longrightarrow$ **fully differential** distributions
3. **Cancel** the $\frac{1}{\epsilon}$ poles stemming from the phase space integration against the poles of the **loop correction**

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Fully general solution?

- **Phase space singularities** of the real radiation
- Explicit **virtual poles**

} Known **independent** on the hard subprocess

⟶ A general procedure seems to be practicable, although non trivial to implement

- ↗ **Subtraction**
- ↘ **Slicing**

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→ **Subtraction** ⟵ This talk

→ **Slicing**

Actual implementation: the subtraction strategy

$F(x)$ arbitrary complicated function

$$I = \lim_{\epsilon \to 0} \left[\int_0^1 \frac{dx}{x} x^\epsilon F(x) - \frac{1}{\epsilon} F(0) \right]$$

Compute *I* without relying on the analytic evaluation of the integral

$$I = \lim_{\epsilon \to 0} \Bigg[\underbrace{\int_0^1 \frac{dx}{x} x^\epsilon \Big(F(x) - F(0) \Big)}_{\text{Regulated, finite for } \epsilon \to 0} + \underbrace{\int_0^1 \frac{dx}{x} x^\epsilon F(0) - \frac{1}{\epsilon} F(0)}_{\text{Extract } 1/\epsilon \text{ pole}} \Bigg]$$

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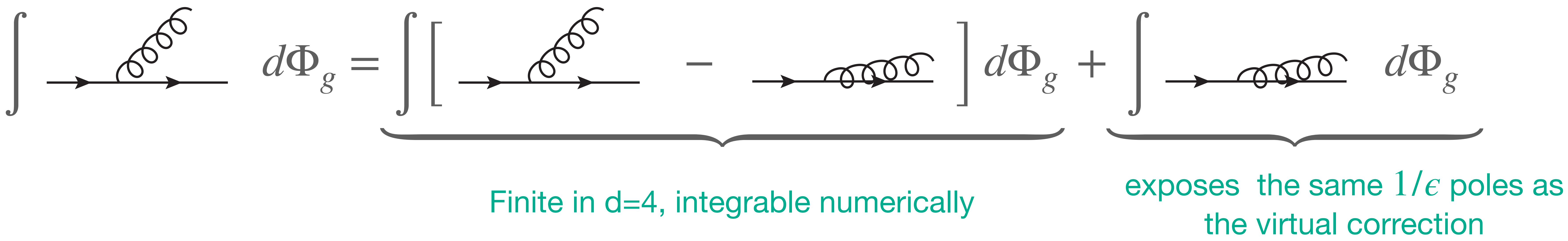
Use a **subtraction scheme** to make poles from **real radiation explicit**



Well established schemes at NLO

- **Catani-Seymour (CS)** *[9602277]*
- **Frixione-Kunst-Signer (FKS)** *[9512328]*
- Nagy-Soper *[0308127]*

What about the NNLO?

Many schemes are available: Antenna *[Gehermann-De Ridder et al. 0505111]*, ColorfullNNLO *[Del Duca et al. 1603.08927]*, Nested-soft-collinear subtraction *[Caola et al. 1702.01352]*, Residue subtraction *[Czakon 1005.0274]*

New strategies have been explored: Geometric IR subtraction *[Herzog 1804.07949]*, Unsubtraction *[Sborlini et al. 1608.01584]*, FDR *[Pittau, 1208.5457]*, Universal Factorisation *[Sterman 2008.12293]*

None of the existing subtraction schemes satisfies **all the '5 criteria'** *[Melnikov, Amplitude 2019]*

1) **Physical transparency**
2) **Generality**
3) **Locality**
4) **Analyticity**
5) **Efficiency**

Local Analytic Sector Subtraction

Go **back to NLO** to implement a new scheme featuring **key properties** that can be exported at **NNLO**.

(This talk: massless partons, FSR only, arbitrary number of FS particles)

$$\frac{d\sigma^{\mathrm{NLO}}}{dX} = \lim_{d\to4}\left\{ \int d\Phi_n \, V_n \delta_n + \int d\Phi_{n+1} \, R_{n+1}\delta_{n+1} \right\}$$

X IR safe observable

$$\frac{d\sigma^{\mathrm{NLO}}_{ct}}{dX} = \int d\Phi_{n+1} \, \overline{K}_{n+1}$$

Counterterm

$$I_n = \int d\Phi_{\mathrm{rad}} \, \overline{K}_{n+1}$$

Integrated Counterterm

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Properties of the scheme:

Analytically calculable (possibly with standard techniques)

Minimal structure and **simple integration**

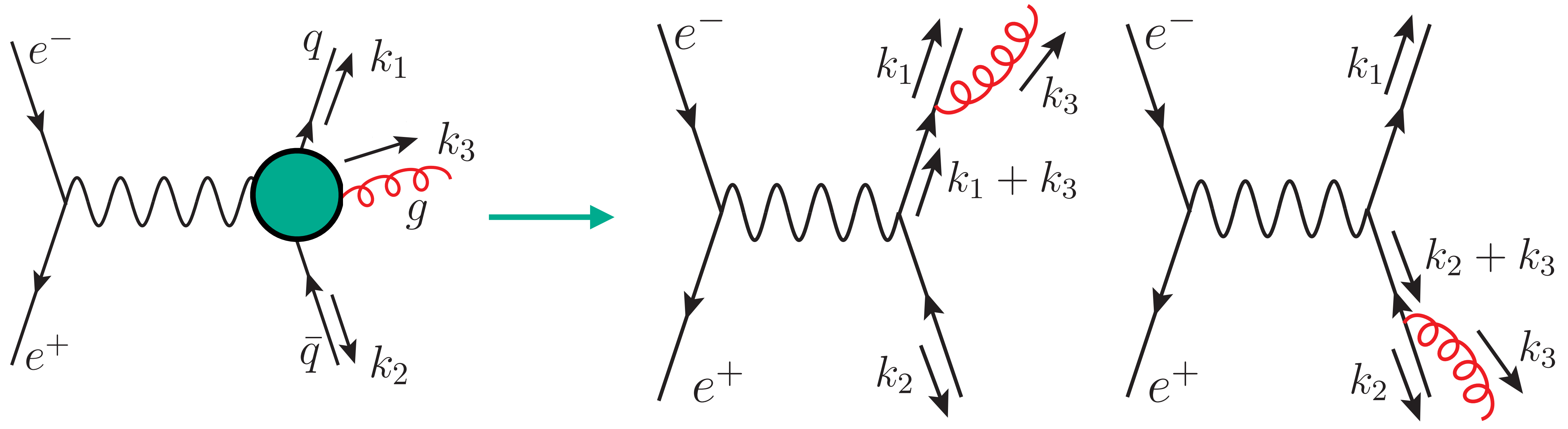
Require:

Choose an **optimise parametrisation** of the phase space

Organise all the overlapping **singularities** and choose an **appropriate kinematics**

Ingredients of the subtraction

- *Phase space partitioning (FKS) : multiple singular configuration that overlap*



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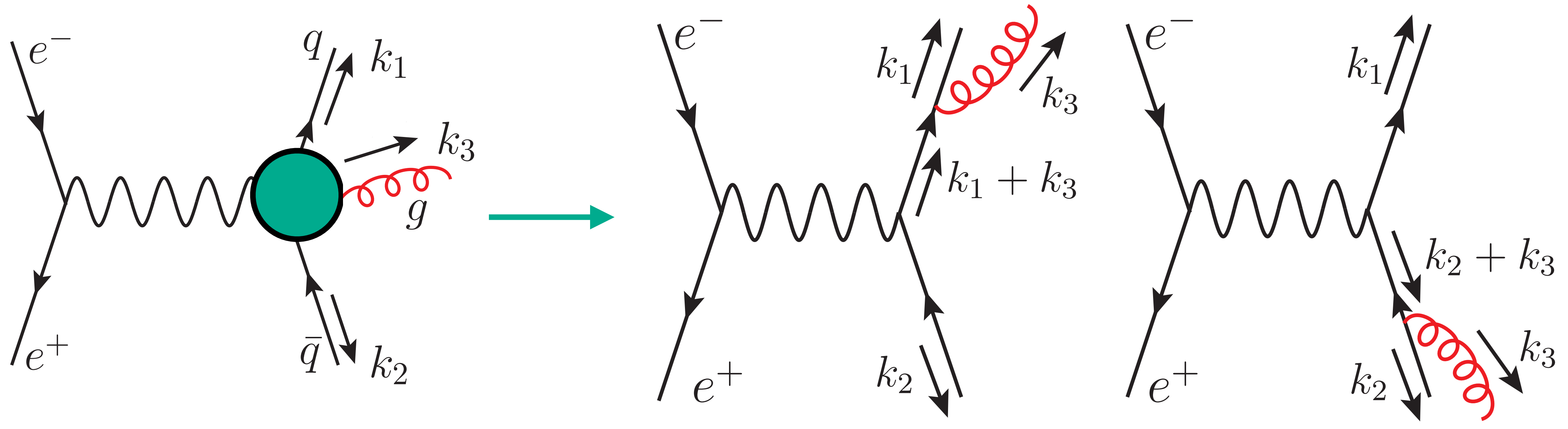

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$$R \;\to\; \infty \quad \begin{cases} E_3 \to 0 & \to\; S_3 \\ \vec{n}_1 \parallel \vec{n}_3 & \to\; C_{13} = C_{31} \\ \vec{n}_2 \parallel \vec{n}_3 & \to\; C_{23} = C_{32} \end{cases}$$

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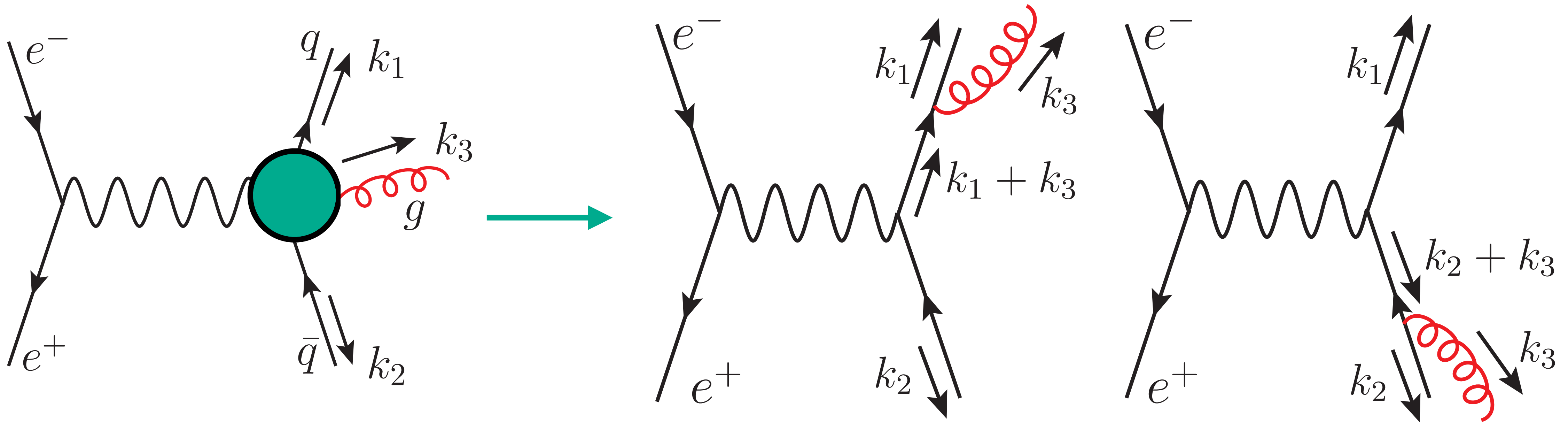

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Sector functions $\mathcal{W}_{ij}$:

At most one soft and/or two collinear partons in each sector

$$R = \sum_{i,j} R\,\mathcal{W}_{ij} = R\,\mathcal{W}_{31} + R\,\mathcal{W}_{32} + \ldots$$

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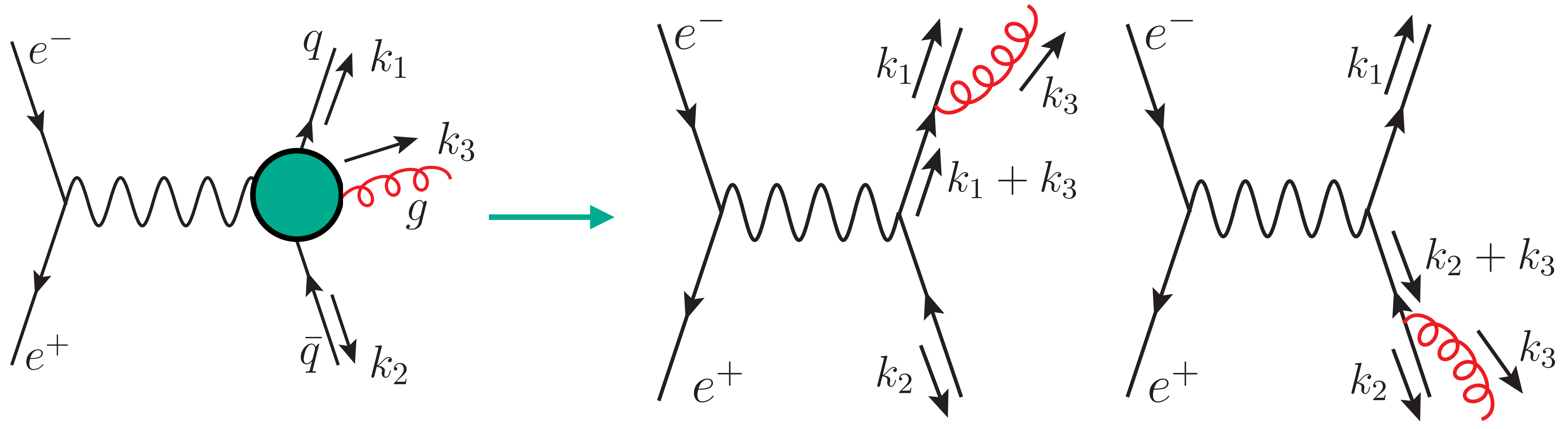

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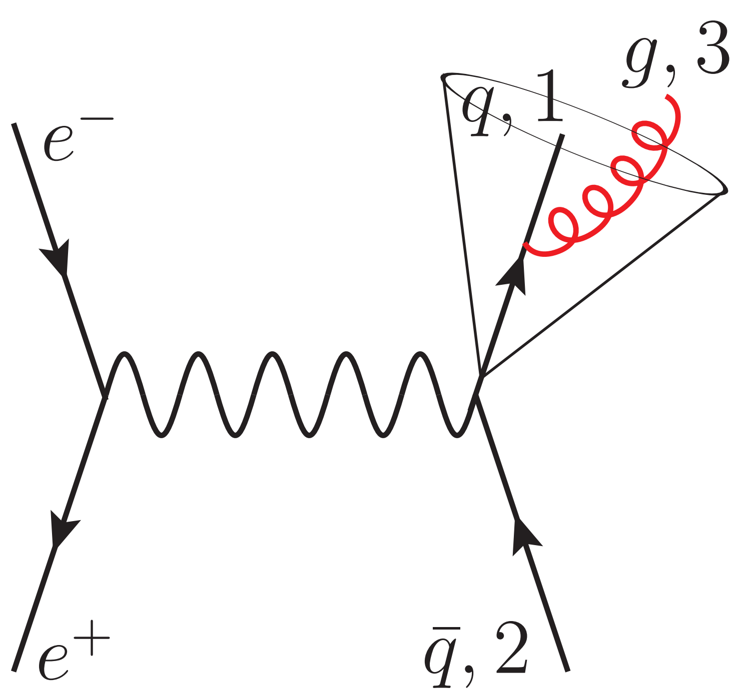

Damp: $\vec{n}_2 \parallel \vec{n}_3$

Enhance: $\vec{n}_1 \parallel \vec{n}_3$

$\mathcal{W}_{31} \sim \frac{1}{s_{31}}$

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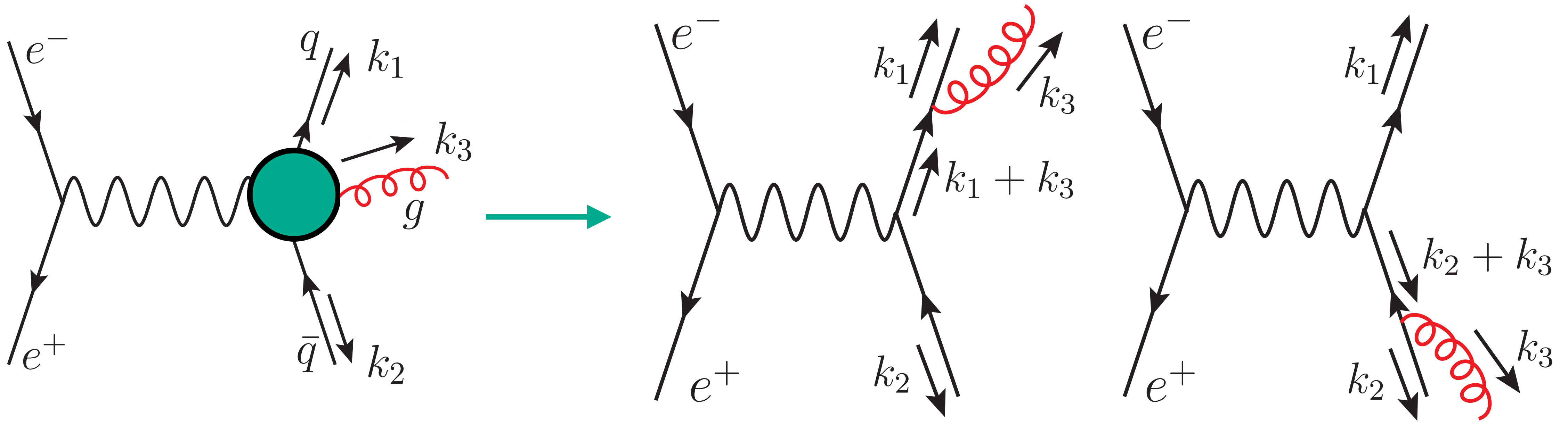

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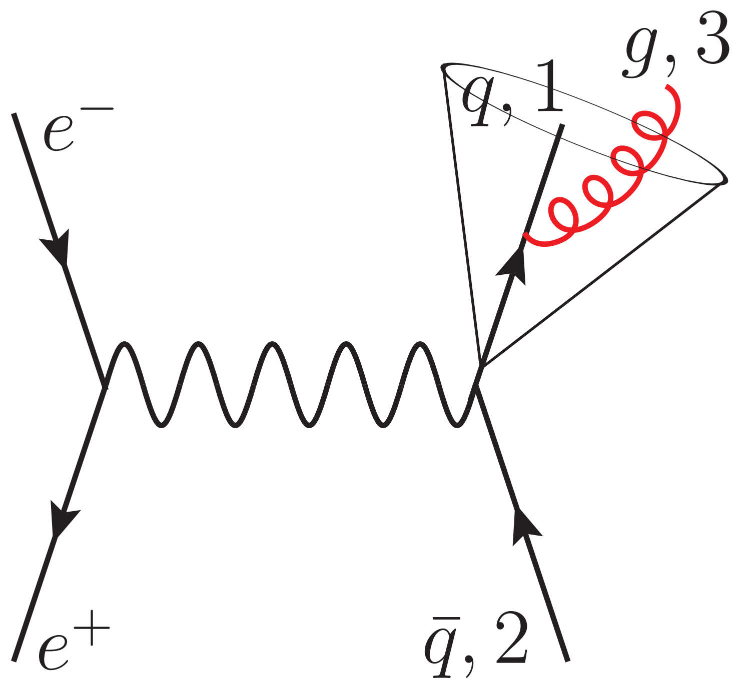

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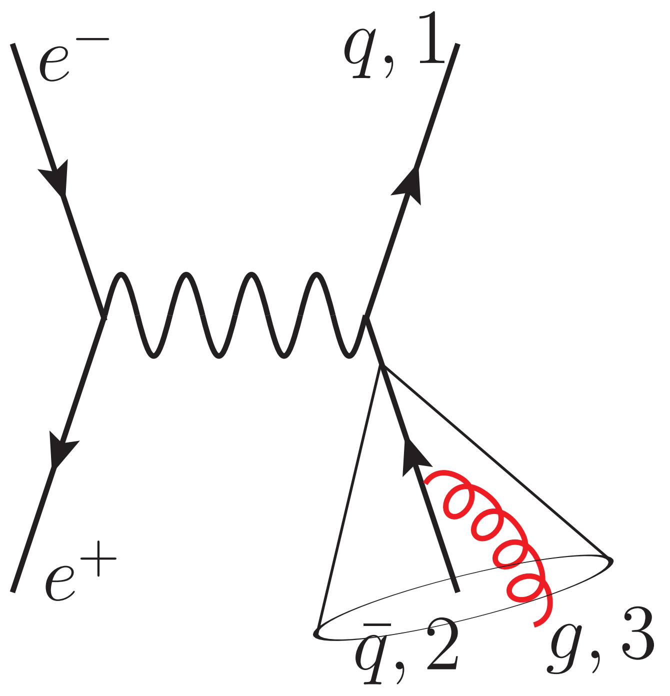

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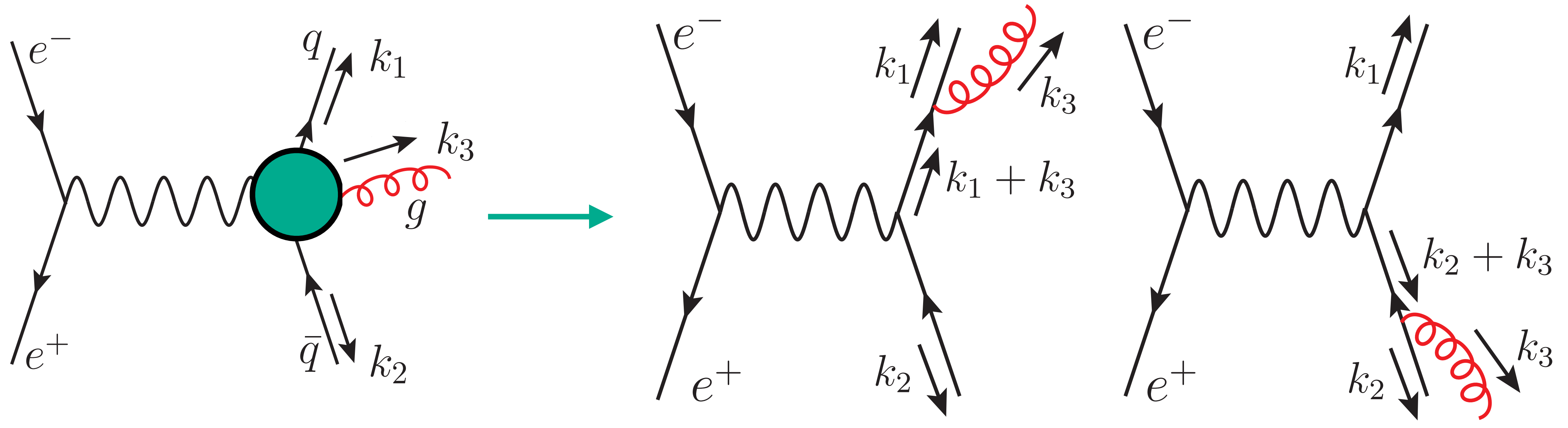
- *Phase space partitioning (FKS) : multiple singular configuration that overlap*



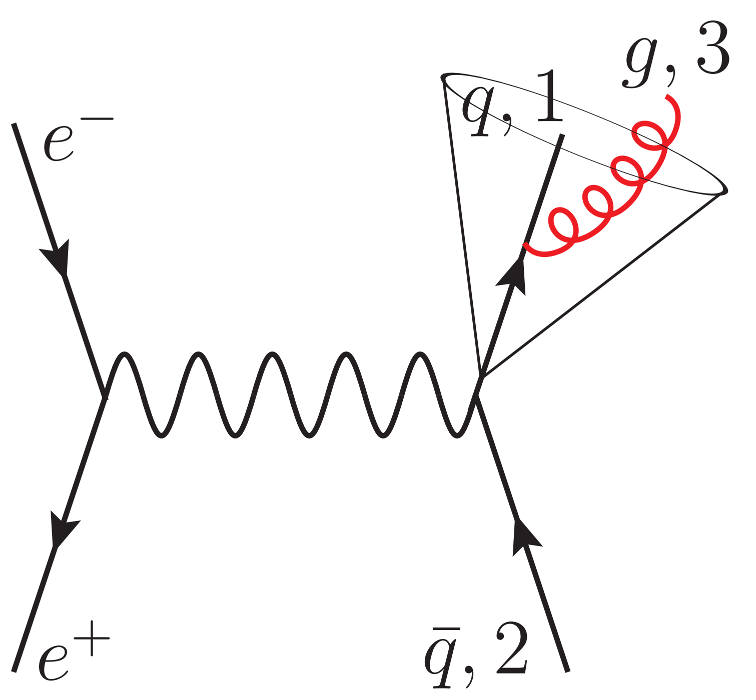
$$R \sim \frac{1}{(k_1+k_3)^2} + \frac{1}{(k_2+k_3)^2} \sim \frac{1}{E_1 E_3 (1-\vec{n}_1 \cdot \vec{n}_3)} + \frac{1}{E_2 E_3 (1-\vec{n}_2 \cdot \vec{n}_3)}$$

$$R \to \infty \quad \begin{cases} E_3 \to 0 & \to S_3 \\ \vec{n}_1 \parallel \vec{n}_3 & \to C_{13} = C_{31} \\ \vec{n}_2 \parallel \vec{n}_3 & \to C_{23} = C_{32} \end{cases}$$

Sector functions $\mathscr{W}_{ij}$:

At most one soft and/or two collinear partons in each sector

$$R = \sum_{i,j} R\,\mathscr{W}_{ij} = R\,\mathscr{W}_{31} + R\,\mathscr{W}_{32} + \ldots$$



Damp: $\vec{n}_2 \parallel \vec{n}_3$

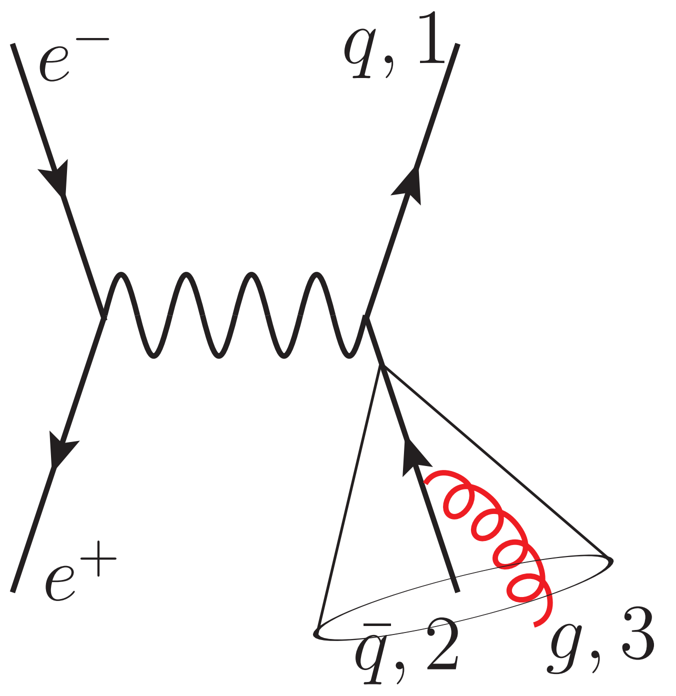
Enhance: $\vec{n}_1 \parallel \vec{n}_3$

$\mathscr{W}_{31} \sim \frac{1}{s_{31}}$

Damp: $\vec{n}_1 \parallel \vec{n}_3$

Enhance: $\vec{n}_2 \parallel \vec{n}_3$

$\mathscr{W}_{32} \sim \frac{1}{s_{32}}$



preserving Lorentz invariance

Ingredients of the subtraction

- *Momentum mapping (CS):*

  Factorise the phase space $d\Phi_{n+1} = d\overline{\Phi}_n \, d\overline{\Phi}_{\rm rad}$

  On-shell particle **conserving momentum** in the entire PS

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On-shell particle **conserving momentum** in the entire PS

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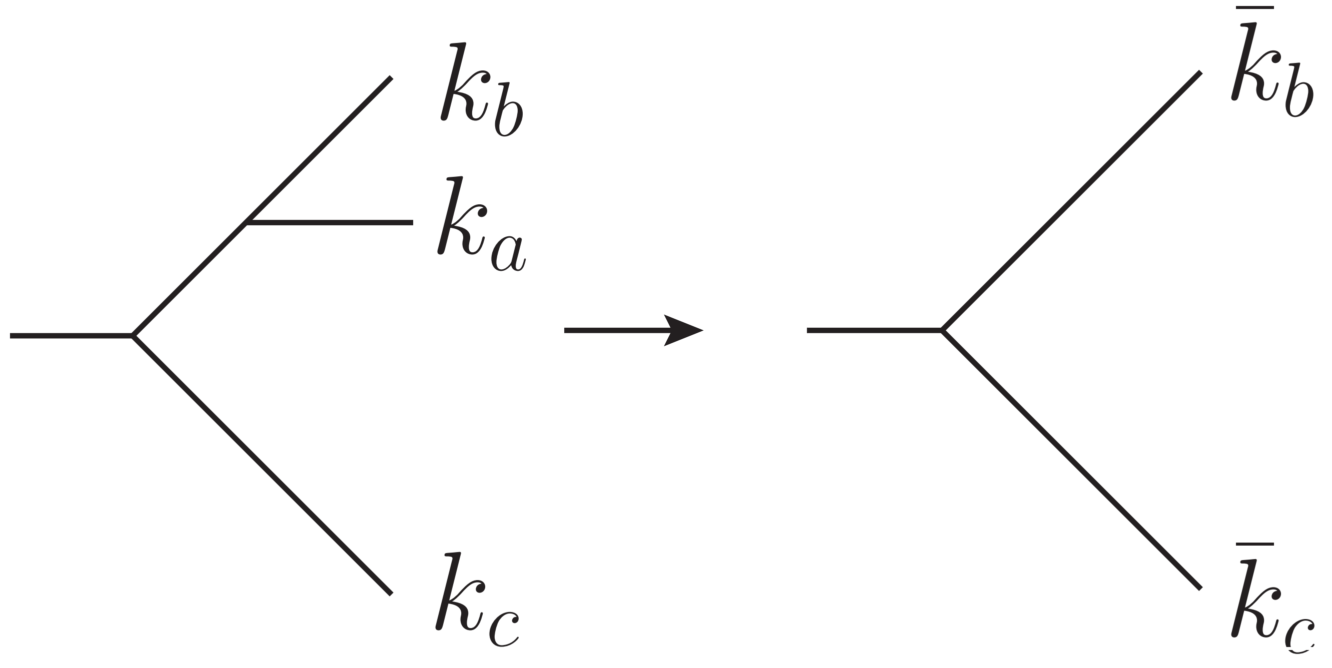

Mapped kinematics $\{\bar{k}\}^{(abc)} = \left\{ \{k\}_{\not{a}\not{b}\not{c}}, \bar{k}_b^{(abc)}, \bar{k}_c^{(abc)} \right\}$

$$\bar{k}_b^{(abc)} + \bar{k}_c^{(abc)} = k_a + k_b + k_c$$

Different ways to combine momenta, depending on the **choice** of the dipole (abc)

→ Freedom to choose the momenta to **simplify the integration**

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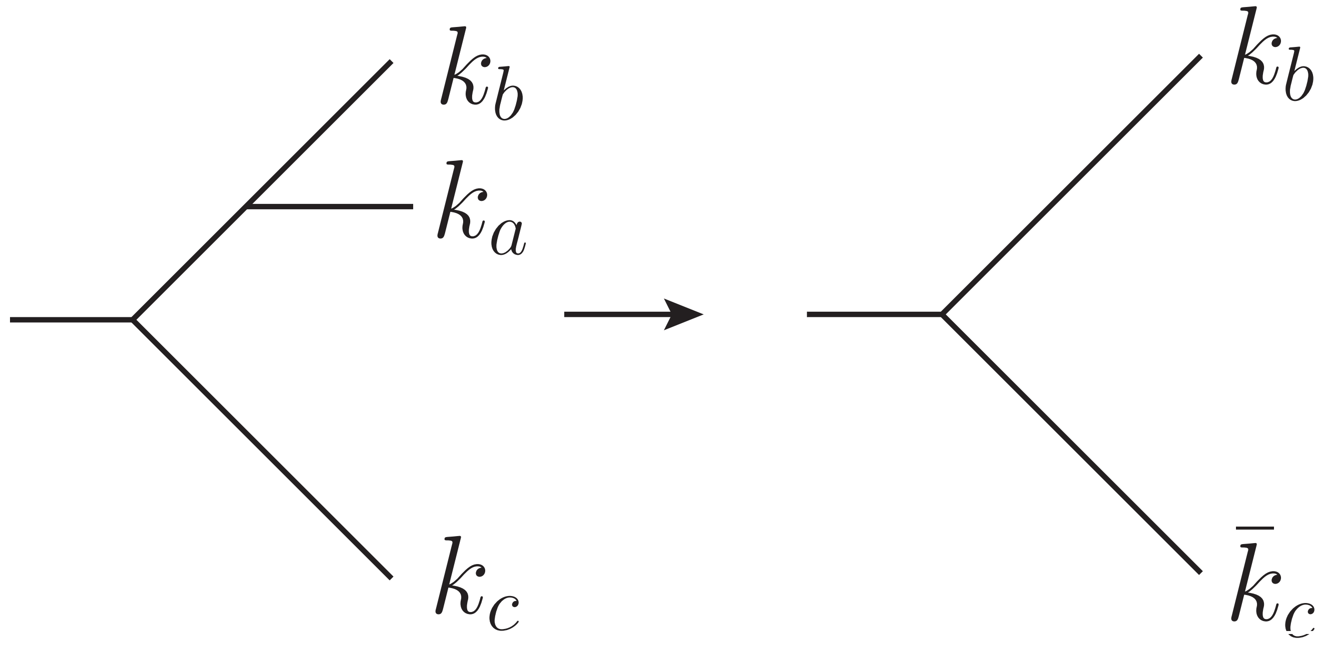

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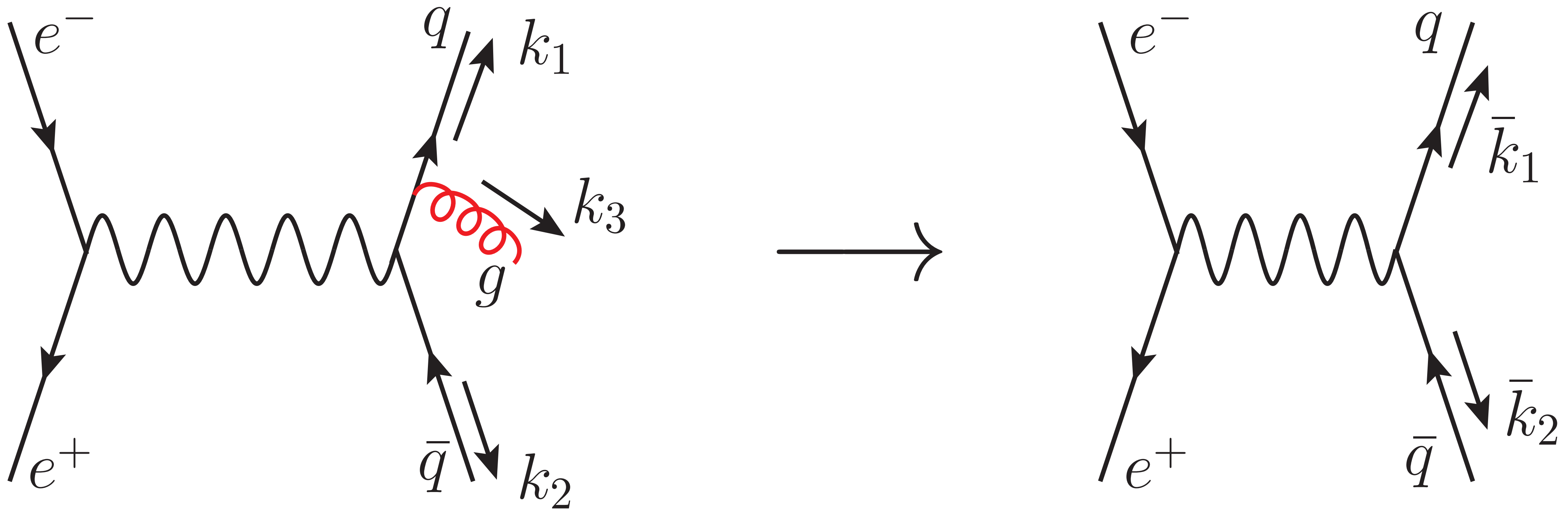
Different ways to combine momenta, depending on the **choice** of the dipole (abc)

→ Freedom to choose the momenta to **simplify the integration**

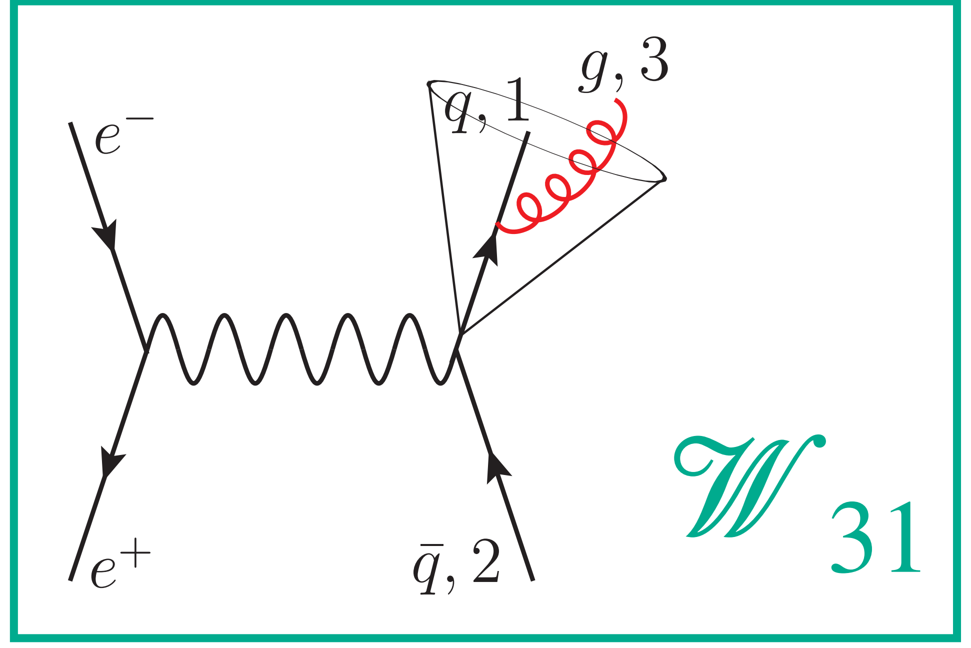
$$k_1, k_2, k_3, k_i^2 = 0$$

$$\bar{k}_2^{(312)} = \frac{s_{312}}{s_{32} + s_{12}} k_2$$

$$\bar{k}_1^{(312)} = k_3 + k_1 - \frac{s_{31}}{s_{32} + s_{12}} k_2$$



Ingredients of the subtraction



- *Candidate counterterm:*

Defined **sector by sector** as the collection of all the contributing limits (correct multiplicity!)

iterative definition $\left(1 - \overline{\mathbf{S}}_3\right)\left(1 - \overline{\mathbf{C}}_{13}\right) R\,\mathcal{W}_{31} = \text{finite}$

$$\overline{K}_{31} = \left[\overline{\mathbf{S}}_3 + \overline{\mathbf{C}}_{13}\left(1 - \overline{\mathbf{S}}_3\right)\right] R\,\mathcal{W}_{31} \;\rightarrow\; R\,\mathcal{W}_{31} - \overline{K}_{31} = \text{finite}$$

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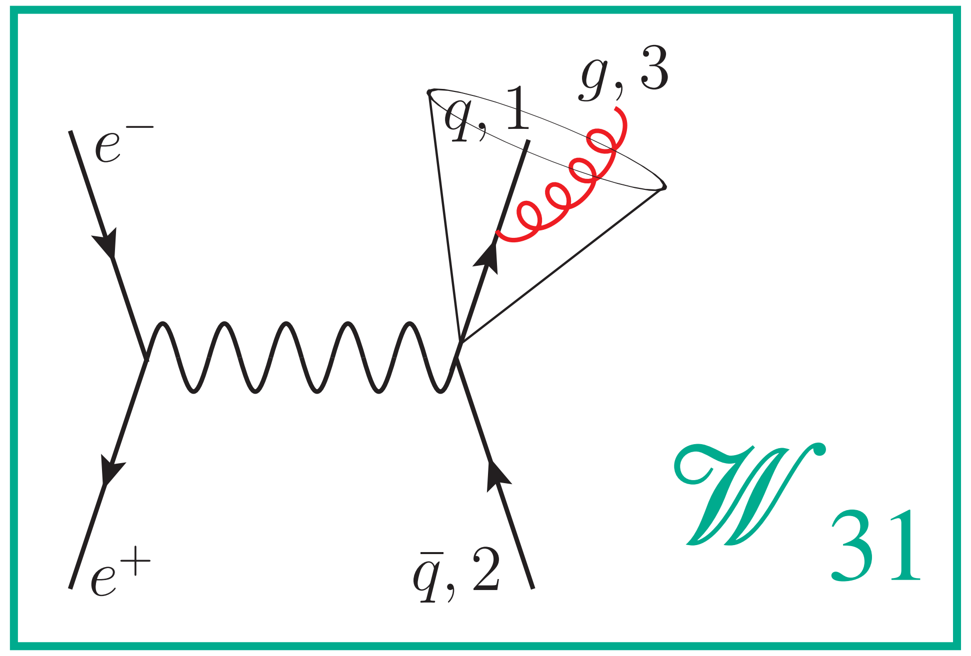

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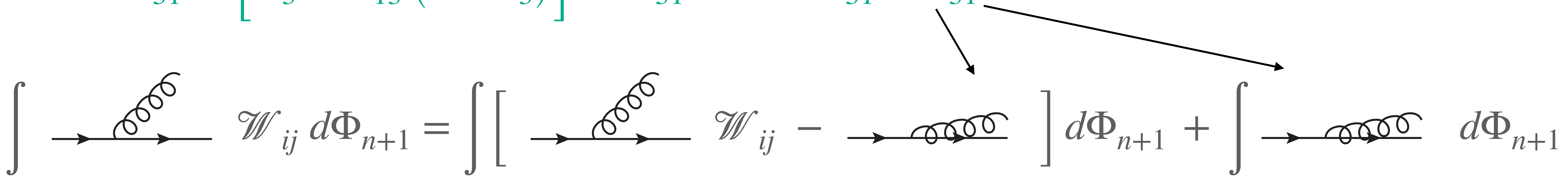

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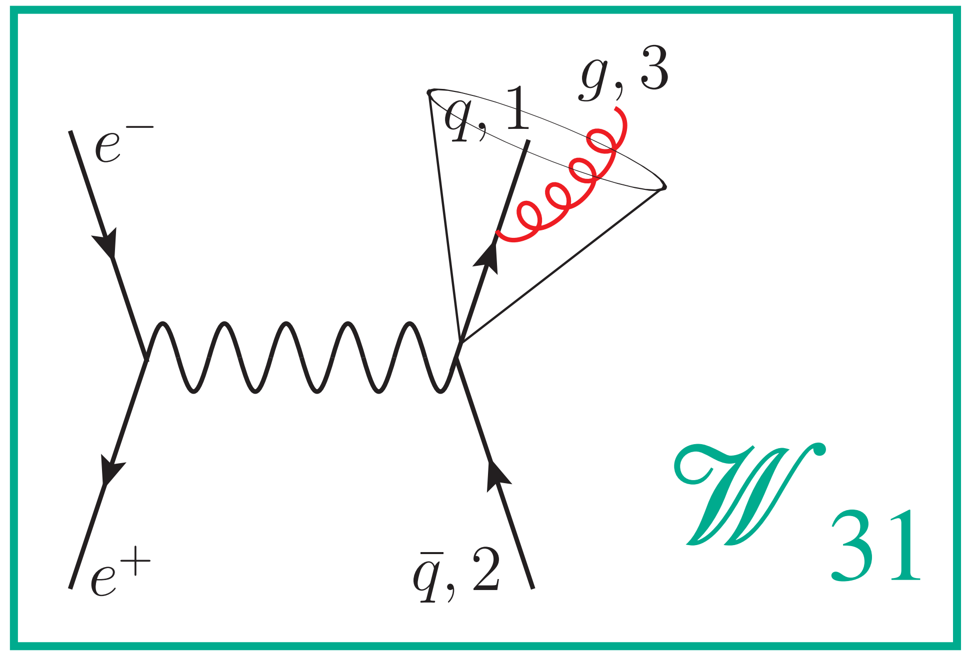

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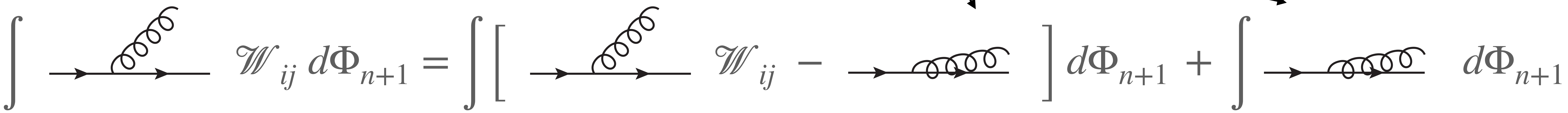

Featuring **optimised remapping** for integration $\{k_{n+1}\} \rightarrow \{k_n\}^{(abc)}$

(abc) **according to the invariants** appearing in the kernel

$$\overline{\mathbf{S}}_i R(\{k\}) \propto \sum_{c,d\neq i} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}\left(\{k\}^{(icd)}\right) \longrightarrow \text{Different mapping for each contribution}$$

$$\overline{\mathbf{C}}_{ij} R(\{k\}) \propto \frac{1}{s_{ij}} P_{ij}^{\mu\nu} B_{\mu\nu}\left(\{k\}^{(ijr)}\right) \longrightarrow \text{Single mapping}$$

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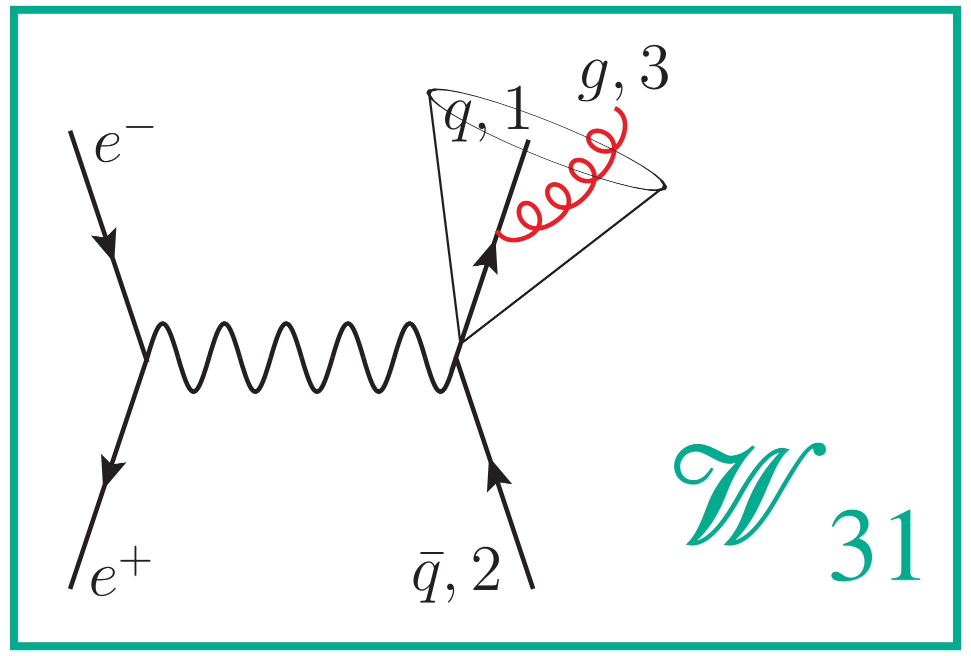

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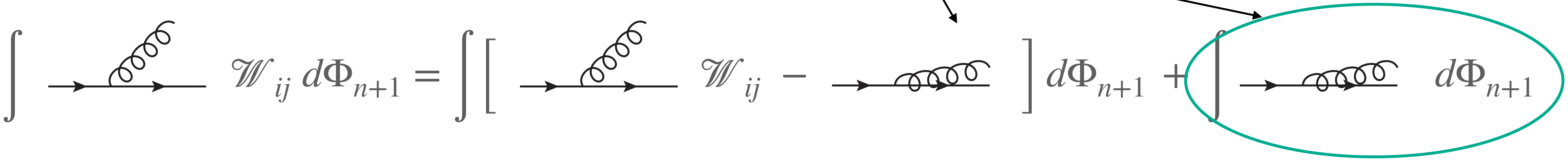

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Single mapping

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- *Analytic integration:*

Parametrisation of the phase space

$$d\Phi_{n+1} = d\Phi_n^{(abc)} \times d\Phi_{\mathrm{rad}}\left(s_{bc}^{(abc)}; y, z, \phi\right) \qquad s_{ab} = y s_{bc}^{(abc)}, \quad s_{ac} = z(1-y) s_{bc}^{(abc)}, \quad s_{bc} = (1-z)(1-y) s_{bc}^{(abc)}$$

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Freedom to **adapt** the **parametrisation to the kernel** ⟶ Exact analytic integration

$$I^s \propto \sum_{c,d} \int d\Phi_{\text{rad}}\left(s_{cd}^{(icd)}; y, z, \phi\right) \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}\left(\{\bar{k}\}^{(icd)}\right) \propto \sum_{c,d} B_{cd}\left(\{\bar{k}\}^{(icd)}\right) \frac{(4\pi)^{\epsilon-2}\,\Gamma(1-\epsilon)\,\Gamma(2-\epsilon)}{\epsilon^2 \Gamma(2-3\epsilon)}$$

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Lesson from NLO

- **Partition functions** are useful tools to organise the singularities, as in **FKS**
- Adapt **CS mappings** to the singular kernels ⟶ simplifications in **analytic counterterm integration**

Key **properties at NLO:** **sector approach** and **minimal structure** from **FKS,** **Lorentz invariance** and **analytic integration** from **CS**

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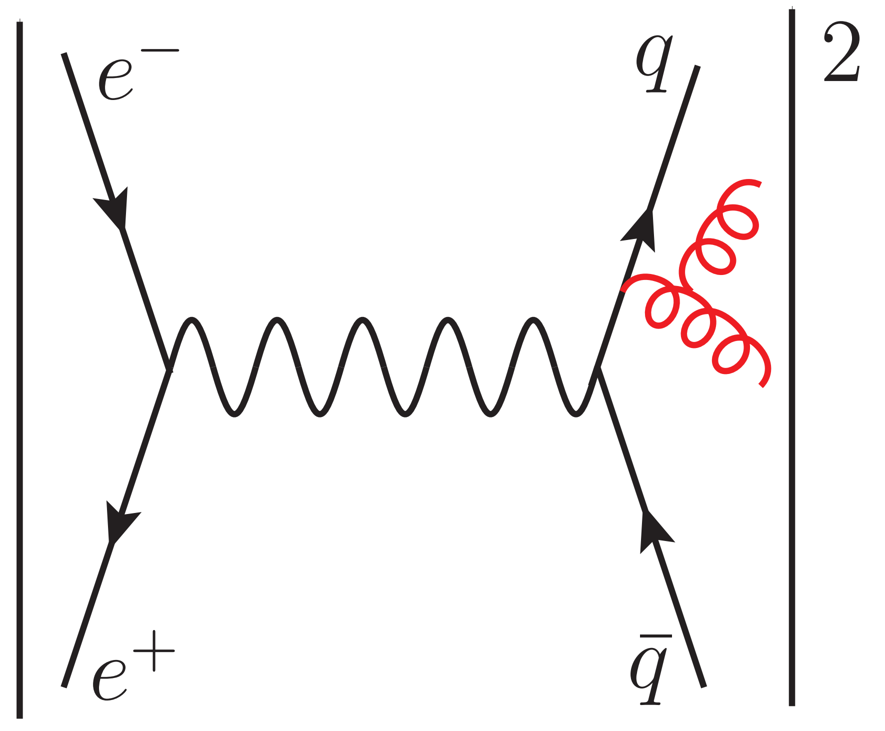
Key **properties at NLO:** **sector approach** and **minimal structure** from **FKS**, **Lorentz invariance** and **analytic integration** from **CS** ⟶ **NNLO?**

NNLO generalities

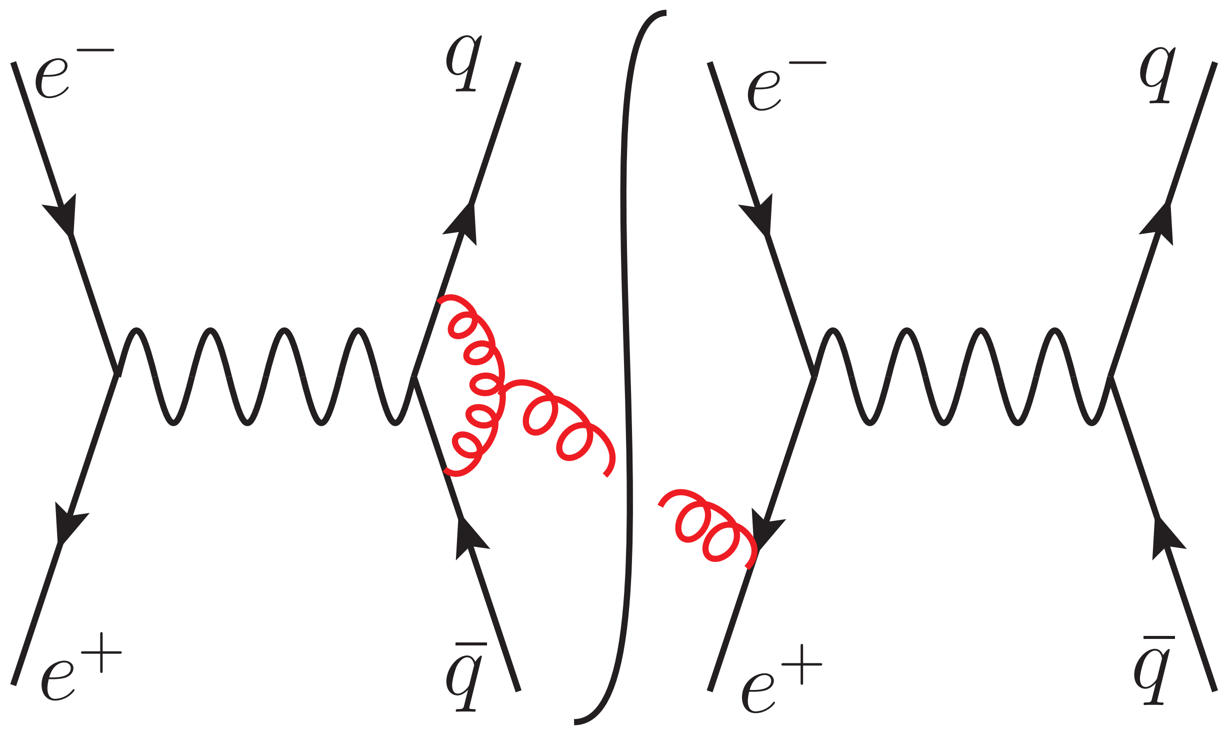
$$\frac{d\sigma^{\mathrm{N^2LO}}}{dX} = \int d\Phi_{n+2}\, RR_{n+2}\, \delta_{n+2}(X) + \int d\Phi_{n+1}\, RV_{n+1}\, \delta_{n+1}(X) + \int d\Phi_n\, VV_n\, \delta_n(X)$$



Finite in *d=4*

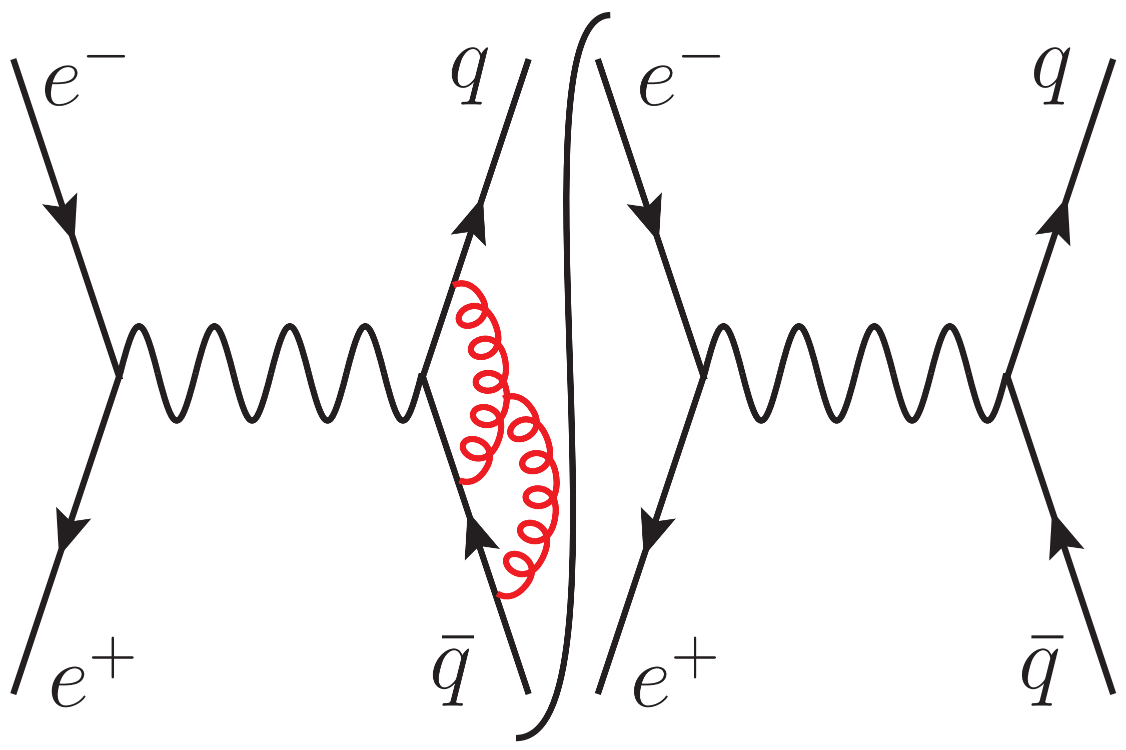
Up to 4 singularities in double-radiation PS

(up to two soft and/or collinear emissions)



Up to $1/\epsilon^2$ (loop nature)

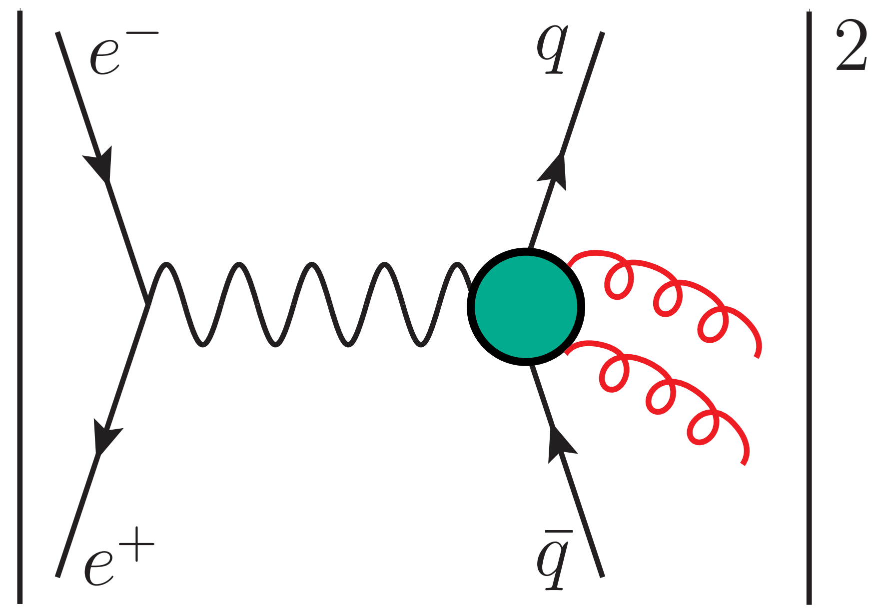
Up to 2 singularities in single-radiation PS



Up to $1/\epsilon^4$ (loop nature)

The double real: main problems and solutions

Many different **singular configurations overlap**



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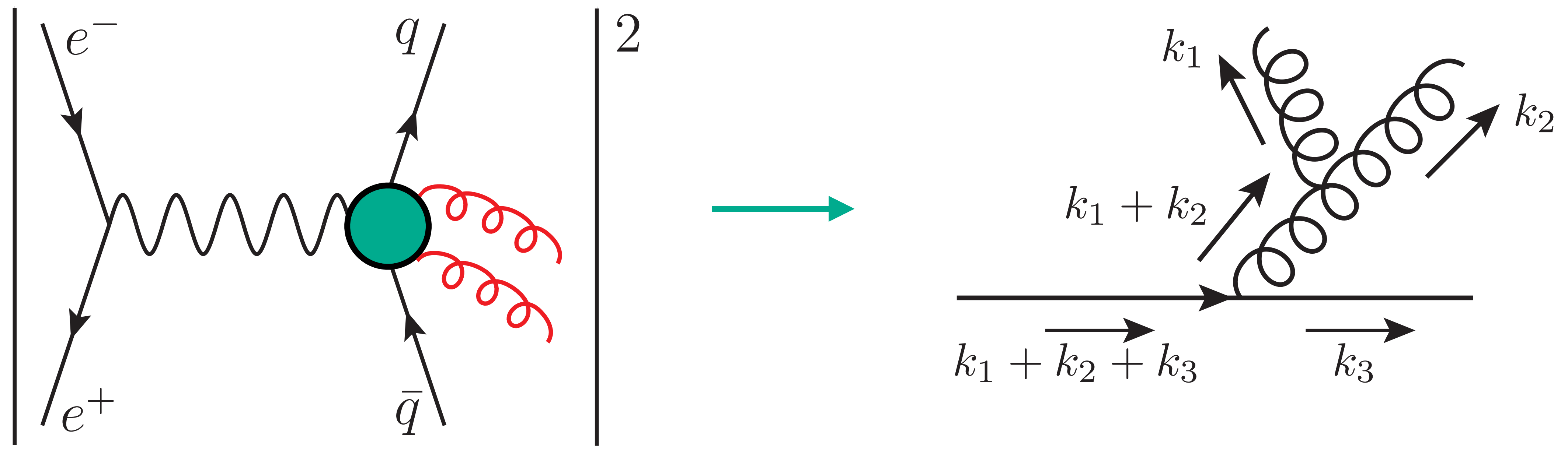

$$RR \sim \frac{1}{(k_1+k_2)^2\,(k_1+k_2+k_3)^2} \sim \frac{1}{E_1\,E_2\,(1-\vec{n}_1\cdot\vec{n}_2)}\,\frac{1}{E_1E_2(1-\vec{n}_1\cdot\vec{n}_2)+E_1E_3(1-\vec{n}_1\cdot\vec{n}_3)+E_2E_3(1-\vec{n}_2\cdot\vec{n}_3)}$$

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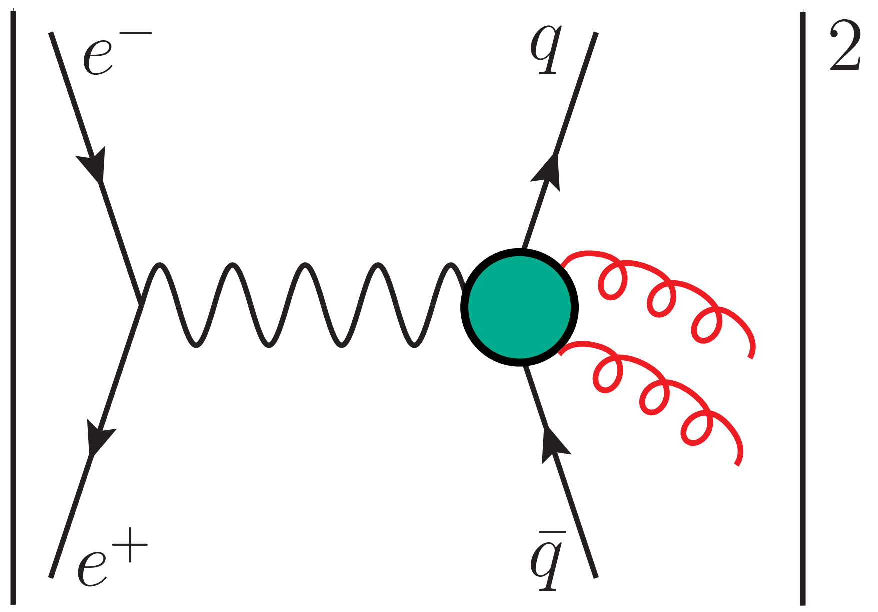

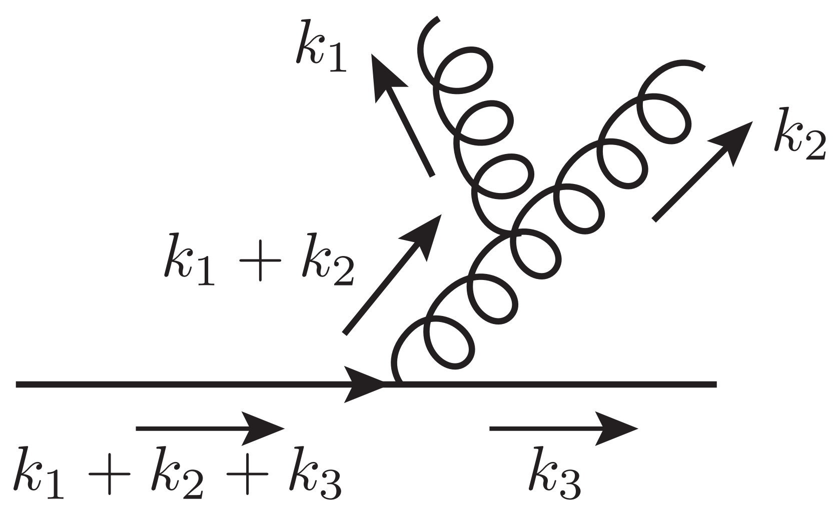

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$$RR \;\rightarrow\; \infty \quad \begin{cases} E_1 \to 0 \\ E_2 \to 0 \\ E_1, E_2 \to 0 \\ \vec{n}_1 \parallel \vec{n}_2 \\ \vec{n}_1 \parallel \vec{n}_2 \parallel \vec{n}_3 \end{cases}$$

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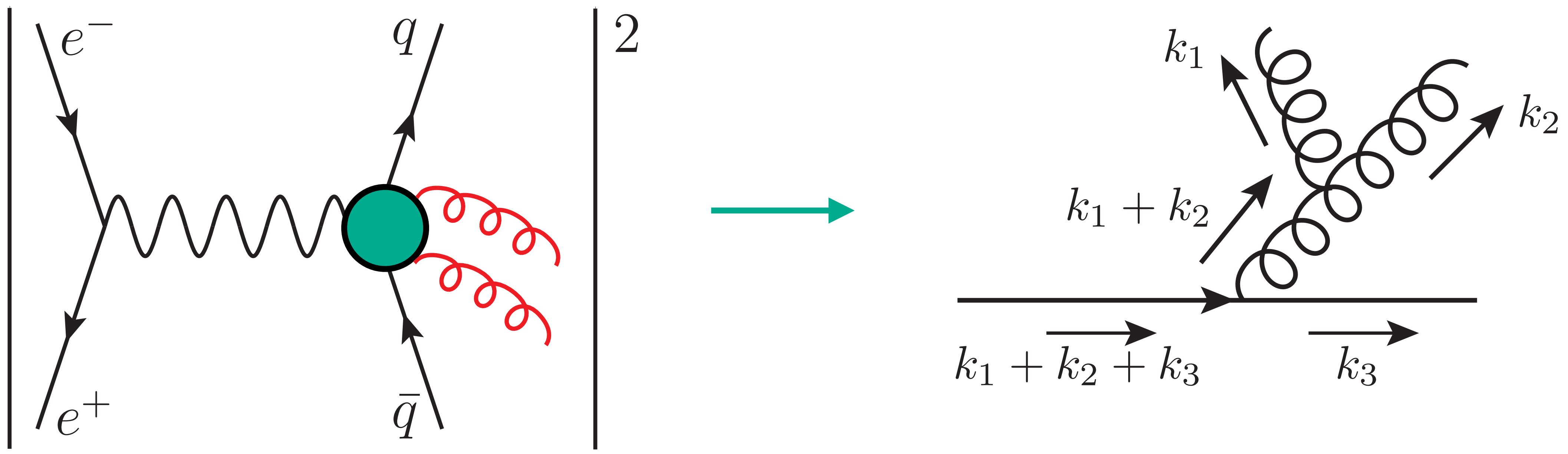

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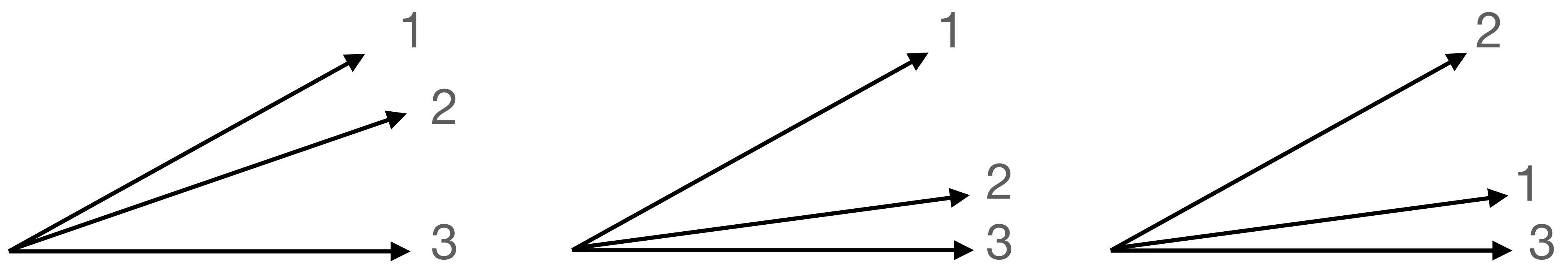
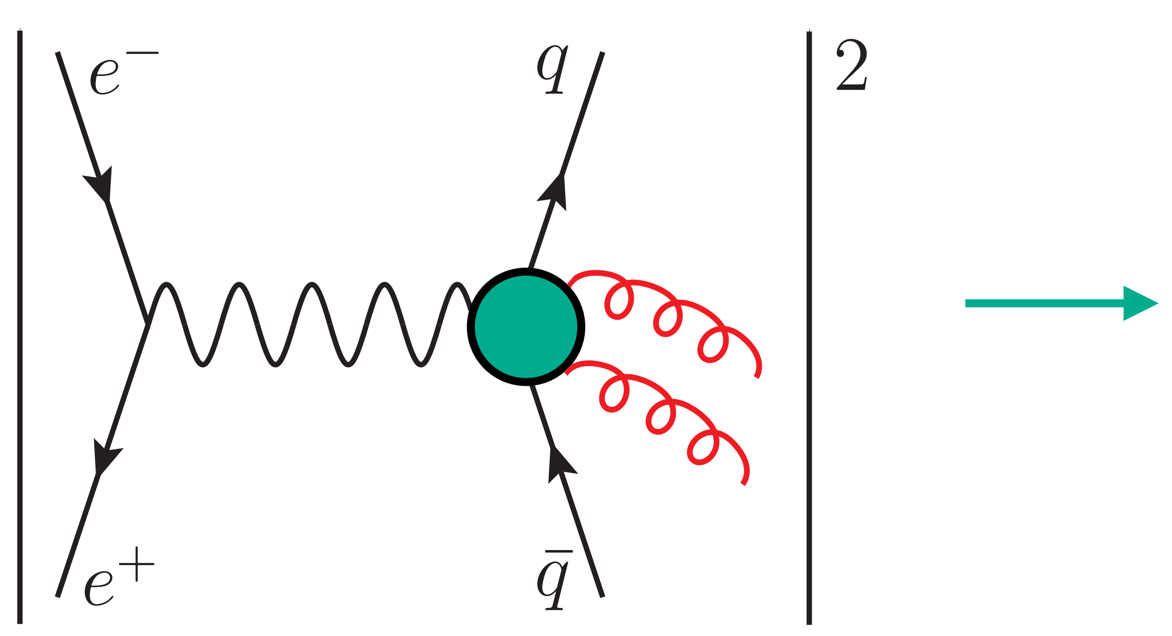
Includes **strongly ordered** configurations

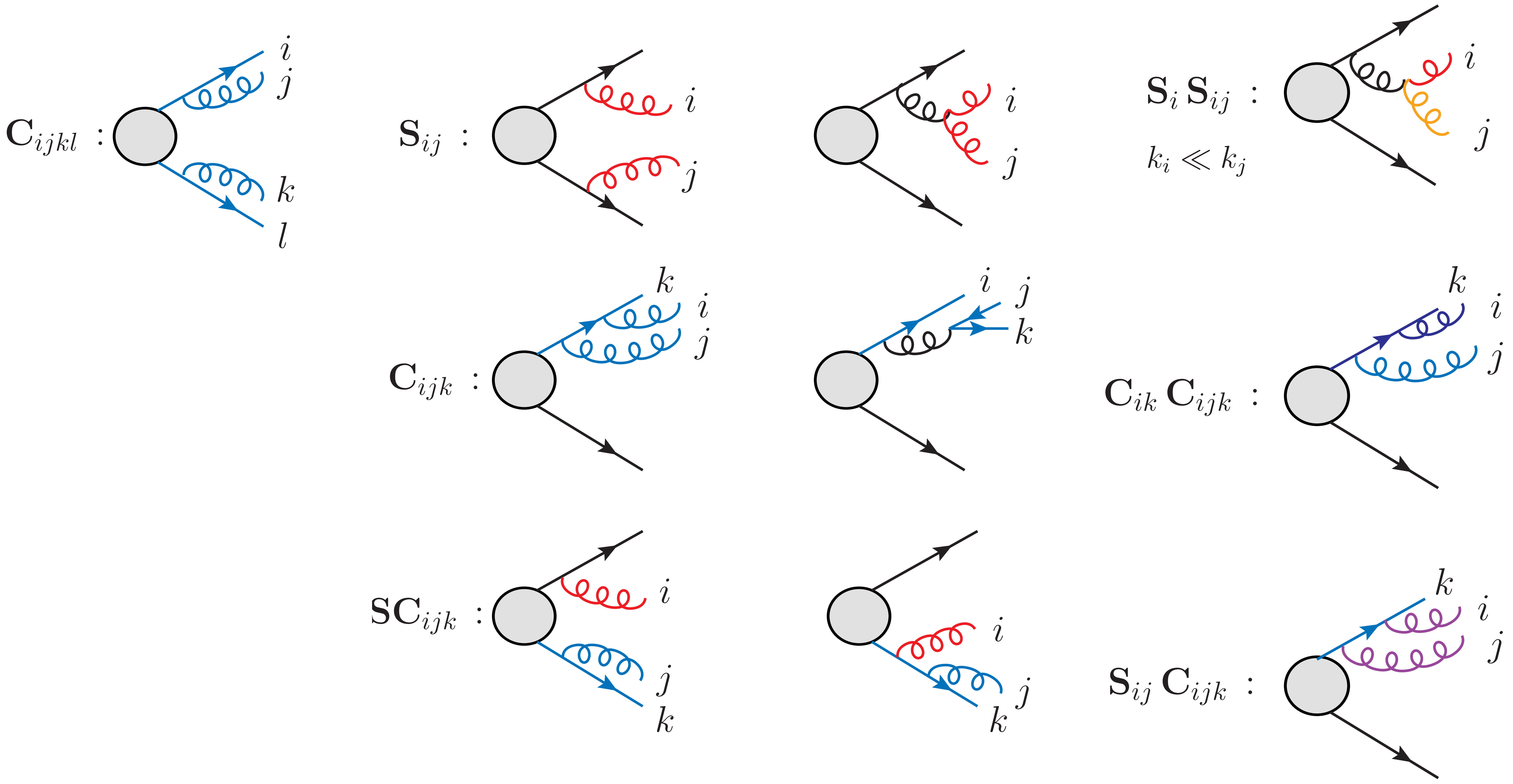


$\vec{n}_1\cdot\vec{n}_2 < \vec{n}_1\cdot\vec{n}_3$ $\quad\vec{n}_2\cdot\vec{n}_3 < \vec{n}_1\cdot\vec{n}_3$ $\quad\vec{n}_1\cdot\vec{n}_3 < \vec{n}_2\cdot\vec{n}_3$

The double real: main problems and solutions

Many different **singular configurations overlap**





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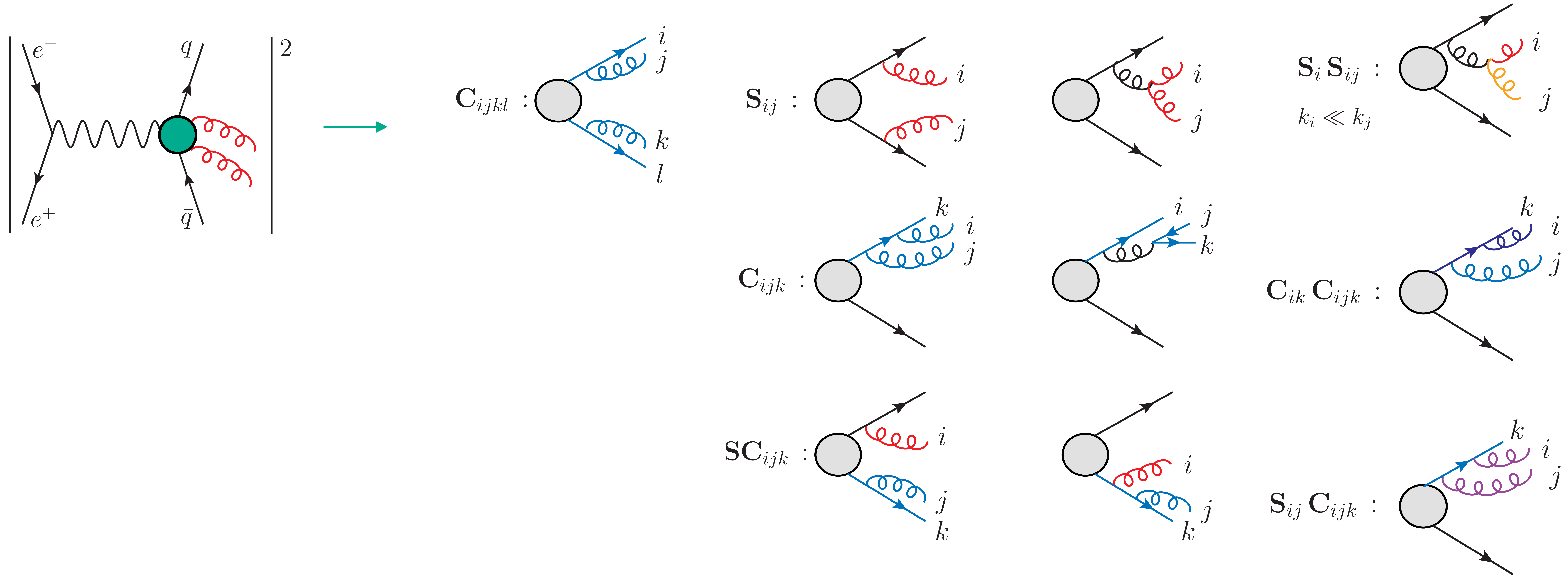
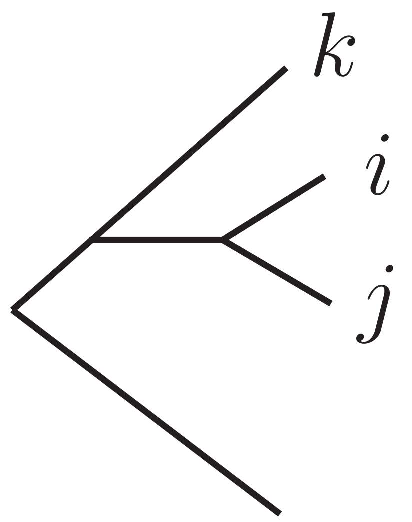


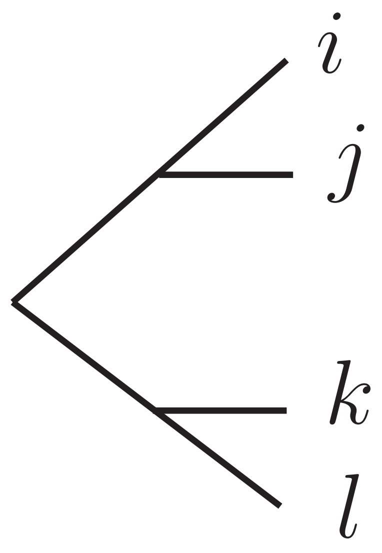
→ **Reorganisation** of the singular limits → **sectors + iterative definition**

The double real: main problems and solutions

Partition of the PS: different topologies to select the minimum number of singular configurations each



$$\mathscr{W}_{ijjk}: \underbrace{\mathbf{S}_i \quad \mathbf{C}_{ij}}_{\text{single unresolved}} \quad \underbrace{\mathbf{S}_{ij} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk}}_{\text{double unresolved}}$$



$$\mathscr{W}_{ijkl}: \quad \mathbf{S}_i\,, \mathbf{C}_{ij}\,, \mathbf{S}_{ik}\,, \mathbf{C}_{ijkk}\,, \mathbf{SC}_{ikl}\,, \mathbf{SC}_{kij}$$

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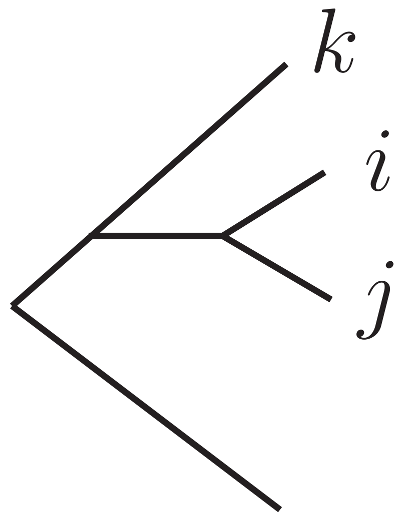

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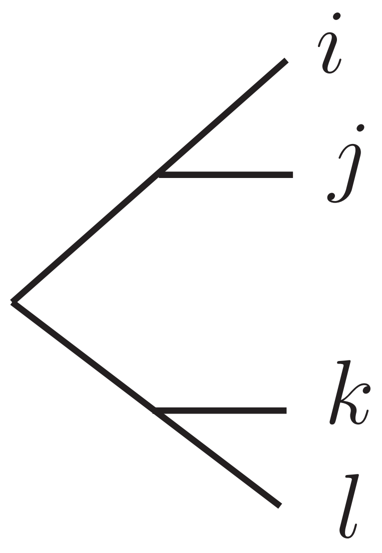
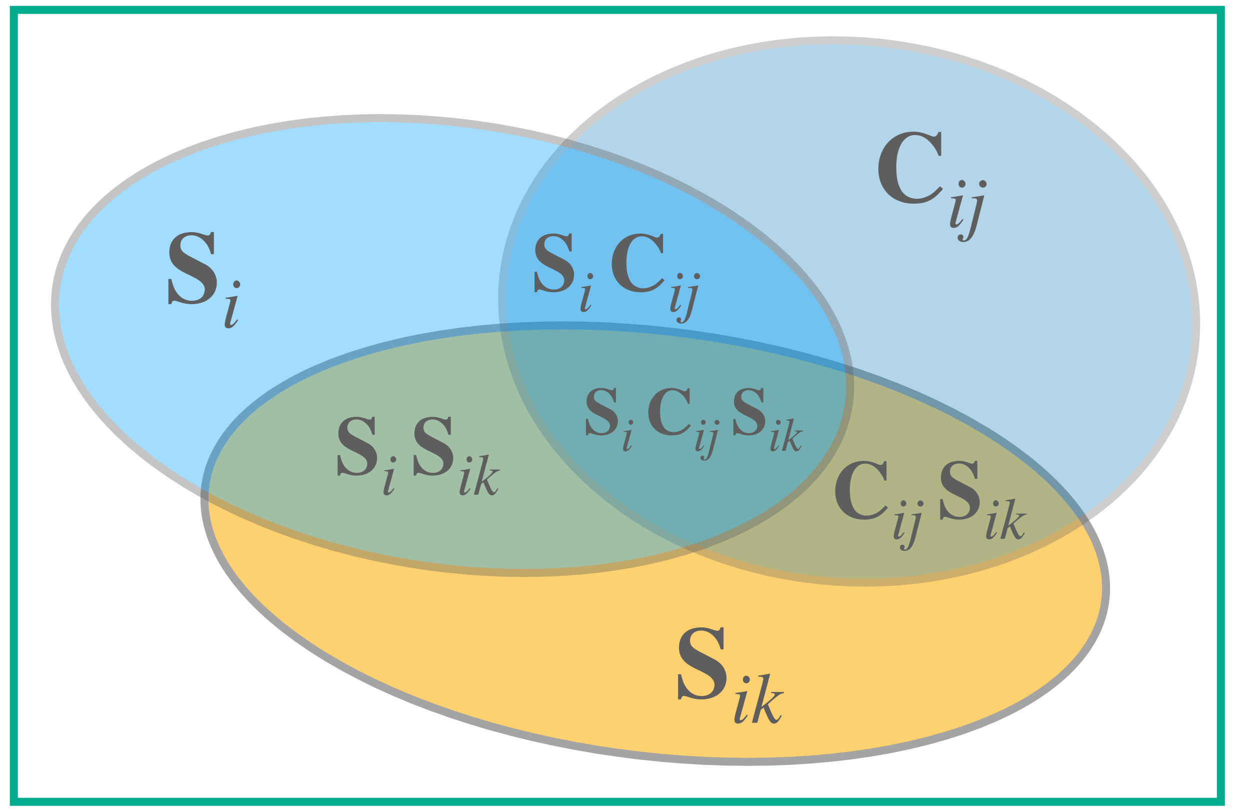


$$\mathcal{W}_{ijkl}: \mathbf{S}_i, \mathbf{C}_{ij}, \mathbf{S}_{ik}, \mathbf{C}_{ijkk}, \mathbf{SC}_{ikl}, \mathbf{SC}_{kij}$$

Iterative construction of the counterterm sector-by-sector

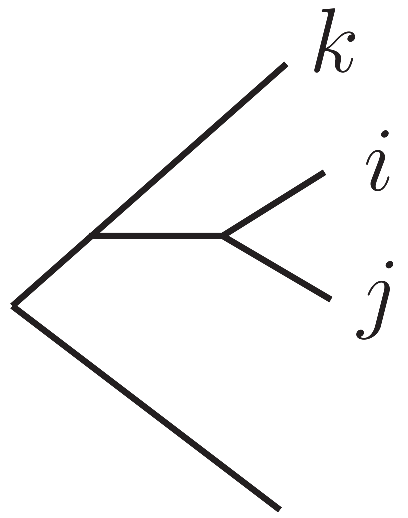
$$(1-\mathbf{S})(1-\mathbf{C})(1-\mathbf{S})(1-\mathbf{C})(1-\mathbf{SC})\, RR\,\mathcal{W}$$

$$= \left(1 - \mathbf{S} - \mathbf{C} + \mathbf{S}\,\mathbf{C} - \mathbf{S} + \mathbf{S}\,\mathbf{S} + \mathbf{C}\,\mathbf{S} + \dots\right) RR\,\mathcal{W} \quad = \text{finite}$$

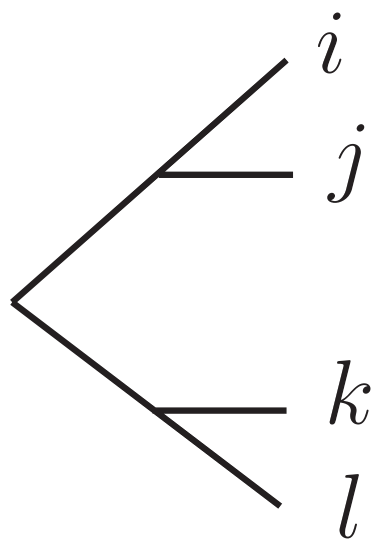


The double real: main problems and solutions

Partition of the PS: different topologies to select the minimum number of singular configurations each



$$\mathscr{W}_{ijjk} : \underbrace{\mathbf{S}_i \quad \mathbf{C}_{ij}}_{\text{single unresolved}} \quad \underbrace{\mathbf{S}_{ij} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk}}_{\text{double unresolved}}$$



$$\mathscr{W}_{ijkl} : \mathbf{S}_i , \mathbf{C}_{ij} , \mathbf{S}_{ik} , \mathbf{C}_{ijkk} , \mathbf{SC}_{ikl} , \mathbf{SC}_{kij}$$

Iterative construction of the counterterm sector-by-sector

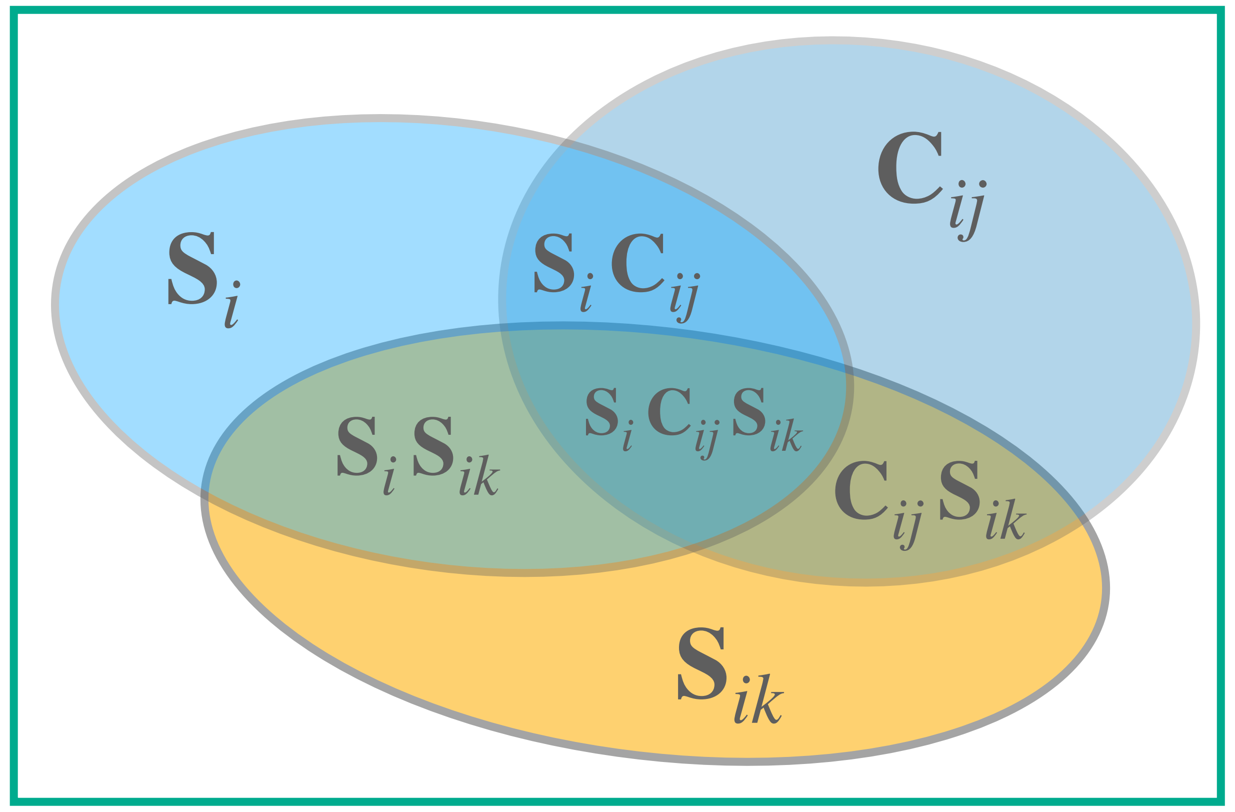
$$(1-\mathbf{S})(1-\mathbf{C})(1-\mathbf{S})(1-\mathbf{C})(1-\mathbf{SC})\, RR\,\mathscr{W}$$

$$= \Big(1 \underbrace{- \mathbf{S} - \mathbf{C} + \mathbf{S}\,\mathbf{C}}_{K^{(1)}} \underbrace{- \mathbf{S}}_{K^{(2)}} + \underbrace{\mathbf{S}\,\mathbf{S} + \mathbf{C}\,\mathbf{S}}_{K^{(12)}} + \ldots \Big)\, RR\,\mathscr{W} \quad = \text{finite}$$

$K^{(1)}$: single unresolved

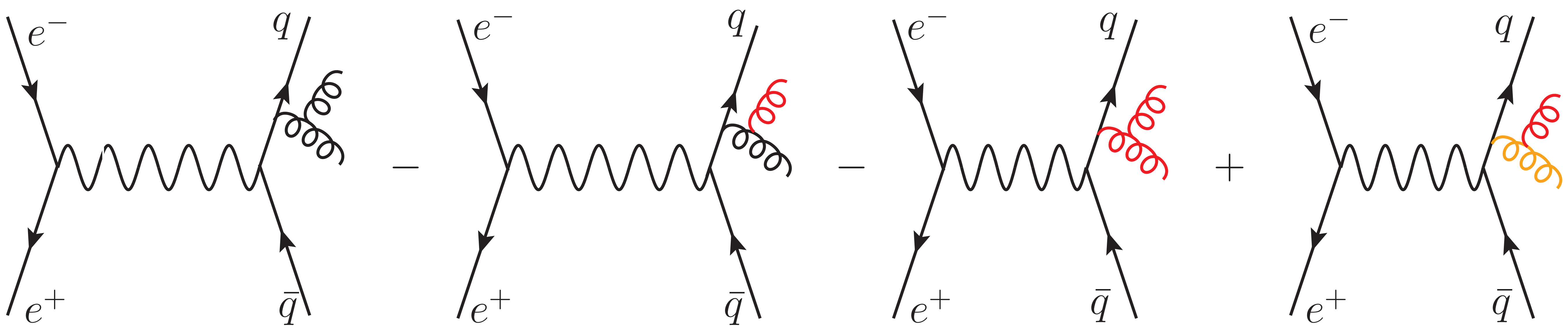
$K^{(2)}$: double unresolved

$K^{(12)}$: double unresolved, strongly ordered



The double real: main problems and solutions

Transparent physical interpretation

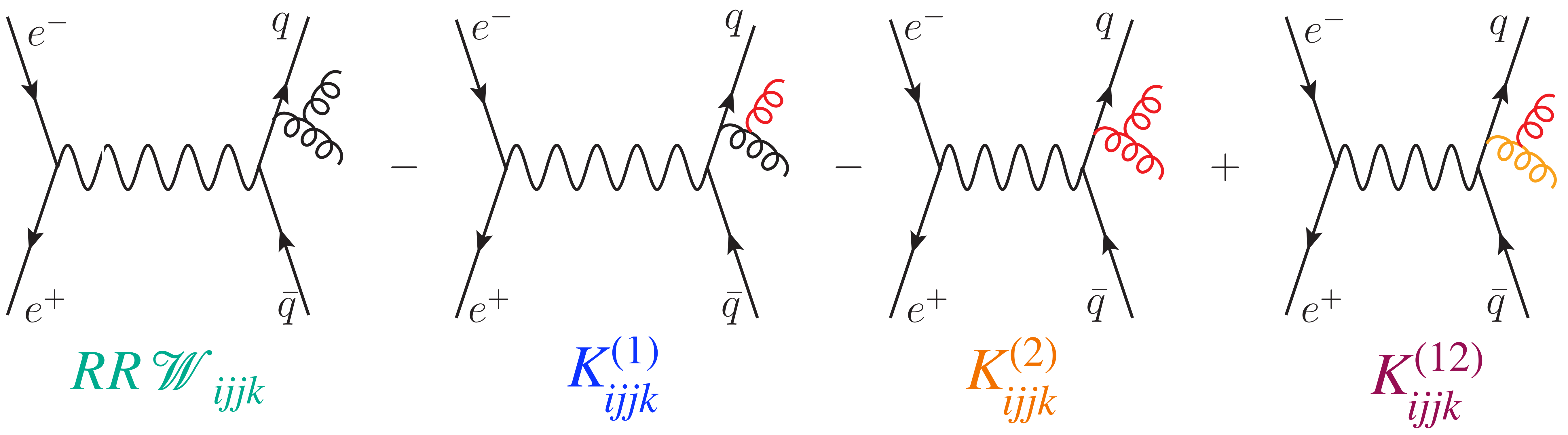


$RR\,\mathscr{W}_{ijjk}$ $\quad K^{(1)}_{ijjk}$ $\quad K^{(2)}_{ijjk}$ $\quad K^{(12)}_{ijjk}$

$$\int d\Phi_{n+2}\Big[\,RR_{n+2}\,\delta_{n+2} - K^{(1)}\delta_{n+1} - \big(K^{(2)} - K^{(12)}\big)\delta_n\,\Big]$$

The double real: main problems and solutions

Transparent physical interpretation



$$\int d\Phi_{n+2}\Big[\, RR_{n+2}\,\delta_{n+2} - K^{(1)}\delta_{n+1} - \left(K^{(2)} - K^{(12)}\right)\delta_n \,\Big]$$

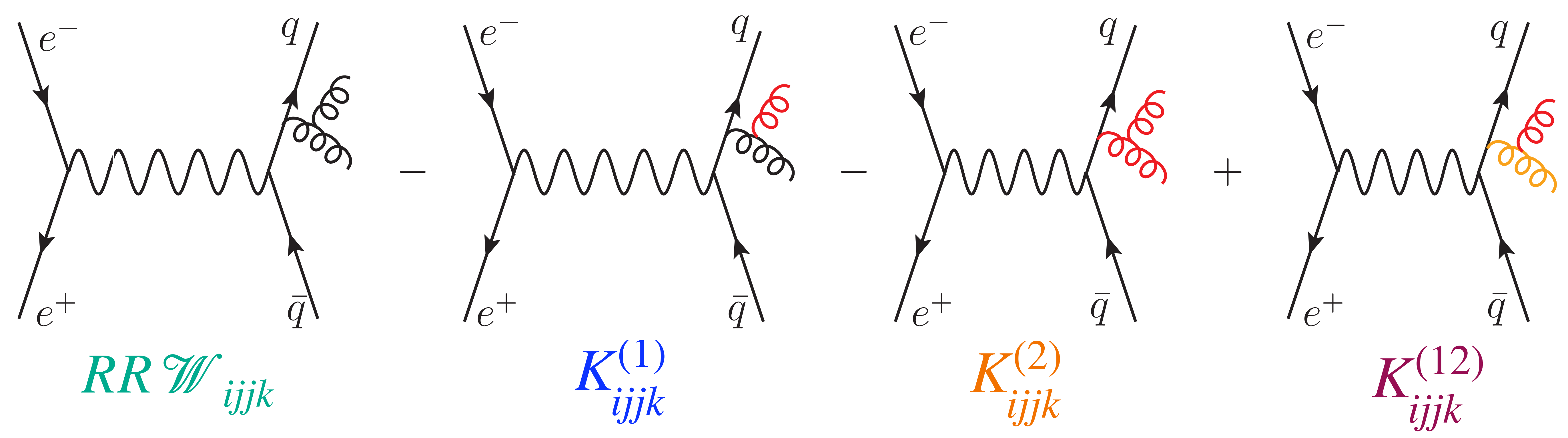
Next efforts:

1. **choose** an optimised **mapping**
2. **integrate** the counterterm over the **appropriate unresolved phase space**

$$I^{(1)} = \int d\Phi_{\mathrm{rad},1}\, K^{(1)}\,, \qquad I^{(2)} = \int d\Phi_{\mathrm{rad},2}\, K^{(2)}\,, \qquad I^{(12)} = \int d\Phi_{\mathrm{rad},1}\, K^{(12)}\,,$$

The double real: main problems and solutions

Transparent physical interpretation



$$\int d\Phi_{n+2}\Big[\, RR_{n+2}\,\delta_{n+2} - K^{(1)}\delta_{n+1} - \big(K^{(2)} - K^{(12)}\big)\delta_n \Big]$$

Next efforts:

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↓

NNLO complexity: highly non trivial!

Double real singular kernels:

Universal NNLO splitting [Catani, Grazzini 9903516,9810389] [Campbell, Glover 9710255]

$$\mathbf{S}_{ij}\, RR\big(\{k\}\big) \propto \sum_{c,d\neq i,j} \Bigg[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not i \not j}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not i \not j}\big) \Bigg]$$

$$I^{(ij)}_{cd} = 2\, T_R\, I^{(q\bar{q})(ij)}_{cd} - 2\, C_A\, I^{(gg)(ij)}_{cd}$$

$$I^{(q\bar{q})(ij)}_{cd} = \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}^2 (s_{ic} + s_{jc})(s_{id} + s_{jd})}$$

$$I^{(gg)(ij)}_{cd} = \frac{(1-\epsilon)(s_{ic}s_{jd} + s_{id}s_{jc}) - 2 s_{ij}s_{cd}}{s_{ij}^2 (s_{ic} + s_{jc})(s_{id} + s_{jd})} + s_{cd}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}s_{ic}s_{jd}s_{id}s_{jc}} \left[1 - \frac{1}{2}\, \frac{s_{ic}s_{jd} + s_{id}s_{jc}}{(s_{ic} + s_{jc})(s_{id} + s_{jd})}\right]$$

$$\mathbf{C}_{ijk}\, RR\big(\{k\}\big) \propto \frac{1}{s_{ijk}^2}\, P^{\mu\nu}_{ijk}\left(s_{ir}, s_{jr}, s_{kr}\right) B_{\mu\nu}\big(\{k\}_{\not i \not j \not k}, k_{ijk}\big)$$

$$P^{\mu\nu}_{ijk} B_{\mu\nu} = P_{ijk}\, B + Q^{\mu\nu}_{ijk}\, B_{\mu\nu}$$

$$P^{(3g)}_{ijk} = C_A^2 \left\{ \frac{(1-\epsilon)s_{ijk}^2}{4 s_{ij}^2} \left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}} \right)^2 + \frac{s_{ijk}}{s_{ij}} \left[4\, \frac{z_i z_j - 1}{z_{ij}} + \frac{z_i z_j - 2}{z_k} + \frac{(1 - z_k z_{ij})^2}{z_i z_k z_{jk}} + \frac{5}{2}\, z_k + \frac{3}{2} \right] + \frac{s_{ijk}^2}{2 s_{ij} s_{ik}} \left[\frac{2 z_i z_j z_{ik}(1 - 2z_k)}{z_k z_{ij}} + \frac{1 + 2z_i(1 + z_i)}{z_{ik} z_{ij}} + \frac{1 - 2 z_i z_{jk}}{z_j z_k} + 2 z_j z_k + z_i(1 + 2z_i) - 4 \right] + \frac{3(1-\epsilon)}{4} \right\} + perm\,.$$

$$Q^{(3g)\mu\nu}_{ijk} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \left\{ \left[\frac{2z_j}{z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} \right) \frac{1}{s_{ik}} \right] \tilde{k}_i^2\, q_i^{\mu\nu} + \left[\frac{2z_i}{z_k} \frac{1}{s_{ij}} - \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_j^2\, q_j^{\mu\nu} - \left[\frac{2 z_i z_j}{z_{ij} z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}} \right) \frac{1}{s_{ik}} \right] \tilde{k}_k^2\, q_k^{\mu\nu} \right\} + perm\,.$$

Double real singular kernels:

Universal NNLO splitting *[Catani, Grazzini 9903516,9810389] [Campbell, Glover 9710255]*

$$\mathbf{S}_{ij}\, RR\big(\{k\}\big) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I_{cd}^{(i)}\, I_{ef}^{(j)}\, B_{cdef}\big(\{k\}_{\not i \not j}\big) + I_{cd}^{(ij)}\, B_{cd}\big(\{k\}_{\not i \not j}\big) \right]$$

$$I_{cd}^{(ij)} = 2\, T_R\, I_{cd}^{(q\bar{q})(ij)} - 2\, C_A\, I_{cd}^{(gg)(ij)}$$

$$I_{cd}^{(q\bar{q})(ij)} = \frac{s_{ic}s_{jd} + s_{id}s_{jc} - s_{ij}s_{cd}}{s_{ij}^2(s_{ic} + s_{jc})(s_{id} + s_{jd})}$$

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$$\mathbf{C}_{ijk}\, RR\big(\{k\}\big) \propto \frac{1}{s_{ijk}^2}\, P_{ijk}^{\mu\nu}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not i \not j \not k}, k_{ijk}\big)$$

$$P_{ijk}^{\mu\nu} B_{\mu\nu} = P_{ijk}\, B + Q_{ijk}^{\mu\nu}\, B_{\mu\nu}$$

$$P_{ijk}^{(3g)} = C_A^2 \left\{ \frac{(1-\epsilon)s_{ijk}^2}{4s_{ij}^2}\left(\frac{s_{jk}}{s_{ijk}} - \frac{s_{ik}}{s_{ijk}} + \frac{z_i - z_j}{z_{ij}}\right)^2 + \frac{s_{ijk}}{s_{ij}}\left[4\,\frac{z_iz_j - 1}{z_{ij}} + \frac{z_iz_j - 2}{z_k} + \frac{(1 - z_kz_{ij})^2}{z_iz_kz_{jk}} + \frac{5}{2}\,z_k + \frac{3}{2}\right] + \frac{s_{ijk}^2}{2s_{ij}s_{ik}}\left[\frac{2z_iz_jz_{ik}(1-2z_k)}{z_kz_{ij}} + \frac{1 + 2z_i(1+z_i)}{z_{ik}z_{ij}} + \frac{1 - 2z_iz_{jk}}{z_jz_k} + 2z_jz_k + z_i(1+2z_i) - 4\right] + \frac{3(1-\epsilon)}{4} \right\} + perm\,.$$

$$Q_{ijk}^{(3g)\mu\nu} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \left\{ \left[\frac{2z_j}{z_k}\frac{1}{s_{ij}} + \left(\frac{z_jz_{ik}}{z_kz_{ij}} - \frac{3}{2}\right)\frac{1}{s_{ik}}\right]\tilde{k}_i^2\, q_i^{\mu\nu} + \left[\frac{2z_i}{z_k}\frac{1}{s_{ij}} - \left(\frac{z_jz_{ik}}{z_kz_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}}\right)\frac{1}{s_{ik}}\right]\tilde{k}_j^2\, q_j^{\mu\nu} - \left[\frac{2z_iz_j}{z_{ij}z_k}\frac{1}{s_{ij}} + \left(\frac{z_jz_{ik}}{z_kz_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}}\right)\frac{1}{s_{ik}}\right]\tilde{k}_k^2\, q_k^{\mu\nu} \right\} + perm\,.$$

Key problem: several **different invariants** combined into **non-trivial** and various **structures**, to be integrated over a **6-dim PS.**

Key solution: split the **different structures** according to the contributing Lorentz invariants and **tune the mapping !**

Double real singular kernels:

Universal NNLO splitting *[Catani, Grazzini 9903516,9810389] [Campbell, Glover 9710255]*

$$\mathbf{S}_{ij}\, RR\big(\{k\}\big) \propto \sum_{c,d\neq i,j} \Bigg[\sum_{e,f\neq i,j} I^{(i)}_{cd}\, I^{(j)}_{ef}\, B_{cdef}\big(\{k\}_{\not i \not j}\big) + I^{(ij)}_{cd}\, B_{cd}\big(\{k\}_{\not i \not j}\big) \Bigg]$$

$$I^{(ij)}_{cd} = 2\, T_R\, I^{(q\bar q)(ij)}_{cd} - 2\, C_A\, I^{(gg)(ij)}_{cd}$$

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$$Q^{(3g)\mu\nu}_{ijk} = C_A^2\, \frac{s_{ijk}}{s_{ij}} \left\{ \left[\frac{2 z_j}{z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} \right) \frac{1}{s_{ik}} \right] \tilde k_i^2\, q_i^{\mu\nu} + \left[\frac{2 z_i}{z_k} \frac{1}{s_{ij}} - \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_k} + \frac{z_i}{z_{ij}} \right) \frac{1}{s_{ik}} \right] \tilde k_j^2\, q_j^{\mu\nu} - \left[\frac{2 z_i z_j}{z_{ij} z_k} \frac{1}{s_{ij}} + \left(\frac{z_j z_{ik}}{z_k z_{ij}} - \frac{3}{2} - \frac{z_i}{z_j} + \frac{z_i}{z_{ik}} \right) \frac{1}{s_{ik}} \right] \tilde k_k^2\, q_k^{\mu\nu} \right\} + perm\,.$$

How the results look like:

$$\int d\Phi_{n+2}\, \overline{\mathbf{C}}_{ijk}\, RR = \int d\Phi_n\big(\bar k^{(ijrk)}\big)\, J_{cc}\big(\bar s^{ijkr}_{kr}\big)\, B\big(\bar k^{(ijrk)}\big)$$

$$J^{(3g)}_{cc}(s) = \left(\frac{\alpha_s}{2\pi}\right)^2 \left(\frac{s}{\mu^2}\right)^{-2\epsilon} C_A^2 \left[\frac{15}{\epsilon^4} + \frac{63}{\epsilon^3} + \left(\frac{853}{3} - 22\,\pi^2 \right) \frac{1}{\epsilon^2} + \left(\frac{10900}{9} - \frac{275}{3}\pi^2 - 376\,\zeta_3 \right) \frac{1}{\epsilon} + \frac{180739}{36} - \frac{3736}{9}\pi^2 - 1555\,\zeta_3 + \frac{41}{10}\pi^4 + \mathcal{O}(\epsilon) \right]$$

The subtraction: general pattern

The subtraction: general pattern

$$\frac{d\sigma^{\rm NNLO}}{dX} = \int d\Phi_n \Big[\underbrace{VV_n + I^{(2)} + I^{(\rm RV)}}_{\text{finite in d=4 and in } \Phi_n} \Big] \delta_n$$

$$+ \int d\Phi_{n+1} \Big[\underbrace{\underbrace{\left(RV_{n+1} + I^{(1)}\right)}_{\text{finite in d=4, singular in } \Phi_{n+1}} \delta_{n+1} - \underbrace{\left(\overline{K}^{(\rm RV)} + I^{(12)}\right)}_{\text{finite in d=4, singular in } \Phi_{n+1}} \delta_n}_{\text{finite in d=4 and in } \Phi_{n+1}} \Big]$$

$$+ \int d\Phi_{n+2} \Big[\underbrace{RR_{n+2}\, \delta_{n+2} - \overline{K}^{(1)} \delta_{n+1} - \left(\overline{K}^{(2)} - \overline{K}^{(12)}\right) \delta_n}_{\text{finite in d=4 and in } \Phi_{n+2}} \Big]$$

Take home message

1. **Phenomenology** requires **higher order corrections**

2. To obtain fully **differential results** a **subtraction scheme** is needed

3. **Local Analytic Sector Subtraction** is designed to address the fundamental requirements for an **optimal subtraction scheme**.

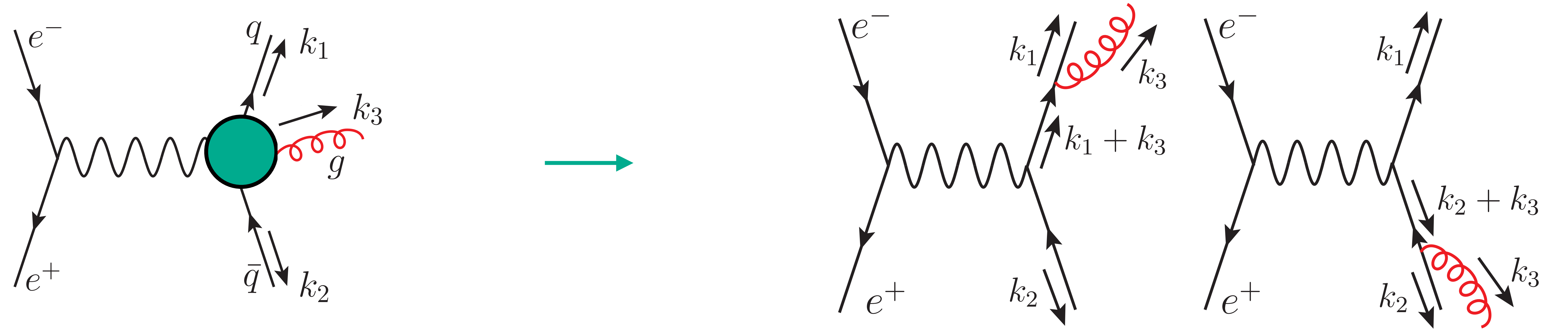
4. The main **building blocks** of the schemes are now **available** for an **arbitrary number of final state partons** (partition, integrated counterterm, mappings, …)

Thank you for your attention!

Backup

Ingredients of the subtraction

- *Phase space partitioning (FKS) : multiple singular configuration that overlap*



Sector functions at NLO

Sector functions $\mathcal{W}_{ij}$:

1) Select the minimum number of singularities

$$\mathbf{S}_i \mathcal{W}_{ab} = 0\,, \qquad \forall i \neq a \qquad\qquad \mathbf{C}_{ij} \mathcal{W}_{ab} = 0\,, \qquad \forall a, b \notin \{i, j\}\,.$$

2) Sum properties

$$\sum_{i,j \neq i} \mathcal{W}_{ij} = 1 \qquad \mathbf{S}_i \sum_{j \neq i} \mathcal{W}_{ij} = 1\,, \qquad \mathbf{C}_{ij} \sum_{a,b \in \{ij\}} \mathcal{W}_{ab} = 1\,.$$

3) Explicit form

$$CM : q^\mu = (\sqrt{s}, \vec{0})\,, \qquad e_i = \frac{s_{qi}}{s}\,, \qquad \omega_{ij} = \frac{s\, s_{ij}}{s_{qi}\, s_{qj}}\,, \qquad \mathcal{W}_{ij} = \frac{\sigma_{ij}}{\sum_{k, l \neq k} \sigma_{kl}}\,, \qquad \sigma_{ij} = \frac{1}{e_i\, \omega_{ij}}$$

$$\mathbf{S}_i \mathcal{W}_{ab} = \delta_{ia} \frac{1/\omega_{ab}}{\sum_{c \neq a} 1/\omega_{ac}}\,, \qquad \mathbf{C}_{ij} \mathcal{W}_{ab} = \left(\delta_{ia}\delta_{jb} + \delta_{ib}\delta_{ja}\right) \frac{e_b}{e_a + e_b}$$

Sector functions at NLO

Sum over sectors before integration

Sector functions **sum rules** ⟶ Summing over sectors $\overline{K}$ becomes **independent** of $\mathcal{W}$

$$\overline{K} = \sum_{i,j} \overline{K}_{ij} \propto \overline{\mathbf{S}}_i R \Big[\overbrace{\sum_j \overline{\mathbf{S}}_i \mathcal{W}_{ij}}^{=1} \Big] + \overline{\mathbf{C}}_{ij} R \Big[\overbrace{\overline{\mathbf{C}}_{ij}(\mathcal{W}_{ij} + \mathcal{W}_{ji})}^{=1} \Big] - \overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} R \Big[\overbrace{\overline{\mathbf{S}}_i \overline{\mathbf{C}}_{ij} \mathcal{W}_{ij}}^{=1} \Big]$$

$$\Longrightarrow \overline{K} = \sum_i \overline{\mathbf{S}}_i R + \sum_{i,j \neq i} \overline{\mathbf{C}}_{ij} \big(1 - \overline{\mathbf{S}}_i\big) R$$

Remarks:

1. The integrated counterterm has to **match the poles of V**, which is **not split** into sectors

2. The sector functions would have made the **integration** much **more involved**

Integration of NLO soft kernel

$$d\Phi_{n+1} = d\Phi_n^{(abc)}\, d\Phi_{\rm rad}^{(abc)} = d\Phi_n^{(abc)} \times d\Phi_{\rm rad}\left(s_{bc}^{(abc)}; y, z, \phi\right)$$

Catani-Seymour parameters: $s_{ab} = y s_{bc}^{(abc)}, \quad s_{ac} = z(1-y) s_{bc}^{(abc)}, \quad s_{bc} = (1-z)(1-y) s_{bc}^{(abc)}$

Radiative phase space: $d\Phi_{\rm rad}^{(abc)} \propto \left(s_{bc}^{(abc)}\right)^{1-\epsilon} \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, (1-y)\left[(1-y)^2\, y\, (1-z)\, z\right]^{-\epsilon}$

Kernel to integrate: $\bar{\mathbf{S}}_i\, R\left(\{k\}\right) \propto \sum_{c,d\neq i} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}\left(\{k\}^{(icd)}\right)$

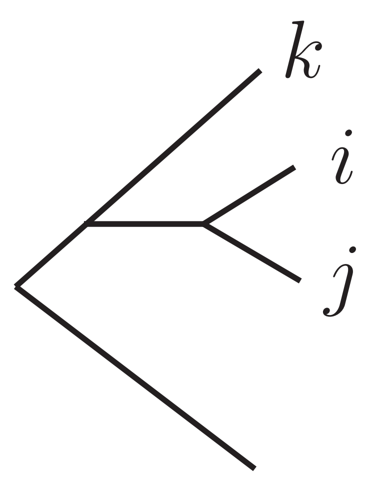
$$I^s \propto \sum_{c,d\neq i} \int d\Phi_{\rm rad}^{(icd)} \frac{s_{cd}}{s_{ic}\, s_{id}} B_{cd}\left(\{k\}^{(icd)}\right) = \sum_{c,d\neq i} \left(s_{bc}^{(abc)}\right)^{-\epsilon} \int_0^\pi d\phi\, \sin^{-2\epsilon}\phi \int_0^1 dy \int_0^1 dz\, (1-y)\left[(1-y)^2\, y\, (1-z)\, z\right]^{-\epsilon} \frac{1-z}{z} B_{cd}\left(\{k\}^{(icd)}\right)$$

$$= \sum_{c,d\neq i} \left(s_{bc}^{(abc)}\right)^{-\epsilon} \frac{(4\pi)^{\epsilon-2}\,\Gamma(1-\epsilon)\,\Gamma(2-\epsilon)}{\epsilon^2\Gamma(2-3\epsilon)} B_{cd}\left(\{k\}^{(icd)}\right)$$

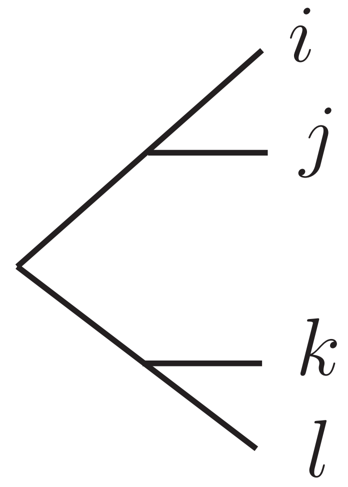
General remarks:

1. Different parametrisation for the soft and for the hard-collinear counterterm
2. Each contribution to the soft is parametrised differently to simplify the integration

Sector functions at NNLO



$\mathcal{W}_{ijjk}\,, \qquad i \neq j \neq k$



$\mathcal{W}_{ijkj}\,, \qquad i \neq j \neq k \neq l$

Singularities selected: $\mathcal{W}_{abcd} \quad \begin{cases} a, c & \to \textit{soft} \\ ab, cd & \to \textit{collinear} \end{cases}$

Possible realisation of the desired properties:

$$\mathcal{W}_{abcd} = \frac{\sigma_{abcd}}{\sigma}, \qquad \sigma = \sum_{a, b \neq a} \sum_{\substack{c \neq a \\ d \neq a, c}} \sigma_{abcd} \qquad \Longrightarrow \qquad \sum_{a, b \neq a} \sum_{\substack{c \neq a \\ d \neq a, c}} \mathcal{W}_{abcd} = 1\,, \qquad \sigma_{abcd} = \frac{1}{(e_a\, w_{ab})^{\alpha}} \frac{1}{(e_c + \delta_{bc}\, e_a)\, w_{cd}}, \quad \alpha > 1$$

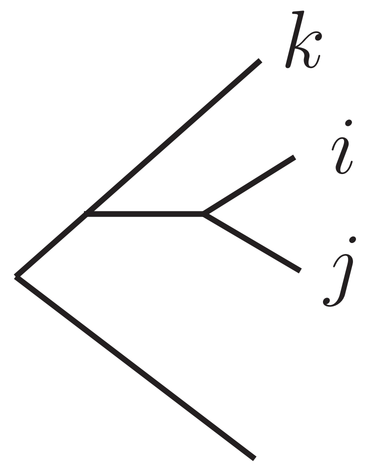
Limits selected by the topologies:

$$\mathcal{W}_{ijjk} \;:\; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ij} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk}$$

$$\mathcal{W}_{ijkl} \;:\; \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ik} \quad \mathbf{C}_{ijkl} \quad \mathbf{SC}_{ikl}, \quad \mathbf{SC}_{kij}$$

NNLO sector functions factorise into products of NLO-type sector function under single-unresolved limits.

Identification of the counterterm in a given topology



$\mathscr{W}_{ijjk}$ $\quad \mathbf{S}_i \quad \mathbf{C}_{ij} \quad \mathbf{S}_{ij} \quad \mathbf{C}_{ijk} \quad \mathbf{SC}_{ijk}$

$$\underbrace{\left(1-\mathbf{S}_i\right)\left(1-\mathbf{C}_{ij}\right)}_{1-\mathbf{L}^{(1)}_{ij}} \ \underbrace{\left(1-\mathbf{S}_{ij}\right)\left(1-\mathbf{C}_{ijk}\right)\left(1-\mathbf{SC}_{ijk}\right)}_{1-\mathbf{L}^{(2)}_{ijjk}} RR\,\mathscr{W}_{ijjk} = \text{finite}$$

According to how the partons become unresolved we define:

$$K^{(1)}_{ijjk} \qquad K^{(2)}_{ijjk} \qquad K^{(12)}_{ijjk}$$

$$K^{(\mathbf{1})}_{ijjk} = \left[\mathbf{S}_i + \mathbf{C}_{ij}\left(1-\mathbf{S}_i\right)\right] RR\,\mathscr{W}_{ijjk}$$

$$K^{(\mathbf{2})}_{ijjk} = \left[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\left(1-\mathbf{S}_{ij}\right) + \mathbf{SC}_{ijk}\left(1-\mathbf{S}_{ij}\right)\left(1-\mathbf{C}_{ijk}\right)\right] RR\,\mathscr{W}_{ijjk}$$

$$K^{(\mathbf{12})}_{ijjk} = \left\{\left[\mathbf{S}_i + \mathbf{C}_{ij}\left(1-\mathbf{S}_i\right)\right]\left[\mathbf{S}_{ij} + \mathbf{C}_{ijk}\left(1-\mathbf{S}_{ij}\right) + \mathbf{SC}_{ijk}\left(1-\mathbf{S}_{ij}\right)\left(1-\mathbf{C}_{ijk}\right)\right]\right\} RR\,\mathscr{W}_{ijjk}$$

Singular structure of the RR

Under fundamental limits, the RR factorise into: (universal kernel) x (lower multiplicity matrix elements)

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I_{cd}^{(i)} I_{ef}^{(j)} B_{cdef}\big(\{k\}_{\not{i}\not{j}}\big) + I_{cd}^{(ij)} B_{cd}\big(\{k\}_{\not{i}\not{j}}\big) \right]$$

$$\mathbf{C}_{ijk}\, RR(\{k\}) \propto \frac{1}{s_{ijk}^2} P_{ijk}^{\mu\nu}\big(s_{ir}, s_{jr}, s_{kr}\big)\, B_{\mu\nu}\big(\{k\}_{\not{i}\not{j}\not{k}}, k_{ijk}\big)$$

$$\mathbf{C}_{ijkl}\, RR(\{k\}) \propto \frac{1}{s_{ij}\, s_{kl}} P_{ij}^{\mu\nu}\big(s_{ir}, s_{jr}\big)\, P_{kl}^{\rho\sigma}\big(s_{kr'}, s_{lr'}\big)\, B_{\mu\nu\rho\sigma}\big(\{k\}_{\not{i}\not{j}\not{k}\not{l}}, k_{ij}, k_{kl}\big)$$

$$\mathbf{SC}_{ijk}\, RR(\{k\}) = \mathbf{CS}_{jki} RR(\{k\}) \propto \frac{1}{s_{jk}} \sum_{c,d\neq i} P_{jk}^{\mu\nu}\, I_{cd}^{(i)}\, B_{\mu\nu}^{cd}\big(\{k\}_{\not{i}\not{j}\not{k}}, k_{jk}\big)$$

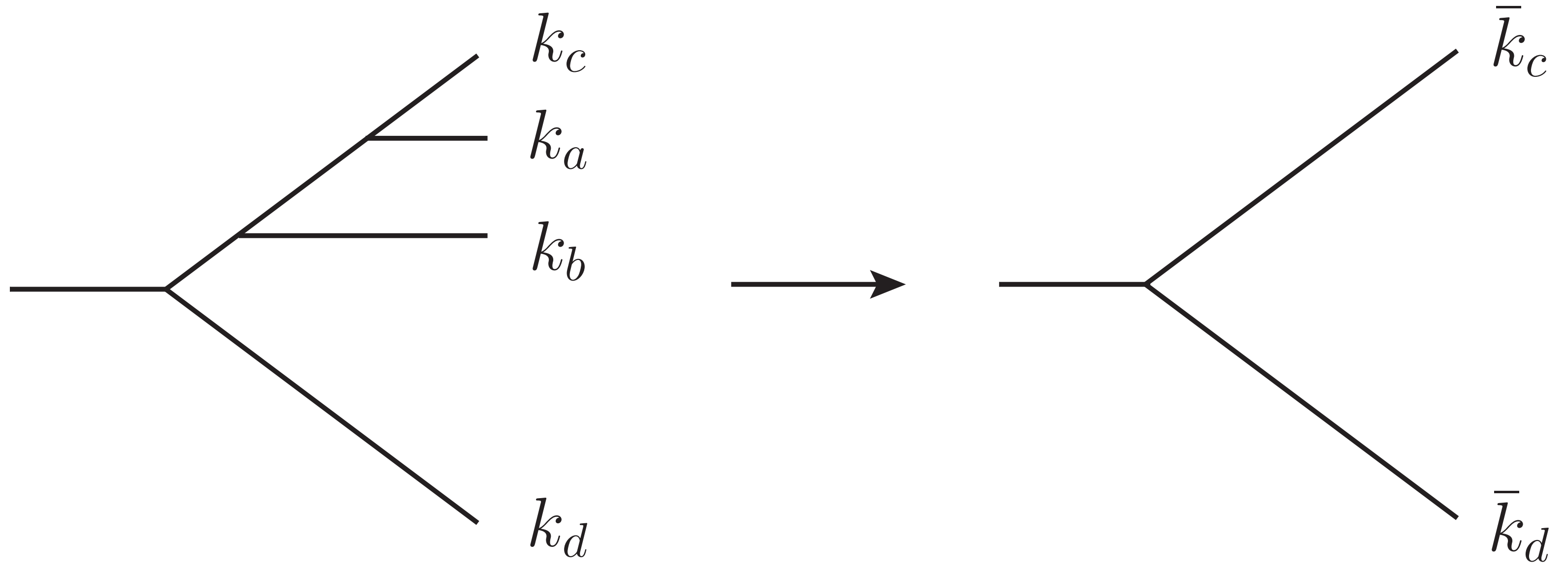
Born-level kinematics does not satisfy the mass-shell condition and momentum conservation

→ Momentum mapping needed!

NNLO momentum mapping

Two different kind of mapping to treat different kernels and simplify the integration

1. One-step mapping



$$\{\bar{k}_n^{(abcd)}\} = \{k_{\not{a}\not{b}\not{c}\not{d}}, \bar{k}_c^{(abcd)}, \bar{k}_d^{(abcd)}\}$$

$$d\Phi_{n+2} = d\Phi_n^{(abcd)} \cdot d\Phi_{\mathrm{rad},2}\left(\bar{s}_{cd}^{(abcd)}; y, z, \phi, y', z', x'\right)$$

$$\int d\Phi_{\mathrm{rad},2} \propto \left(\bar{s}_{cd}^{(abcd)}\right)^{2-2\epsilon} \int_0^1 dw' \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi\, (\sin\phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \left[w'(1-w')\right]^{-1/2-\epsilon} \left[y'(1-y')^2\, z'(1-z')\, y^2(1-y)^2\, z(1-z)\right]^{-\epsilon} (1-y')\, y(1-y)$$

2. Two-step mapping

$$\{\bar{k}_n^{(acd,bef)}\} = \{\bar{k}^{(acd)}_{\not{a}\not{b}\not{e}\not{f}}, \bar{k}_e^{(acd,bef)}, \bar{k}_f^{(acd,bef)}\}$$

$$d\Phi_{n+2} = d\Phi_n^{(acd,bef)} \cdot d\Phi_{\mathrm{rad},2}^{(acd,bef)} = d\Phi_n^{(acd,bef)} \cdot d\Phi_{\mathrm{rad},1}\left(\bar{s}_{ef}^{(acd,bef)}; y, z, \phi\right) \cdot d\Phi_{\mathrm{rad},1}\left(\bar{s}_{cd}^{(acd)}; y', z', \phi'\right)$$

$$d\Phi_{\mathrm{rad},2}^{(acd,bef)} \propto \left(\bar{s}_{cd}^{(acd,bef)} \bar{s}_{ef}^{(acd,bef)}\right)^{1-\epsilon} \int_0^\pi d\phi' \, (\sin\phi')^{-2\epsilon} \int_0^1 dy' \int_0^1 dz' \int_0^\pi d\phi\, (\sin\phi)^{-2\epsilon} \int_0^1 dy \int_0^1 dz \left[y'(1-y')^2\, z'(1-z')\, y(1-y)^2\, z(1-z)\right]^{-\epsilon} (1-y')(1-y)$$

Integration over the double phase space: example

Starting from the limit

$$\mathbf{S}_{ij}\, RR(\{k\}) \propto \sum_{c,d\neq i,j} \left[\sum_{e,f\neq i,j} I_{cd}^{(i)}\, I_{ef}^{(j)}\, B_{cdef}\big(\{k\}_{\not i \not j}\big) + I_{cd}^{(ij)}\, B_{cd}\big(\{k\}_{\not i \not j}\big)\right]$$

We are free to map each term of the sum separately, adapting the choice to the invariants appearing in the kernel

$$\bar{\mathbf{S}}_{ij}\, RR(\{k\}) \propto \sum_{\substack{c\neq i,j \\ d\neq i,j,c}} \Bigg[\sum_{\substack{e\neq i,j,c,d \\ f\neq i,j,c,d}} I_{cd}^{(i)}\, \bar{I}_{ef}^{(j)(icd)} B_{cdef}\Big(\big\{\bar{k}^{(icd,jef)}\big\}\Big) + 4 \sum_{e\neq i,j,c,d} I_{cd}^{(i)}\, \bar{I}_{ed}^{(j)(icd)} B_{cded}\Big(\big\{\bar{k}^{(icd,jed)}\big\}\Big)$$
$$+ 2\, I_{cd}^{(i)}\, I_{cd}^{(j)} B_{cdcd}\Big(\big\{\bar{k}^{(ijcd)}\big\}\Big) + \Big(I_{cd}^{(ij)} - \frac{1}{2} I_{cc}^{(ij)} - \frac{1}{2} I_{dd}^{(ij)}\Big) B_{cd}\Big(\big\{\bar{k}^{(ijcd)}\big\}\Big)\Bigg]$$

The PS parametrisation follows the mapping structure

$$I^{(2)}_{\mathrm{SS},cdef} = \int d\Phi_{\mathrm{rad},2}\, I_{cd}^{(i)}\, \bar{I}_{ef}^{(j),(icd)} = \int d\bar{\Phi}^{(icd,jef)}_{\mathrm{rad}}\, \bar{I}_{ef}^{(j),(icd)} \int d\Phi^{(icd)}_{\mathrm{rad}}\, I_{cd}^{(i)}$$

$$= \left[\frac{(4\pi)^{\epsilon-2}}{\big(\bar{s}_{cd}^{(icd,jef)}\big)^{\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} \right] \left[\frac{(4\pi)^{\epsilon-2}}{\big(\bar{s}_{ef}^{(icd,jef)}\big)^{\epsilon}} \frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\epsilon^2\,\Gamma(2-3\epsilon)} \right]$$