The Cardy-like limit of the $\mathcal{N}=1$ superconformal index

Marco Fazzi

INFN Milano-Bicocca
Based on 2012.15208 & 2103.15853
with Antonio Amariti (INFN Milano) and Alessia Segati (Milano Uni)

[see also Cassani-Komargodski 2104.01464
& Arabi Ardehali-Murthy 2104.13932]
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Here focus on $d=4$ (field theory dimension).
A lot of work in $d=3$, some results in $d=5,6$
This talk:

Black hole entropy & superconformal index

New general formula for the Cardy-like limit

Examples (including & beyond $\mathcal{N}=4$ super-YM)
The problem: counting the microstates for susy black holes (BHs); possibly identify them
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$$S_{BH} = k_B \log n$$
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$n$ microscopic \textbf{dof} concentrated \textbf{on the boundary} ($A_{\text{BH}}$ dependence)
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[Strominger-Vafa, ...]
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In AdS$_5$, 1/16-BPS BH embedded in AdS$_5 \times S^5$ vacuum of string theory

[Gutowski-Reall,...]
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Extremal: zero temp
“1/16-BPS”: 2 out of 32 supercharges

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$\mathbb{R} \times S^3$
conformal boundary

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$\mathbb{R} \times S^3$
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3 electric charges $q_a$
2 angular momenta $j_i$
(constrained by susy)

[Gutowski-Reall,…]
AdS/CFT: weakly-coupled string theory on $\text{AdS}_5 \times S^5 \leftrightarrow 4d \mathcal{N}=4 \ \text{SU}(N) \ \text{super-YM}$

[Maldacena]
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[C. Maldacena]

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BH microstates correspond to states in CFT
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[Maldacena]

CFT provides definition of microscopical theory, with its dof. 
**BH microstates** correspond to **states in CFT**

Ensemble of states in dual $\mathcal{N}=4$ on $\mathbb{R} \times S^3$?

Susy & charges suggest BH entropy should be given by 
**$1/16$-BPS states of $\mathcal{N}=4$ super-YM** w/ spins $j_i$ & $\text{U}(1)$ charges $q_a$
For more general $\mathcal{N} = 1$ SCFTs, count 1/4-susy states (1 supercharge $Q$)
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When $M_{d-1} = S^{d-1}$, enumeration given by superconformal index

[Römelsberger, Kinney-Maldacena-Minwalla-Raju, …]
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for $\text{AdS}_5$ BHs, entropy correctly reproduced by log of superconformal index (*) of $d=4 \mathcal{N}=4$ SYM at large $N$ ($\sim 1/\sqrt{G_{\text{Newton}}}$) & for complex chemical potentials


(*) Legendre transform of log of index
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$(S_{\text{BH}} = \log n)$

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\[ S_{\text{BH}} \sim -\frac{i}{2} N^2 \frac{\Delta_1 \Delta_2 \Delta_3}{\omega_1 \omega_2} \]

real order-$N^2$ quantity $\sim a, \mathcal{N}=4$ central charge

associated w/ charge operators $Q_a, a=1,2,3$

associated w/ ang. momentum operators $J_1, J_2$

\(^{(*)}\) Legendre transform of log of index
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for AdS$_5$ BHs, entropy correctly reproduced by log of superconformal index$^($ for $d=4 \ \mathcal{N}=4$ SYM at large $N$ ($\sim 1/\sqrt{G_{\text{Newton}}}$) & for complex chemical potentials


$S_{\text{BH}} = \log n$

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The superconformal index of $\mathcal{N}=4$ super-YM

$$
\mathcal{I}_{sc} = \operatorname{Tr}|_{Q=0} (-1)^{F} e^{i(\Delta a Q_a + \omega_i J_i)}
$$
The superconformal index of $\mathcal{N}=4$ super-YM

\[ \mathcal{I}_{sc} = \text{Tr}\big|_{Q=0} (-1)^F e^{i(\Delta_a Q_a + \omega_i J_i)} \]

generalization of Witten index $\text{Tr}_{H} (-1)^F$ (bosons: $F=0$, fermions $F=1$)
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counts 1/16-BPS states w/ sign: different from counting ALL 1/16-BPS states (entropy: hard problem)
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\[ \mathcal{I}_{sc} = \text{Tr}_{Q=0} (-1)^F e^{i(\Delta_a Q_a + \omega_i J_i)} \]

preserved supercharge $Q$

(complex)

$\Delta_a, \omega_i$ chemical potentials
The superconformal index of $\mathcal{N}=4$ super-YM

Computation via susy localization as $Z_{S^3 \times S^1}$. Agreement with gravity result for AdS$_5$ BH in two limits:
The superconformal index of $\mathcal{N}=4$ super-YM

Computation via \textit{susy localization} as $Zs^3 x S^1$. Agreement with gravity result for AdS$_5$ BH in two limits:

$$\log I_{sc} \sim -\frac{i}{27} \frac{16 a(\Delta)}{\omega_1 \omega_2} \quad \text{large-} N \text{ limit from exact evaluation of superconformal index via “Bethe Ansätz” formula } I_{sc} = \sum Z H^{-1}$$

[Closset-Kim-Willett, Benini-Milan, Benini-Colombo-Soltani-Zaffaroni-Zhang]
The superconformal index of $\mathcal{N}=4$ super-YM

Composed via super localization as $Z_{\mathbb{S}^3\times \mathbb{S}^1}$. Agreement with gravity result for AdS$_5$ BH in two limits:

$$\log I_{sc} \sim N \to \infty \ -i \frac{16 a(\vec{\Delta})}{27 \omega_1 \omega_2} = -i \frac{N^2 \Delta_1 \Delta_2 \Delta_3}{2 \omega_1 \omega_2}$$

large-$N$ limit from exact evaluation of superconformal index via “Bethe Ansätze” formula $I_{sc} = \sum Z H^{-1}$

[Closset-Kim-Willett, Benini-Milan, Benini-Colombo-Soltani-Zaffaroni-Zhang]

$$\log I_{sc} \sim |\omega_i| \to 0 \ 4 \pi^2 i \frac{3 \omega_1 + 3 \omega_2 \pm 2\pi(3c - 5a)}{27 \omega_1 \omega_2} (3c - 5a) + \frac{4\pi^2}{\omega_1 \omega_2} \left( \frac{\omega_1 + \omega_2 \pm 2\pi}{\omega_1 \omega_2} (a - c) + O(|\omega_i|^0) \right)$$

“Cardy-like limit”: small-$|\omega_i|$ limit from matrix model representation of superconformal index

[Choi-Kim-Nahmgoong, CaboBizet-Cassani-Martelli-Murthy]
The superconformal index of $\mathcal{N}=4$ super-YM

$$\mathcal{I}_{sc} = \text{Tr} \left|_{Q=0} (-1)^F e^{i(\Delta_a Q_a + \omega_i J_i)} \right.$$ 

Computed via **susy localization** as $Z^3 s^1 x s^1$. Agreement with gravity result for AdS$_5$ BH in two limits:

$$\log \mathcal{I}_{sc} \sim_{N \to \infty} -i \frac{16 a(\bar{\Delta})}{27 \omega_1 \omega_2} = -i \frac{N^2}{2} \frac{\Delta_1 \Delta_2 \Delta_3}{\omega_1 \omega_2}$$

**large-$N$ limit** from exact evaluation of superconformal index via “Bethe Ansatz” formula $I_{sc} = \sum Z H^{-1}$

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“Cardy-like limit”: small-$|\omega|$ limit from matrix model representation of superconformal index

[Choi-Kim-Nahmgoong, CaboBizet-Cassani-Martelli-Murthy]
New general formula when $\omega_1 = \omega_2 = \tau$

[Amariti-MF-Segati '21]

\[
\log I_{sc} \sim 4\pi i \frac{\pm 12\tau^2 - 6\tau \pm 1}{27\tau^2} (3c - 2a) + 4\pi i \frac{\mp 5\tau + 2}{3\tau} (c - a) + \log \Gamma_Z
\]
New general formula when $\omega_1 = \omega_2 \equiv \tau$

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Captures $\tau^{-2}, \tau^{-1}, \tau^0$ orders. Valid at finite rank $N$ of gauge group.
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Captures $\tau^{-2}$, $\tau^{-1}$, $\tau^0$ orders. Valid at finite rank $N$ of gauge group

Valid for generic $\mathcal{N}=1$ SCFT with $G=\text{ABCD}$ gauge group

Valid for holographic & non-holographic theories ($a=c$ or $a\neq c$ at large $N$)
New general formula when $\omega_1 = \omega_2 \equiv \tau$

Valid for \textit{generic} $\mathcal{N}=1$ SCFT with $G=ABCD$ gauge group

Valid for \textit{holographic} & \textit{non-holographic} theories ($a=c$ or $a\neq c$ at large $N$)

Finite \textit{log $\Gamma_Z$ correction}: \textbf{minimal charge} of matter under center $Z(G)$
(order of character lattice modulo Weyl symmetry)

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[ Amariti-MF-Segati '21 ]

$$\log \mathcal{I}_{sc} \mid_{|\tau| \to 0} \sim 4\pi i \left( \frac{\pm 12\tau^2 - 6\tau + 1}{27\tau^2} (3c - 2a) + \frac{\mp 5\tau + 2}{3\tau} (c - a) \right) + \log \Gamma_Z$$

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Valid for generic $\mathcal{N}=1$ SCFT with $G=ABCD$ gauge group

Valid for holographic & non-holographic theories ($a=c$ or $a\neq c$ at large $N$)

Finite log $\Gamma_Z$ correction: minimal charge of matter under center $Z(G)$ (order of character lattice modulo Weyl symmetry)

explains & expands previous results

$\mathcal{N}=4$ super-YM (adj matter): $\Gamma_Z = \text{dim } Z(G)$


toric SU($N$) quivers (bifundamental matter): $\Gamma_Z = N$

[GonzalezLezcano-Hong-Liu-PandoZayas]
\[ I_{sc}(\tau, \Delta) = \frac{(q; q)_{\infty}^{2rk_G}}{|\text{Weyl}(G)|} \int \prod_{i=1}^{rk_G} du_i \frac{\prod_{I=1}^{n_{\chi}} \prod_{\rho_I} \tilde{\Gamma}(\rho_I(\bar{u}) + \Delta_I)}{\prod_{\alpha} \tilde{\Gamma}(\alpha(\bar{u})))} \equiv \frac{1}{|\text{Weyl}(G)|} \int \prod_{i=1}^{rk_G} du_i e^{S_{\text{eff}}(\bar{u}; \tau, \Delta)} \]
One-slide proof: index as matrix model

\[ q = e^{i \tau} \]

\[ I_{sc}(\tau, \Delta) = \frac{(q; q)^{2rk_G}}{|\text{Weyl}(G)|} \left( \prod_{i=1}^{rk_G} du_i \right) \left( \prod_{l=1}^{n_\chi} \prod_{\rho_l} \tilde{\Gamma}(\rho_l(\bar{u}) + \Delta_I) \right) \]

\[ \equiv \frac{1}{|\text{Weyl}(G)|} \left( \prod_{i=1}^{rk_G} du_i \right) e^{S_{\text{eff}}(\bar{u}; \tau, \Delta)} \]

\( e^{2\pi i u_i} \in S^1 \)

\( u_i \) gauge holonomies

\( \rho_l \) gauge weights

\( \alpha \) gauge roots

\( \Delta_I = \nu_l(\xi) + R_l v_R \)

\( \nu_l \) flavor weights;

\( \xi \) flavor holonomies

\( v_R \) R-sym chem potential

\( R_l \) R-charges
One-slide proof: index as matrix model

\[ \mathcal{I}_{\text{sc}}(\tau, \Delta) = \frac{(q, q)_\infty^{2r_k G}}{|\text{Weyl}(G)|} \int \prod_{i=1}^{r_k G} du_i \frac{\Pi_{I=1}^{n_x} \rho_I \Gamma(\rho_I(\bar{u}) + \Delta_I)}{\Pi_\alpha \Gamma(\alpha(u))} \equiv \frac{1}{|\text{Weyl}(G)|} \int \prod_{i=1}^{r_k G} du_i e^{S_{\text{eff}}(\bar{u}; \tau, \Delta)} \]

effective action

\[ \Delta_I = \nu_I(\xi) + R_I v_R \]

matter chemical potentials

EOM of matrix model \( \int [du] e^{S_{\text{eff}}} \) at leading order:
One-slide proof: index as matrix model

\[ I_{\text{sc}}(\tau, \Delta) = \frac{(q; q)^{2rkG}}{|\text{Weyl}(G)|} \int \prod_{i=1}^{rkG} du_i \frac{\prod_{I=1}^{n_X} \prod_{\rho_I} \bar{\Gamma}(\rho_I(\bar{u}) + \Delta_I)}{\prod_{\alpha} \Gamma(\alpha(\bar{u}))} \equiv \frac{1}{|\text{Weyl}(G)|} \int \prod_{i=1}^{rkG} du_i e^{S_{\text{eff}}(\bar{u}; \tau, \Delta)} \]

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EOM of matrix model \[ \int [du] e^{S_{\text{eff}}} \text{ at leading order:} \]

\[ 0 = \frac{\partial S_{\text{eff}}(\bar{u}; \tau, \Delta)}{\partial u_{ia}} = -\frac{i\pi}{\tau^2} \sum_{I=1}^{n_X} \sum_{\rho_I} \frac{\partial \rho_I(\bar{u})}{\partial u_{ia}} B_2(\{\rho_I(\bar{u}) + \Delta_I\}_{\tau}) \]
One-slide proof: index as matrix model

\[ \mathcal{I}_{sc}(\tau, \Delta) = \frac{(q;q)^{2r_k G}}{|\text{Weyl}(G)|} \int_0^{\pi} du \prod_{i=1}^{r_k G} \prod_{\alpha} \frac{\Gamma(\alpha(\bar{u}))}{\Gamma(\rho_I(\bar{u}) + \Delta_I)} \]

\[ \equiv \frac{1}{|\text{Weyl}(G)|} \int_0^{\pi} du e^{S_{\text{eff}}(\bar{u}, \tau, \Delta)} \]

\[ e^{2\pi i u_i} \in S^1 \quad u_i \text{ gauge holonomies} \]

\[ \rho_I \text{ gauge weights} \quad \alpha \text{ gauge roots} \]

\[ \Delta_I = \nu_I(\xi^I) + R_I v_R \quad \nu_I \text{ flavor weights; } \xi^I \text{ flavor holonomies} \]

\[ v_R \text{ R-sym chem potential} \quad R_i \text{ R-charges} \]

\[ n_x \text{ matter fields} \]

\[ 0 = \frac{\partial S_{\text{eff}}(\bar{u}, \tau, \Delta)}{\partial u_{i a}} = -\frac{i\pi}{\tau^2} \sum_{I=1}^{n_x} \sum_{\rho_I} \frac{\partial \rho_I(\bar{u})}{\partial u_{i a}} B_2(\{\rho_I(\bar{u}) + \Delta_I\}_{\tau}) \]

\[ B_n \text{ Bernoulli polynomials} \]

\[ \log \tilde{\Gamma}(u) \sim \frac{B_3(u_{\tau})}{\tau^2} + \frac{B_2(u_{\tau})}{\tau} + B_1(u_{\tau}) + \tau \]
One-slide proof: index as matrix model

\[ I_{sc}(\tau, \Delta) = \frac{\Pi_{1}^{\infty}}{\text{Weyl}(G)} \int \frac{(q; q)_{2rG}^{2 \rho}}{} \prod_{i=1}^{n} du_i \prod_{\rho_i} \tilde{\Gamma}(\rho_i(\bar{u}) + \Delta_1) \frac{1}{|\text{Weyl}(G)|} \int \prod_{i=1}^{rG} du_i \ e^{S_{\text{eff}}(\bar{u}; \tau, \Delta)} \]

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\[ \Delta_1 = \nu_I(\xi) + R_I v_R \]

\[ \nu_I \] flavor weights;

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EOM of matrix model \[ \int [du] \ e^{S_{\text{eff}}} \text{ at leading order:} \]

\[ 0 = \frac{\partial S_{\text{eff}}(\bar{u}; \tau, \Delta)}{\partial u_{i\alpha}} = -\frac{i\pi}{\tau^2} \sum_{I=1}^{n_x} \sum_{\rho_i} \frac{\partial \rho_i(\bar{u})}{\partial u_{i\alpha}} B_2(\{\rho_i(\bar{u}) + \Delta_1\}_\tau) \]

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\[ \log \tilde{\Gamma}(u) \sim \frac{B_3(u_\tau)}{\tau^2} + \frac{B_2(u_\tau)}{\tau} + B_1(u_\tau) + \tau \]
One-slide proof: index as matrix model

\[ I_{sc}(\tau, \Delta) = \frac{(q; q)^{2rG}}{|\text{Weyl}(G)|} \int \prod_{i=1}^{rG} du_i \frac{\prod_{I=1}^{n_x} \hat{\Gamma}(\rho_I(\vec{u}) + \Delta_I)}{\prod_{\alpha} \hat{\Gamma}(\alpha(\vec{u}))} \equiv \frac{1}{|\text{Weyl}(G)|} \int \prod_{i=1}^{rG} du_i e^{S_{\text{eff}}(\vec{u}; \tau, \Delta)} \]

Effective action

Matter chemical potentials

\[ \Delta_I = \nu I(\xi) + R_I v_R \]

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\( u_i \) gauge holonomies

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EOM of matrix model \( \int [du] e^{S_{\text{eff}}} \) at leading order:

\[ 0 = \frac{\partial S_{\text{eff}}(\vec{u}; \tau, \Delta)}{\partial u_{ia}} = -\frac{i\pi}{\tau^2} \sum_{I=1}^{n_x} \sum_{\rho_I} \frac{\partial \rho_I(\vec{u})}{\partial u_{ia}} B_2\left(\{\rho_I(\vec{u}) + \Delta_I\}_{\tau}\right) \]

\[ \log \hat{\Gamma}(u) \big|_{\tau \to 0} \sim \frac{B_3\{u\}_{\tau}}{\tau^2} + \frac{B_2\{u\}_{\tau}}{\tau} + B_1\{u\}_{\tau} + \tau \]

Ansatz for saddle points:

\[ u_{i_a} = u_{*i_a} + v_{i_a} \tau , \quad v_{i_a} \sim O(|\tau|^0) \]
One-slide proof: index as matrix model

\[ \mathcal{I}_{sc}(\tau, \Delta) = \frac{(q; q)^{2 \text{rk}_G}}{|\text{Weyl}(G)|} \int du_i \prod_{I=1}^{\text{rk}_G} \prod_{I} \rho_I(\bar{u}) \Gamma(\alpha(\bar{u})) \prod_{I=1}^{\text{rk}_G} \Delta_I \]

\[ = \frac{1}{|\text{Weyl}(G)|} \int du_i \ e^{S_{\text{eff}}(\bar{u}; \tau, \Delta)} \]

Effective action

Matter chemical potentials

\[ \Delta_I = \nu_I(\xi) + R_I v_R \]

\( q = e^{i \tau} \)

\( e^{2 \pi i u_i} \in S^1 \)

\( u_i \) gauge holonomies

\( \rho_i \) gauge weights

\( \alpha \) gauge roots

\( n_x \) matter fields

\( \mathcal{E}_{\text{O.M.}} \) of matrix model \( \int [du] \ e^{S_{\text{eff}}} \) at leading order:

\[ 0 = \frac{\partial S_{\text{eff}}(\bar{u}; \tau, \Delta)}{\partial u_{ia}} = -\frac{i \pi}{\tau^2} \sum_{I=1}^{n_x} \sum_{\rho_I} \frac{\partial \rho_I(\bar{u})}{\partial u_{ia}} B_2(\{\rho_I(\bar{u}) + \Delta_I\}_\tau) \]

\[ \log \Gamma(u) \mid_{\tau \to 0} \sim B_3(\{u\}_\tau) + \frac{B_2(\{u\}_\tau)}{\tau} + B_1(\{u\}_\tau) + \tau \]

Ansatz for saddle points:

\[ u_{ia} = u_{*ia} + v_{ia} \tau \]

\( v_{ia} \sim O(1) \)

\( n_x \) matter fields

Captures up to finite terms in \( \tau \)

(goes beyond preexisting results up to \( \tau^{-2}, \tau^{-1} \))

\# inequivalent ways of selecting \( u_{*i} \) given by \( \Gamma_z \)
One-slide proof: index as matrix model

\[ q = e^{i\tau} \]

\[ \mathcal{I}_{sc}(\tau, \Delta) = \frac{(q; q)_{2rG}}{|\text{Weyl}(G)|} \int \prod_{I=1}^{rk_G} du_i \prod_{I=1}^{n_x} \frac{\hat{\Gamma}(\rho_I(\bar{u}) + \Delta_I)}{\hat{\Gamma}(\alpha(\bar{u}))} = \frac{1}{|\text{Weyl}(G)|} \int \prod_{I=1}^{rk_G} du_i e^{S_{\text{eff}}(\bar{u}; \tau, \Delta)} \]

effective action

matter chemical potentials

\[ \Delta_I = \nu_I(\xi) + R_I v_R \]

\[ e^{2\pi i u_i} \in S^1 \]

\[ u_i \text{ gauge holonomies} \]

\[ \rho_I \text{ gauge weights} \]

\[ \alpha \text{ gauge roots} \]

\[ n_x \text{ matter fields} \]

EOM of matrix model \( \int [du] e^{S_{\text{eff}}} \) at leading order:

\[ 0 = \frac{\partial S_{\text{eff}}(\bar{u}; \tau, \Delta)}{\partial u_{i_a}} = -\frac{i\pi}{\tau^2} \sum_{I=1}^{n_x} \sum_{\rho_I} \frac{\partial \rho_I(\bar{u})}{\partial u_{i_a}} B_2(\{\rho_I(\bar{u}) + \Delta_I\}) \]

\[ B_n \text{ Bernoulli polynomials} \]

\[ \log \hat{\Gamma}(u) \overset{\tau \to 0}{\sim} B_3(\{u\}) \tau^3 + \frac{B_2(\{u\})}{\tau^2} + B_1(\{u\}) + \tau \]

Ansatz for saddle points:

\[ u_{i_a} = u_{i_a}^* + v_{i_a} \tau, \quad v_{i_a} \sim O(|\tau|^0) \]

captures up to finite terms in \( \tau \)

(goes beyond preexisting results up to \( \tau^{-2}, \tau^{-1} \))

# inequivalent ways of selecting constants \( u_{i_a} \) given by \( \Gamma_Z \)

plug Ansatz back into \( S_{\text{eff}} \) and impose physical constraints on matter chem. potentials \( \Delta_i \):

superconformal index computed by 3d pure Chern-Simons partition function &

dependence on SCFT central charges \( a(\Delta_i), c(\Delta_i) \)
$\mathcal{N}=4$ SYM: USp(2N) and SO(N) gauge groups

[Amariti-MF-Segati ’20]
$\mathcal{N}=4$ SYM: USp($2N$) and SO($N$) gauge groups

[ Amariti-MF-Segati '20 ]

$\Gamma_Z = \dim Z(G) = 2 \text{ or } 4$; careful analysis of saddles:

\[ \begin{array}{c}
\frac{1}{4} \quad \frac{1}{2} \quad \frac{3}{4} \\
0 \quad 1
\end{array} \]

(focus on USp)
$\mathcal{N}=4$ SYM: USp($2N$) and SO($N$) gauge groups

[Amari-MF-Segati '20]

$\Gamma_Z = \dim Z(G) = 2$ or $4$; careful analysis of saddles:

Dominant saddle in BH ‘region’ (constraints on $\Delta$): $u_i = m/2 + v_i \tau^j$; $m = \{0,1\} \Rightarrow \Gamma_Z = 2$
$\mathcal{N}=4$ SYM: USp($2N$) and SO($N$) gauge groups

[Amiriti-MF-Segati '20]

$\Gamma_Z = \dim Z(G) = 2$ or $4$; careful analysis of saddles:

(focus on USp)

Dominant saddle in BH ‘region’ (constraints on $\Delta$): $u_i = m/2 + v_i \tau$; $m = \{0,1\} \Rightarrow \Gamma_Z = 2$
\( \mathcal{N}=4 \) SYM: USp\((2N)\) and SO\((N)\) gauge groups

\[ \Gamma_Z = \dim Z(G) = 2 \text{ or } 4; \text{ careful analysis of saddles:} \]

\[ \begin{align*}
L &\leftarrow 0 \quad 0 \quad L-N \quad 1 \\
0 &\quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{3}{4} \quad u_i
\end{align*} \]

L=N: all \( u_i = 0 \)

L=0: all \( u_i = \frac{1}{2} \)

Dominant saddle in BH ‘region’ (constraints on \( \Delta I \)): \( u_i = m/2 + v_i \tau; m = \{0, 1\} \Rightarrow \Gamma_Z = 2 \)

\[ \log I_{sc}^{USp(2N)} = \frac{i\pi N(2N + 1)}{\tau^2} \prod_{I=1}^{3} \left( \{\Delta_I\}_\tau - \frac{1 + \eta}{2} \right) + \log 2 + O(e^{-1/|\tau|}) \]

with \( N \) holonomies at 0 or \( \frac{1}{2} \).
$\mathcal{N}=4$ SYM: USp(2N) and SO(N) gauge groups

[ Amariti-MF-Segati '20 ]

$\Gamma_Z = \dim Z(G) = 2$ or 4; careful analysis of saddles:

Dominant saddle in BH ‘region’ (constraints on $\Delta$): $u_i = m/2 + v_i \tau$; $m = \{0,1\} \rightarrow \Gamma_Z = 2$

$$\log \mathcal{I}^{\text{USp}(2N)}_{\text{sc}} \bigg|_{|\tau| \rightarrow 0} = -\frac{i\pi N(2N + 1)}{\tau^2} \prod_{I=1}^{3} \left( \{\Delta_I\}_\tau - \frac{1 + \eta}{2} \right) + \log 2 + \mathcal{O}(e^{-1/|\tau|})$$

with $N$ holonomies at 0 or $\frac{1}{2}$
$\mathcal{N}=4$ SYM: USp($2N$) and SO($N$) gauge groups

[ Amariti-MF-Segati ’20 ]

$\Gamma_Z = \dim Z(G) = 2$ or $4$; careful analysis of saddles:

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\( \mathcal{N}=4 \) SYM: USp\((2N)\) and SO\((N)\) gauge groups

\[ \Gamma_Z = \dim Z(G) = 2 \text{ or } 4; \text{ careful analysis of saddles:} \]

(focus on USp)

![Diagram of subdominant saddles](focus on USp)

subdominant saddles:
$\mathcal{N}=4$ SYM: USp($2N$) and SO($N$) gauge groups

Amariti-MF-Segati ’20

$\Gamma_Z = \dim Z(G) = 2$ or $4$; careful analysis of saddles:

(focus on USp)

subdominant saddles: $N$ at $\frac{1}{4}$
$\mathcal{N}=4$ SYM: $\text{USp}(2N)$ and $\text{SO}(N)$ gauge groups

[ Amariti-MF-Segati '20 ]

$\Gamma_Z = \dim Z(G) = 2$ or $4$; careful analysis of saddles:

(focus on USp)

subdominant saddles:
$\mathcal{N}=4$ SYM: USp($2N$) and SO($N$) gauge groups

$\Gamma_Z = \dim Z(G) = 2$ or $4$; careful analysis of saddles:

(focus on USp)

subdominant saddles: $P$ at $0$; $P$ at $\frac{1}{2}$; $N-2P$ at $\frac{1}{4}$
$\mathcal{N}=4$ SYM: $\text{USp}(2N)$ and $\text{SO}(N)$ gauge groups

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(focus on USp)

subdominant saddles: $P$ at 0; $P$ at $\frac{1}{2}$; $N-2P$ at $\frac{1}{4}$

In both cases, $S_{\text{eff}}$ (ie log index) evaluated at subdominant saddle is \textbf{subleading at large $N$}:

hierarchy of saddles very important
$\mathcal{N}=4$ SYM: USp($2N$) and SO($N$) gauge groups

$\Gamma_Z = \dim Z(G) = 2$ or $4$; careful analysis of saddles:

(focus on USp)

In both cases, $S_{\text{eff}}$ (ie log index) evaluated at subdominant saddle is subleading at large $N$:

hierarchy of saddles very important

S-duality between USp and SO (identity of superconformal indices) nontrivially realized on saddles
We’ve looked at more complicated $\mathcal{N}=1$ models:
non-toric (toric: $U(1)^3$ global symmetry), not all ranks equal, non-holographic
(including subleading corrections and finite terms)

[Amariti-MF-Segati ’21]
$N$ D3’s probing $\mathbb{C}^3 / \mathbb{Z}_2 \times \mathbb{Z}_2$
(toric model: $U(1)^3$)

Seiberg-dual nontoric phase

groups not all $SU(N)$; middle is $SU(2N)$
$N$ D3’s probing $\mathbb{C}^3 / \mathbb{Z}_2 \times \mathbb{Z}_2$
(toric model: $U(1)^3$)

Seiberg-dual nontoric phase

Cardy-like limit of superconformal indices computed independently in two phases from $S_{\text{eff}}$ match precisely.
Both given by our new formula with $\Gamma_Z = N$

Nontrivial **check of validity** of our formula
$N$ D3’s probing non-toric threefolds (& different ranks):

Cone over $dP_4$

Laufer’s theory

4 flavor U(1)s

1 flavor U(1)
\( N \) D3’s probing non-toric threefolds (\& different ranks):

- Cone over \( dP_4 \)
- Laufer’s theory

4 flavor \( U(1) \)s

Groups not all \( SU(N) \); some \( SU(2N) \): nontrivial ‘complication’
Non-holographic $\mathcal{N}=1$ theories ($a \neq c$ at large $N$): SQCD

$\mathcal{N}=2$ theories
Non-holographic $\mathcal{N}=1$ theories ($a\neq c$ at large $N$): SQCD

SU($N$) in conformal window

$\mathcal{N}=2$ theories
Non-holographic $\mathcal{N}=1$ theories ($a\neq c$ at large $N$): SQCD

SU($N$) in conformal window \hspace{1cm} \text{adjoint SU($N$)}

$\mathcal{N}=2$ theories
Non-holographic $\mathcal{N}=1$ theories ($a\neq c$ at large $N$): SQCD

SU($N$) in conformal window       adjoint SU($N$)       USp(2$N$)

$\mathcal{N}=2$ theories
Non-holographic $\mathcal{N}=1$ theories ($a \neq c$ at large $N$): SQCD

SU($N$) in conformal window \hspace{1cm} \text{adjoint SU($N$)} \hspace{1cm} \text{USp(2$N$)}

$\mathcal{N}=2$ theories

family of $\mathcal{N}=1$ SU($n$) Lagrangians
enhancing to ( $A_1$, $A_{2n-1}$) Argyres-Douglas $\mathcal{N}=2$ SCFT

[Maruyoshi-Song, ...]
Non-holographic $\mathcal{N}=1$ theories ($a \neq c$ at large $N$): SQCD

$SU(N)$ in conformal window \hspace{1cm} \text{adjoint } SU(N) \hspace{1cm} USp(2N)$

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[Maruysohi-Song, …]

$\mathcal{N}=2$ SCFT: $SU(N)$ w/ hypers $\square$ \hspace{0.2cm} & \hspace{0.2cm} $\square$

[Ennes-Lozano-Naculich-Schnitzer]
Non-holographic $\mathcal{N}=1$ theories ($a \neq c$ at large $N$): SQCD

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[Maruyoshi-Song,…]

$\mathcal{N}=2$ SCFT: SU($N$) w/ hypers

[Ennes-Lozano-Naculich-Schnitzer]

Cardy-like limit of log of SCI matches $S_{BH}$ at large $N$

$$S_{BH} = 2\pi \sqrt{Q_2^2 - Q_\ell^2 - Q_\bar{\ell}^2 + 2Q_1(Q_2 - Q_\ell - Q_\bar{\ell})} - \frac{a}{4}(J_1 + J_2)$$

[Hosseini-Zaffaroni]
Non-holographic $\mathcal{N}=1$ theories ($a \neq c$ at large $N$): SQCD

$\mathcal{N}=2$ theories

family of $\mathcal{N}=1$ SU($n$) Lagrangians

enhancing to ($A_1$, $A_{2n-1}$) Argyres-Douglas $\mathcal{N}=2$ SCFT

[Maruyoshi-Song,…]

$\mathcal{N}=2$ SCFT: SU($N$) w/ hypers $\square$ & $\blacksquare$

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[Hosseini-Zaffaroni]

peculiarity:

$$\Gamma_Z = (3+(-1)^N)/2$$

depends on parity of $N$

(reflected in degeneracy of saddles)
Conclusions

Formula for Cardy-like limit of superconformal index for generic $\mathcal{N}=1$ ABCD SCFTs: extends previous results valid at lowest order and/or for non-generic theories (super-YM, toric)
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Includes leading & subleading contributions (from 3d pure Chern-Simons term) and
finite log correction from minimal charge of matter = degeneracy of matrix model saddle

[3d EFT interpretation of result given by Cassani-Komargodski]
Conclusions

Formula for Cardy-like limit of superconformal index for generic $\mathcal{N}=1$ ABCD SCFTs: extends previous results valid at lowest order and/or for non-generic theories (super-YM, toric)

Includes leading & subleading contributions (from 3d pure Chern-Simons term) and finite log correction from minimal charge of matter = degeneracy of matrix model saddle

[3d EFT interpretation of result given by Cassani-Komargodski]

No ‘rigorous’ proof but can explicitly determine leading & subleading contributions with very general Ansatz $u_i = u_{\star i} + v_i \tau$ for matrix model saddle point
Outlook

AdS/CFT derivation/interpretation of finite log correction (i.e. quantum gravity corrections to asymptotically-AdS BH entropy)

[Bobev-Charles-Gang-Hristov-Reys, Bobev-Charles-Hristov-Reys for AdS$_4$ BHs]

‘Derivation’ of new formula from 3d EFT for SU case: applicable to generic $\mathcal{N}=1$ SCFTs too?

[Cassani-Komargodski for SU case]

Extend formula to 2 different angular momenta: $\omega_1 \neq \omega_2 = \tau$

(Structure of) other subleading saddles?

[ArabiArdehali-Hong-Liu, CaboBizet-Cassani-Martelli-Murthy]

Bethe Ansätz approach in generic case is terra incognita; match large-$N$ limit to Cardy-like limit. Very nontrivial: eg ‘basic solutions’ don’t work for Laufer SU($N$) x SU(2$N$)

[Benini-Colombo-Soltani-Zaffaroni-Zhang for SU($N$) holographic quivers dual to AdS$_5$ x S$^5$]

Beyond $\tau^0$, exponentially suppressed orders in $\tau^{-1}$ vs in $N^{-1}$ from Bethe Ansätz. Match? Meaning?

[Aharony-Benini-Mamroud-Milan]
Thanks
\[ I_{sc}(\tau, \Delta) = \frac{(q; q)^{2rkG}_\infty}{|\text{Weyl}(G)|} \int \prod_{i=1}^{rkG} du_i \prod_{I=1}^{n_x} \prod_{\alpha} \tilde{\Gamma}(\alpha_i(\bar{u})) \prod_{I} \tilde{\Gamma}(\rho_I(\bar{u}) + \Delta_I) \]

\[ \equiv \frac{1}{|\text{Weyl}(G)|} \int \prod_{i=1}^{rkG} du_i e^{S_{\text{eff}}(\bar{u}; \tau, \Delta)} \]

\[ \Delta_I = \nu_I(\vec{\xi}) + R_I \nu_R \]

- \( q = e^{i\tau} \)
- \( u_i \in (0, 1] \)
- \( u_i \sim u_i + 1 \)
- \( u_i \) gauge holonomies
- \( z_i = e^{2\pi i u_i} \in S^1 \)
- \( \rho_i \) gauge weight
- \( \nu_i \) flavor weight; \( \xi \) flavor holonomies
- \( \nu_R \) R-sym chem pot; \( R_i \) R-charge
For $a=1,\ldots,n_G$ gauge groups and $l=1,\ldots,n_X$ matter fields, effective action:

$$S_{\text{eff}}(\vec{u}; \tau, \Delta) = \sum_{l=1}^{n_X} \sum_{I} \log \tilde{\Gamma}(\rho_I(\vec{u}) + \Delta_I) + \sum_{a=1}^{n_G} \sum_{\alpha} \log \theta_0(\alpha_a(\vec{u}); \tau) + \sum_{a=1}^{n_G} 2 r_{k_{G_a}} \log(q; q_\infty)$$

- matter contribution
- gauge
- $q$-Pochhammer
Saddles:

Expand all functions in $S_{\text{eff}}$ for small $\tau$; eg matter fields contribute as

$$\log \tilde{\Gamma}(u) \sim \frac{B_3(u_\tau)}{\tau^2} + \frac{B_2(u_\tau)}{\tau} + \frac{B_1(u_\tau)}{\tau^0} + \tau$$

Bernoulli polynomials $B_n$

$$u_\tau \equiv u - [\text{Re}(u) - \cot(\arg \tau) \text{Im}(u)]^{\tau\text{-modded value}}$$
Saddles:

Expand all functions in $S_{\text{eff}}$ for small $\tau$; eg matter fields contribute as

$$
\log \tilde{\Gamma}(u) \bigg|_{|\tau|\to 0} \sim \frac{B_3(\{u\}_\tau)}{\tau^2} + \frac{B_2(\{u\}_\tau)}{\tau} + \frac{B_1(\{u\}_\tau)}{\tau^0} + \tau
$$

Bernoulli polynomials $B_n$

EOM of matrix model **at leading order**: 

$$
0 = \frac{\partial S_{\text{eff}}(\vec{u}; \tau, \Delta)}{\partial u_{i_a}} = -\frac{i \pi}{\tau^2} \sum_{I=1}^{\mathcal{N}_X} \sum_{\rho_I} \frac{\partial \rho_I(\vec{u})}{\partial u_{i_a}} B_2(\{\rho_I(\vec{u}) + \Delta_I\}_\tau)
$$

**Ansatz** for saddle points of the form:

$$
 u_{i_a} = u_{*i_a} + v_{i_a} \tau , \quad v_{i_a} \sim \mathcal{O}(|\tau|^0)
$$
Saddles:

Expand all functions in $S_{\text{eff}}$ for small $\tau$; eg matter fields contribute as

$$\log \tilde{\Gamma}(u) \bigg|_{\tau \to 0} \sim \frac{B_3(\{u\}_\tau)}{\tau^2} + \frac{B_2(\{u\}_\tau)}{\tau} + \frac{B_1(\{u\}_\tau)}{\tau^0} + \tau$$

Bernoulli polynomials $B_n$

EOM of matrix model \textbf{at leading order}:

$$0 = \frac{\partial S_{\text{eff}}(\vec{u}; \tau, \Delta)}{\partial u_{i\alpha}} = -\frac{i\pi}{\tau^2} \sum_{I=1}^{n_X} \sum_{\rho_I} \frac{\partial \rho_I(\vec{u})}{\partial u_{i\alpha}} B_2(\{\rho_I(\vec{u}) + \Delta_I\}_\tau)$$

\textbf{Ansatz} for saddle points of the form:

$$u_{i\alpha} = u_{*i\alpha} + v_{i\alpha} \tau, \quad v_{i\alpha} \sim \mathcal{O}(|\tau|^0)$$

It \textbf{captures} up to \textbf{finite terms in} $\tau$: goes beyond preexisting results up to $\tau^{-2}$, $\tau^{-1}$.

Number of \textbf{inequivalent} ways of selecting \textbf{constants} $u_{*i}$ given by $\Gamma_Z$
Plug Ansatz back into $S_{\text{eff}}$ and impose physical constraints on matter charges $\Delta_i$.
Plug Ansatz back into $S_{\text{eff}}$ and impose physical constraints on matter charges $\Delta_I$

Superpotential constraint:

$$
\left( \hat{\Delta}_I = \frac{2}{2\tau - \eta} \{\Delta_I\}_\tau \right)
$$

Matter fields in each superpotential term:

$$
\sum_{I \in \mathcal{W}} \hat{\Delta}_I = 2 \quad \Rightarrow \quad \sum_{I \in \mathcal{W}} \{\Delta_I\}_\tau = 2\tau - \eta
$$
Plug Ansatz back into $S_{\text{eff}}$ and impose physical constraints on matter charges $\Delta_I$

Superpotential constraint:

$$\begin{align*}
\hat{\Delta}_I &= \frac{2}{2\tau - \eta} \{\Delta_I\}_\tau \\
\sum_{I \in W} \hat{\Delta}_I &= 2 \quad \Rightarrow \quad \sum_{I \in W} \{\Delta_I\}_\tau = 2\tau - \eta
\end{align*}$$

R-sym anomaly freedom in the $\Delta_I$ variables:

$$T(G) + \sum_{I \in G_a} T(R_I) (\hat{\Delta}_I - 1) = 0$$

Matter fields in each superpotential term

Matter fields charged under a-th gauge group

index of irrep
Plug Ansatz back into $S_{\text{eff}}$ and impose physical constraints on matter charges $\Delta_I$

Superpotential constraint:

Matter fields in each superpotential term

$$\sum_{I \in W} \hat{\Delta}_I = 2 \Rightarrow \sum_{I \in W} \{\Delta_I\}_\tau = 2\tau - \eta$$

R-sym anomaly freedom in the $\Delta_I$ variables:

Matter fields charged under $a$-th gauge group

$$T(G) + \sum_{I \in G_a} T(R_I)(\hat{\Delta}_I - 1) = 0$$

$$\left( \hat{\Delta}_I = \frac{2}{2\tau - \eta} \{\Delta_I\}_\tau \right)$$

Index of irrep

Toric theories
Plug Ansatz back into $S_{\text{eff}}$ and impose physical constraints on matter charges $\Delta_I$

Superpotential constraint:

$$\sum_{I \in W} \hat{\Delta}_I = 2 \Rightarrow \sum_{I \in W} \{\Delta_I\}_\tau = 2\tau - \eta$$

R-sym anomaly freedom in the $\Delta_I$ variables:

$$T(G) + \sum_{I \in G_a} T(\mathcal{R}_I)(\hat{\Delta}_I - 1) = 0$$

**CONSEQUENCE #1:** linear term in holonomies $u_i$ in $S_{\text{eff}}$ vanishes for all ABCD algebras
Plug Ansatz back into $S_{\text{eff}}$ and impose physical constraints on matter charges $\Delta_I$

Superpotential constraint:

$$\sum_{I \in W} \hat{\Delta}_I = 2 \Rightarrow \sum_{I \in W} \{\Delta_I\}_\tau = 2\tau - \eta$$

R-sym anomaly freedom in the $\Delta_I$ variables:

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CONSEQUENCE #1: linear term in holonomies $u_i$ in $S_{\text{eff}}$ vanishes for all ABCD algebras

CONSEQUENCE #2: quadratic term reconstructs 3d $G=$ABCD pure CS partition function at level $-\eta T(G)$

[See also Cassani-Komargodski for 3d EFT interpretation. 3d CS term previously observed in GonzalezLezcano-Hong-Liu-PandoZayas & Amariti-MF-Segati ’20]
GR calculation of BH $\subset$ AdS$_5 \times$ S$^5$ entropy:

$$S_{\text{BH}}(q_a, j_i) = 2\pi \sqrt{q_1 q_2 + q_1 q_3 + q_2 q_3 - \frac{\pi}{4G_N^{(5)} g_{\text{AdS}}^3}} (j_1 + j_2)$$

$$N^2 = \frac{\pi}{2G_N^{(5)} g_{\text{AdS}}^3}$$
Asymptotically AdS$_5$ BH:

Near the horizon:

$$ds^2_{r \sim r_c} \sim -(r - r_c)^2 dt^2 + \frac{dr^2}{(r - r_c)^2} + \text{const} \ ds^2_{\mathcal{M}_{d-1}}$$

Asymptotically AdS$_5$ (with $\mathbb{R} \times M_{d-1}$ conformal boundary):

$$ds^2_{r \to \infty} \sim \frac{dr^2}{r^2} + r^2 (-dt^2 + ds^2_{\mathcal{M}_{d-1}}) + \ldots$$