Generalised Riemann Hypothesis and Brownian Motion

Giuseppe Mussardo
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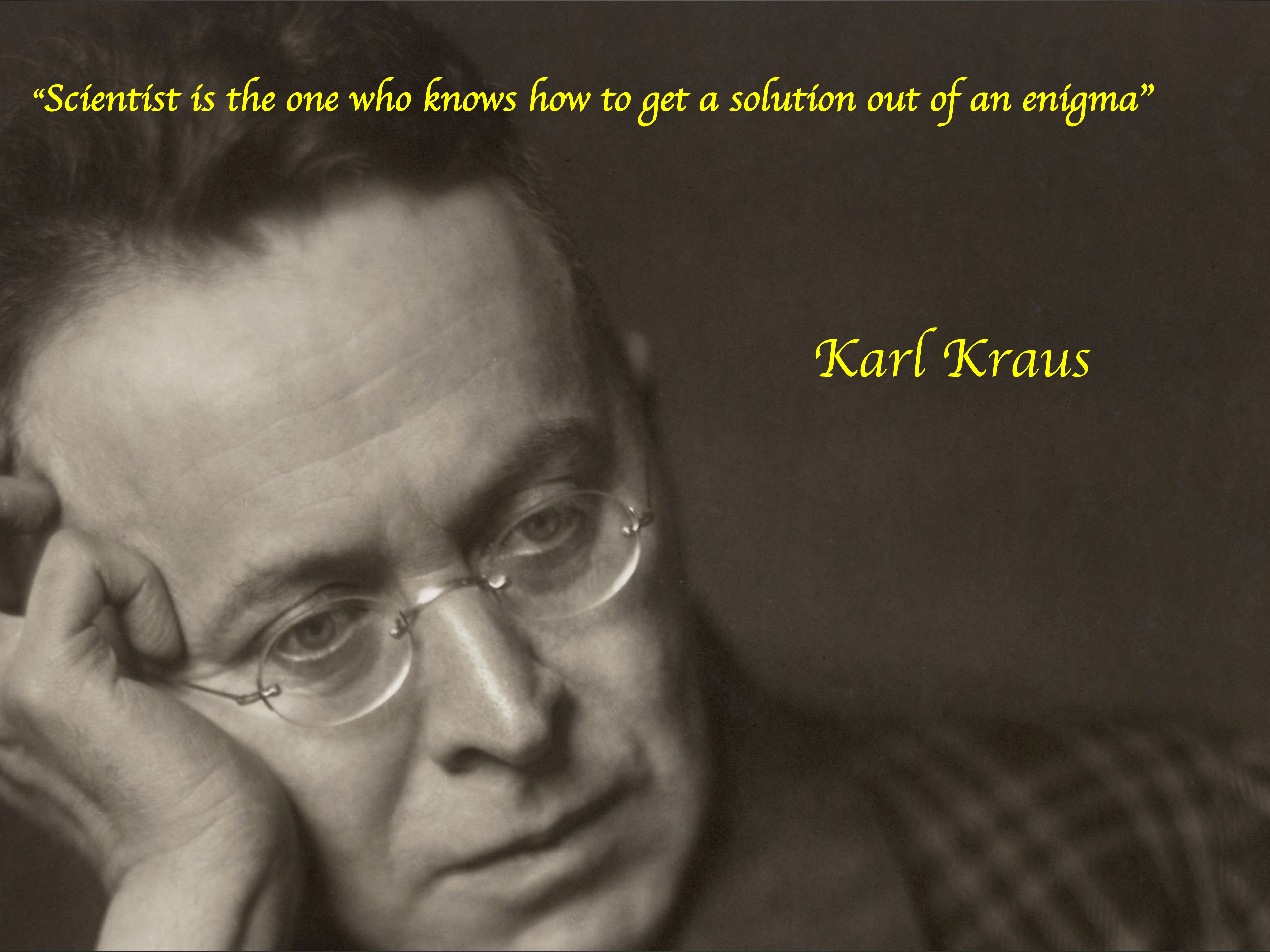

"Scientist is the one who knows how to get a solution out of an enigma"
Karl Kraus

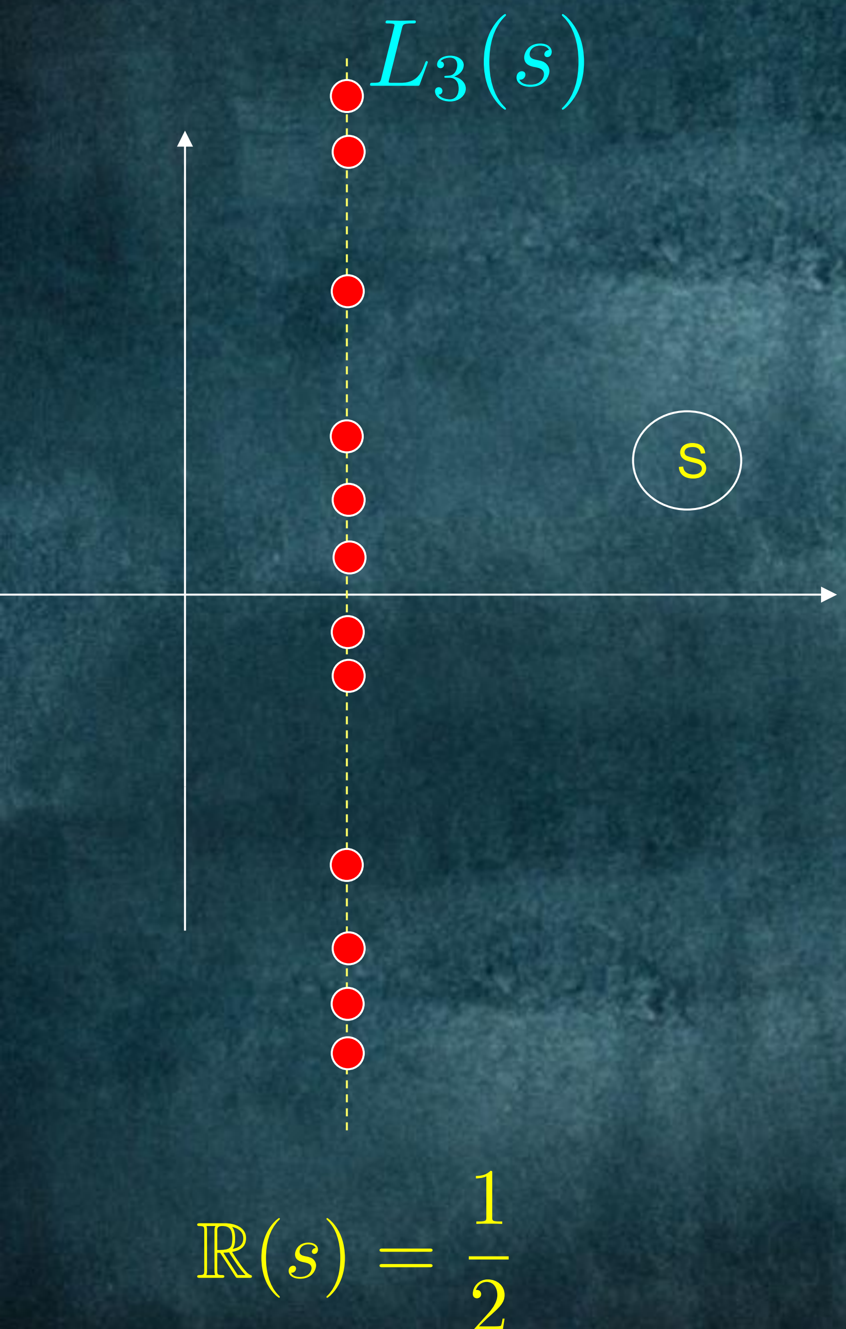
$$\zeta(s)$$

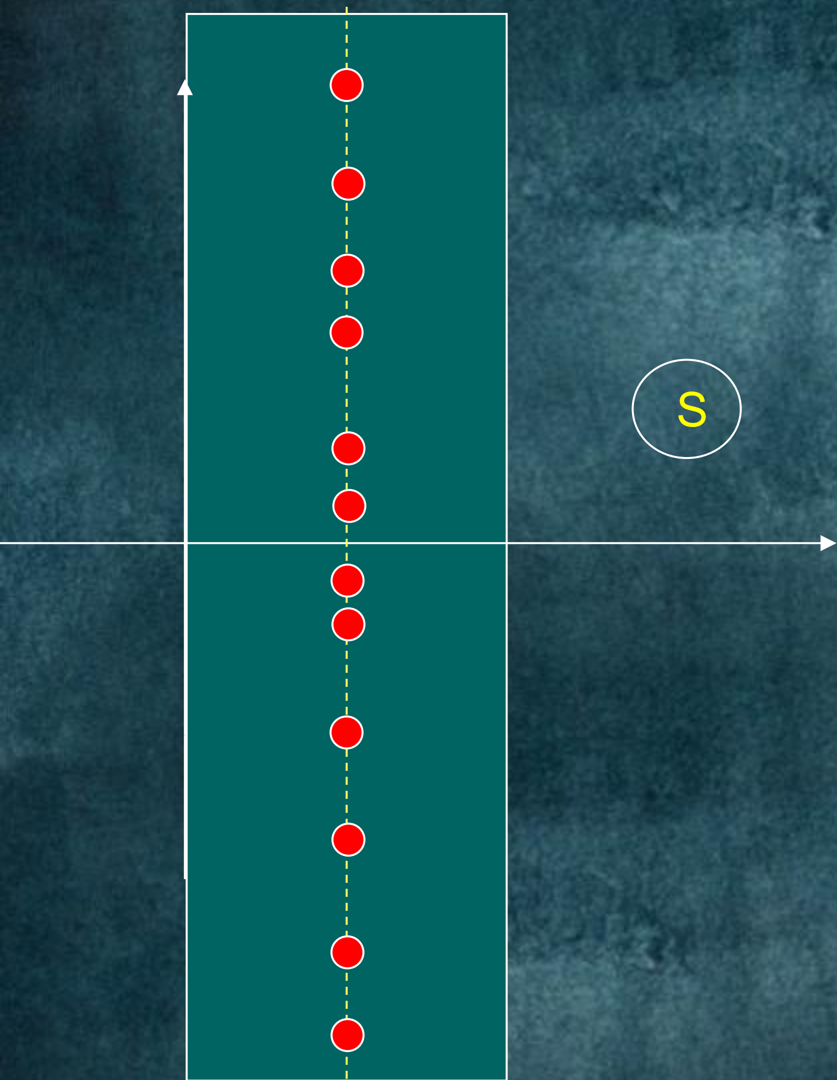
$$\mathbb{R}(s) = \frac{1}{2}$$

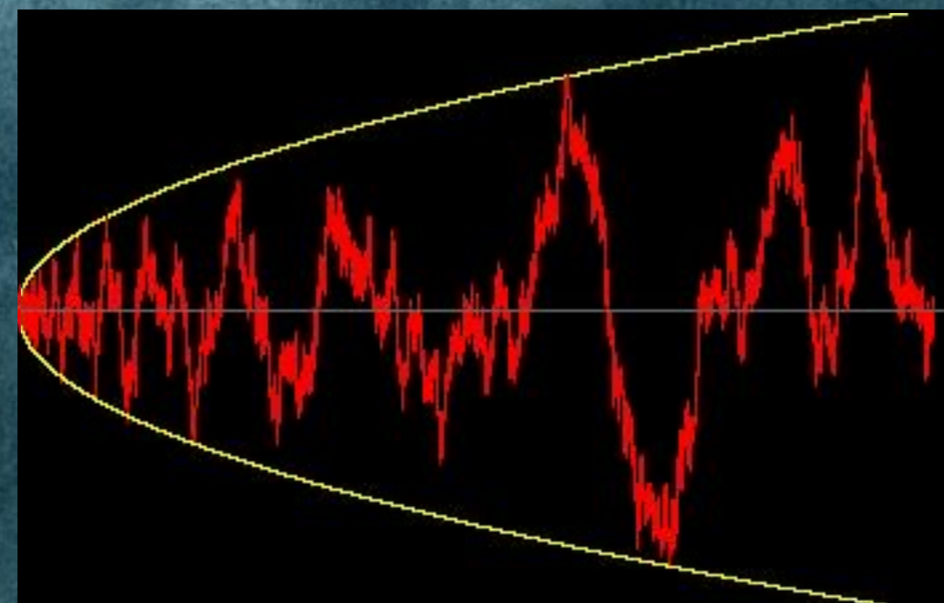
$L_1(s)$

$$\mathbb{R}(s) = \frac{1}{2}$$

$L_2(s)$
s
$\mathbb{R}(s) = \frac{1}{2}$

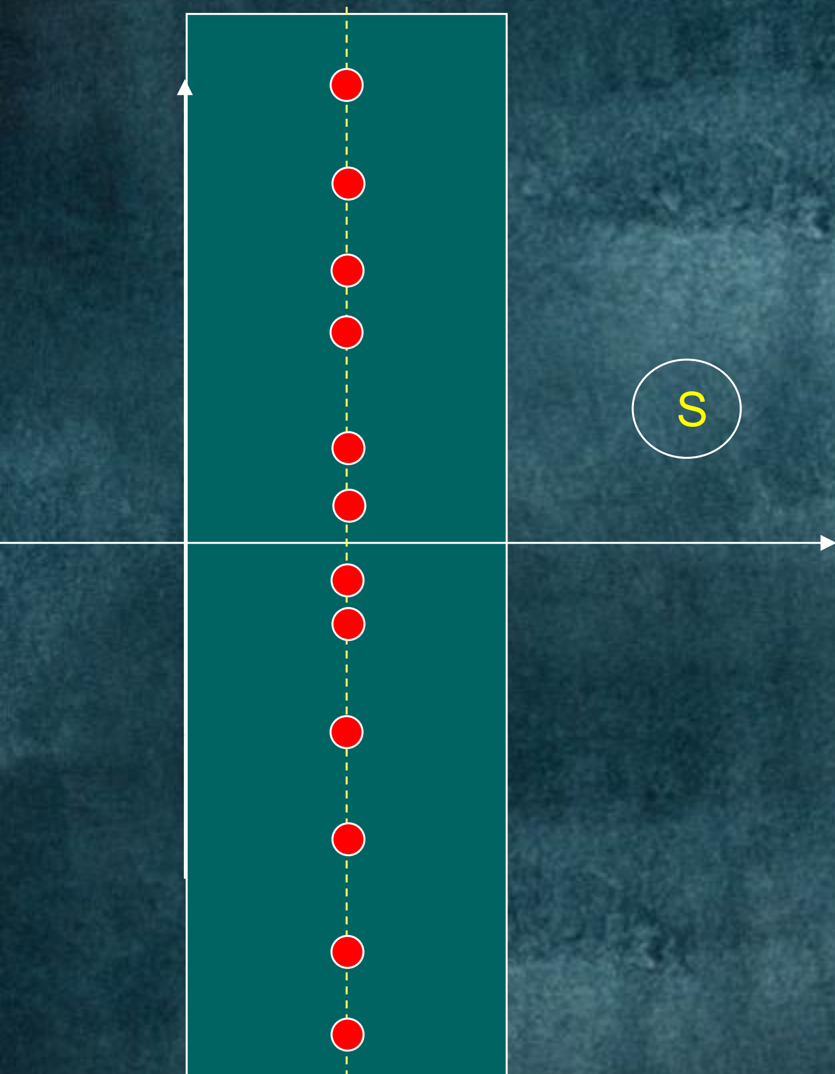

$L_3(s)$
s
$\mathbb{R}(s) = \frac{1}{2}$



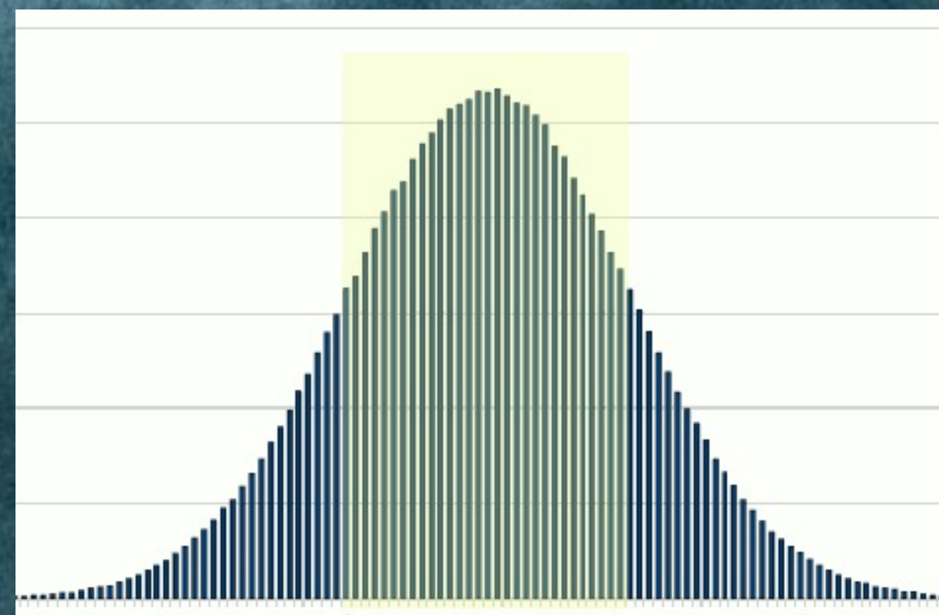


$$\mathbb{R}(s) = \frac{1}{2}$$

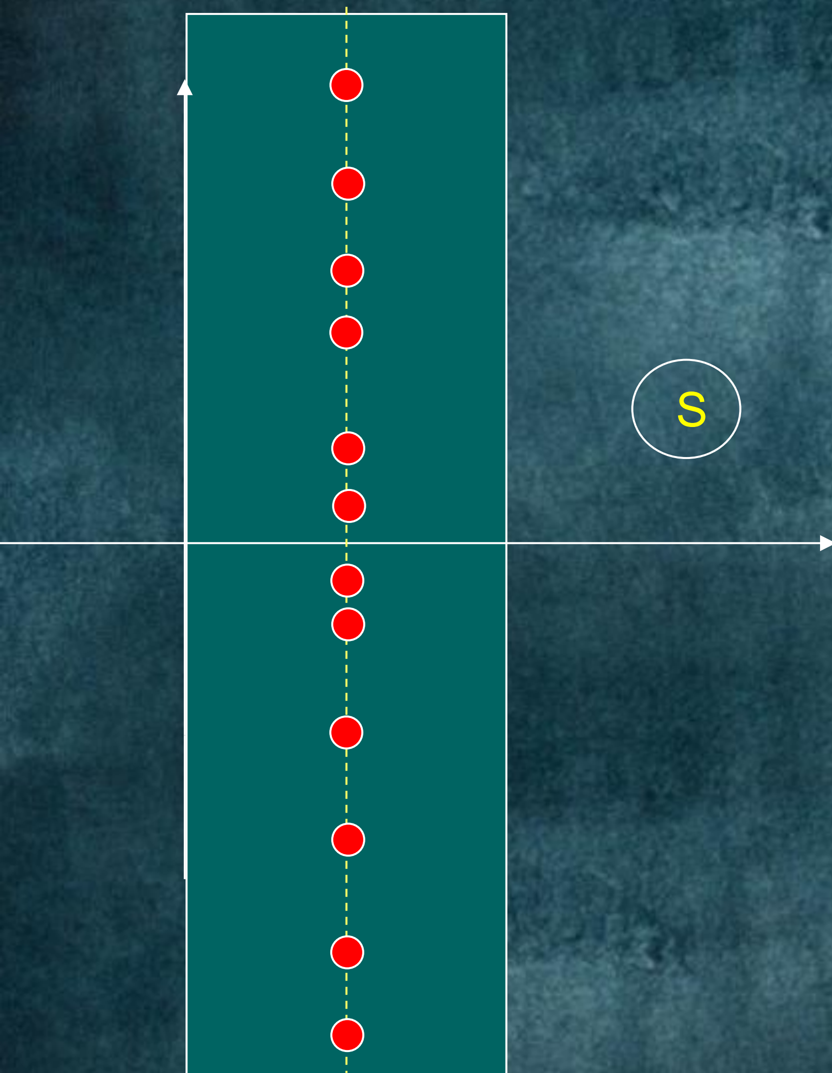
$$C_N \simeq N^{1/2}$$

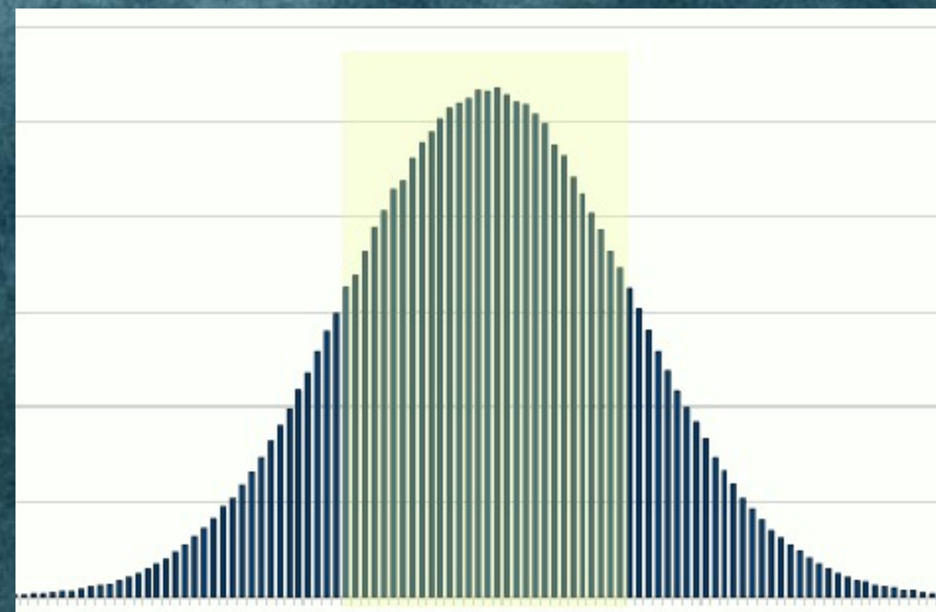


$$\mathbb{R}(s) = \frac{1}{2}$$



$$C_N \simeq N^{1/2+\epsilon}$$





$$\mathbb{R}(s) = \frac{1}{2}$$

$$C_N \simeq (N \log \log N)^{1/2}$$

Theories and Theorems

$$F = m\,a$$

Theories and Theorems

$$\partial_\mu F^{\mu\nu} = J^\nu$$

MAGNÉTISME.


1
2
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17
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SSO
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NNO
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II.

Theories and Theorems

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n}{dx^2} + V(x)\psi_n = E_n\,\psi_n$$


Chas Addams

A paradigmatic example

Square-free numbers

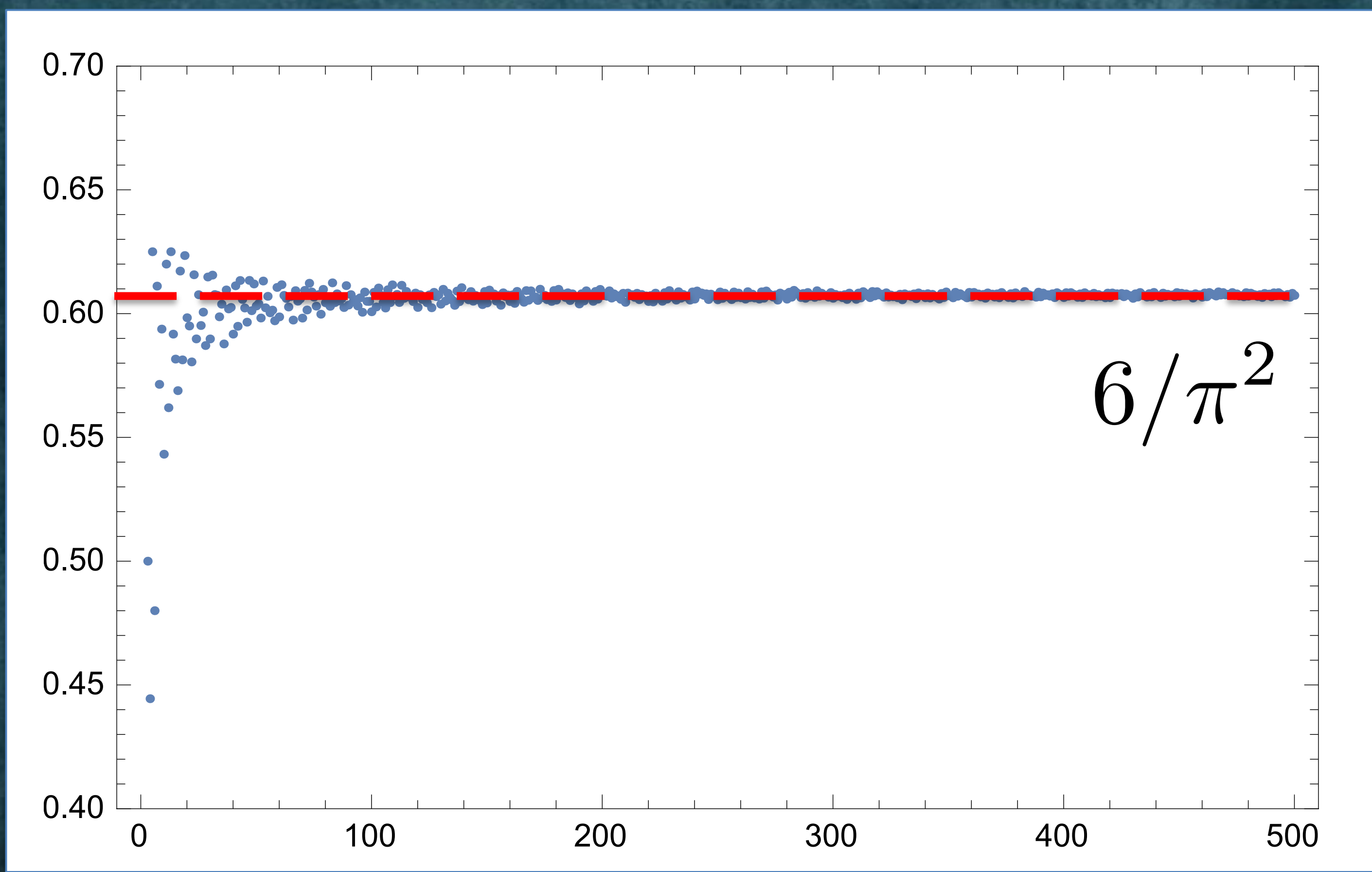
$$f_n = p_1 \cdot p_2 \cdots p_k$$

What is the density of square-free numbers among the integers?

- assuming no correlation between numbers, $1/p$ is the probability that a number x is divisible by the prime p
- hence $(1-1/p^2)$ is the probability that a number x is NOT divisible by the prime p more than one time

$$\mathcal{P} \simeq \prod_p \left(1 - \frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2} = 0.607927...$$

$1, \ , \ , 4, \ , \ , \ , 8, 9, \ , \ , 12, \ , \ , \ , 16, \ , 18, \ , \ , \ , \ , \ , 24, 25, \cdots$



Randomness of Möbius coefficents and brownian motion: growth of the Mertens function and the Riemann Hypothesis

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The validity of the Riemann Hypothesis (RH) on the location of the non-trivial zeros of the Riemann ζ-function is directly related to the growth of the Mertens function $M(x) = \sum_{k=1}^{x} \mu(k)$, where $\mu(k)$ is the Möbius coefficient of the integer k: the RH is indeed true if the Mertens function goes asymptotically as $M(x) \simeq x^{1/2+\epsilon}$. We show that this behavior can be established on the basis of a new probabilistic approach based on the *global* properties of Mertens function. To this aim, we focus the attention on the square-free numbers and we derive a series of probabilistic results concerning the prime number distribution along the series of square-free numbers, the average number of prime divisors, the Erdős-Kac theorem for square-free numbers, etc. These results lead us to the conclusion that the Mertens function is subject to a normal distribution as much as any other random walk, therefore with an asymptotic behaviour given by $x^{1/2+\epsilon}$. This represents a theoretical advance in the field. We also argue how the Riemann Hypothesis implies the Generalised Riemann Hypothesis for the Dirichlet L-functions. Next we study the *local* properties of the Mertens function dictated by the Möbius coefficients restricted to the square-free numbers. Motivated by the natural curiosity to see how close to a purely random walk is any sub-sequence extracted by the sequence of the Möbius coefficients for the square-free numbers, we perform a massive statistical analysis on these coefficients, applying to them a series of randomness tests of increasing precision and complexity: together with several frequency tests within a block, the list of our tests include those for the longest run of ones in a block, the binary matrix rank test, the Discrete Fourier Transform test, the non-overlapping template matching test, the entropy test, the cumulative sum test, the random excursion tests, etc. The successful outputs of all these tests (with a level of confidence of 99% that all the sub-sequences analyzed are indeed random) can be seen as impressive "experimental" confirmations of the brownian nature of the restricted Möbius coefficients and the probabilistic normal law distribution of the Mertens function analytically established earlier. In view of the theoretical probabilistic argument and the large battery of statistical tests, we can conclude that while a violation of the RH is strictly speaking not impossible, it is however ridiculously improbable.

Topics of the seminar

- Randomness vs Determinism
- Riemann's Hypothesis
- Dirichlet characters, $\mathcal{L}$-functions and all that
- The Generalised Riemann Hypothesis
- For Whom the Bell Tolls

Randomness vs Determinism

Number Theory is usually considered as a field rigidily ruled by the “military” deterministic laws of arithmetic, where randomness seems to play no role.

But, is it really so??

I would like to undermine this certainty of yours

Two examples

- Arithmetic tales from the world of π

- The strange realm of large numbers

Law and (dis)order in π

$$\pi = \cfrac{4}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \cdots}}}}}$$

Law and (dis)order in π

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2} \cdots$$

Law and (dis)order in π

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \arctan\left(\frac{1}{F_{2k+1}}\right)$$

F_k is the k-th Fibonacci number

3.141592653589793238462643383279502
88419716939937510582097494459230781
64062862089986280348253421170679821
48086513282306647093844609550582231
72535940812848111745028410270193852
11055596446229489549303819644288109
75665933446128475648233786783165271
20190914564856692346034861045432664
82133936072602491412737245870066063
15588174881520920962829254091715364
36789259036001133053054882046652138
41469519415116094330572703657595919
53092186117381932611793105118548074
46237996274956735188575272489122793

Law and (dis)order in π

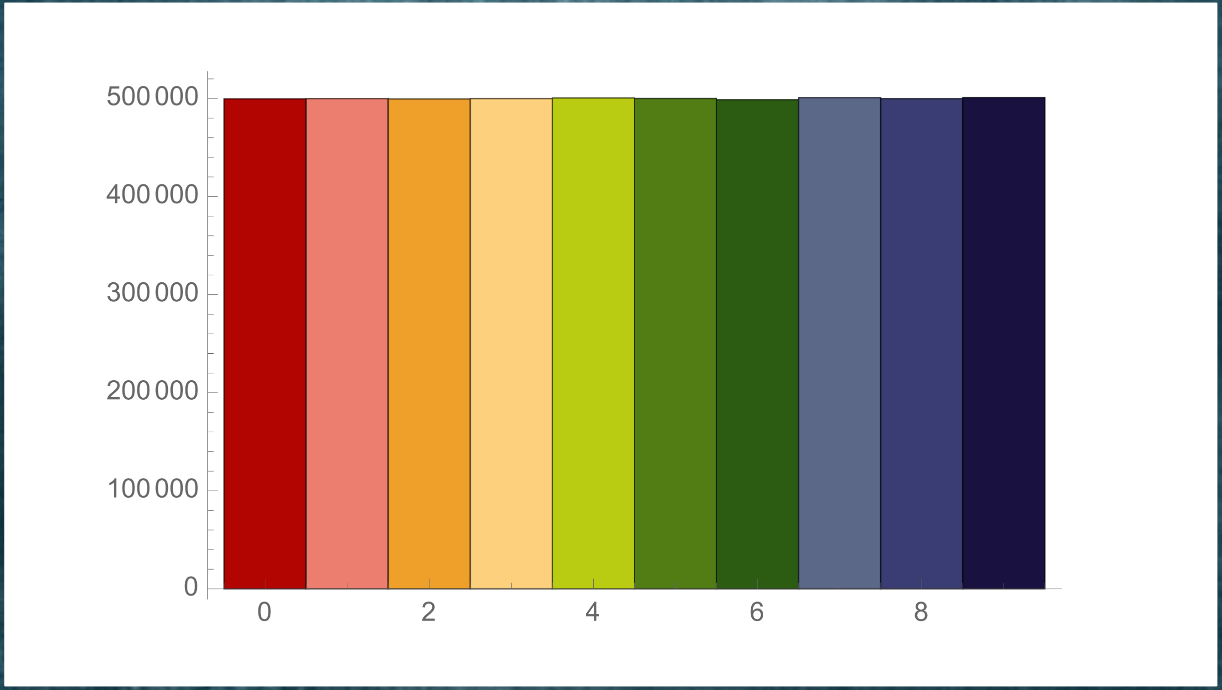
10^d

d →

n ↓

n \ d	2	3	4	5	6	7	8	9	10	11	12
0	8	93	968	9999	99959	999440	9999922	99993942	999967995	10000104750	99999485134
1	8	116	1026	10137	99758	999333	10002475	99997334	1000037790	9999937631	99999945664
2	12	103	1021	9908	100026	1000306	10001092	100002410	1000017271	10000026432	100000480057
3	11	102	974	10025	100229	999964	9998442	99986911	999976483	9999912396	99999787805
4	10	93	1012	9971	100230	1001093	10003863	100011958	999937688	10000032702	100000357857
5	8	97	1046	10026	100359	1000466	9993478	99998885	1000007928	9999963661	99999671008
6	9	94	1021	10029	99548	999337	9999417	100010387	999985731	9999824088	99999807503
7	8	95	970	10025	99800	1000207	9999610	99996061	1000041330	10000084530	99999818723
8	12	101	948	9978	99985	999814	10002180	100001839	999991772	10000157175	100000791469
9	14	106	1014	9902	100106	1000040	9999521	100000273	1000036012	9999956635	99999854780

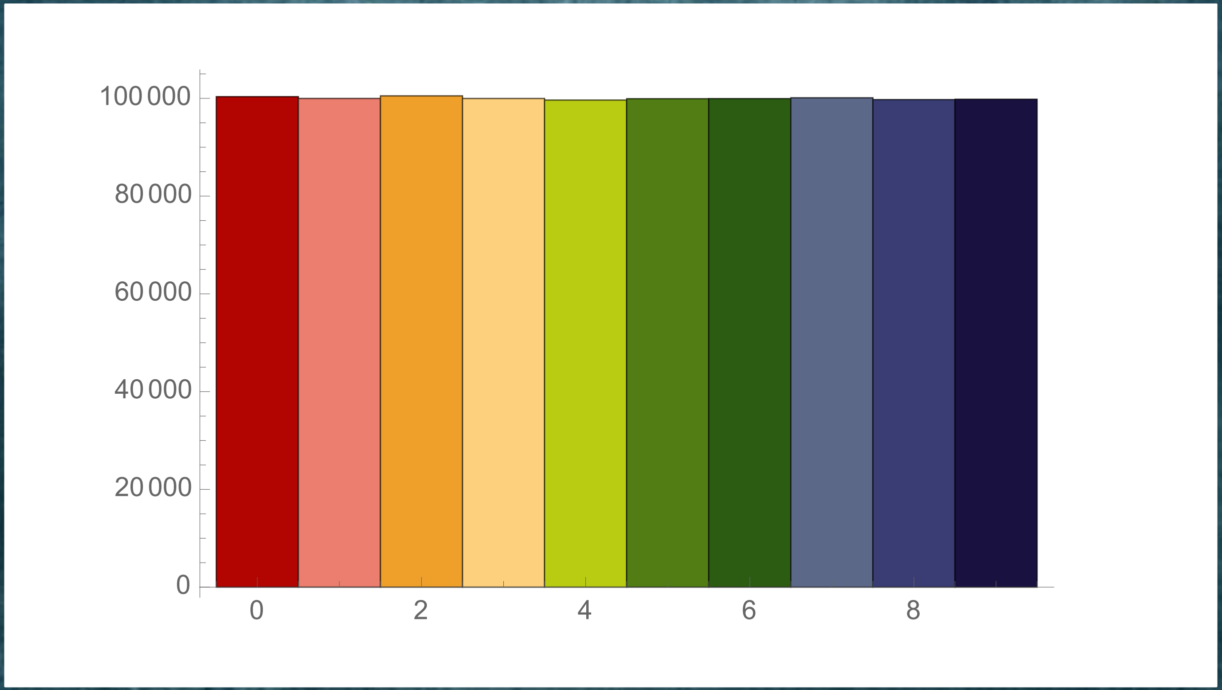
Law and (dis)order in π



First 5.000.000 digits

3.1415[redacted]26535[redacted]97932[redacted]84626[redacted]33832[redacted]9502
8[redacted]41971[redacted]93993[redacted]51058[redacted]09749[redacted]45923[redacted]781
64[redacted]62862[redacted]89986[redacted]80348[redacted]53421[redacted]70679[redacted]21
480[redacted]6513282306647093844609550582231
7253594081284811174502841027019385
2
1105559644622948954930381964428810
9
7566593344612847564823378678316527
1
2019091456485669234603486104543266
4
8213393607260249141273724587006606
3
1558817488152092096282925409171536
4
3678925903600113305305488204665213
8
4146951941511609433057270365759591
9
5309218611738193261179310511854807
4
4623799627495673518857527248912279
3

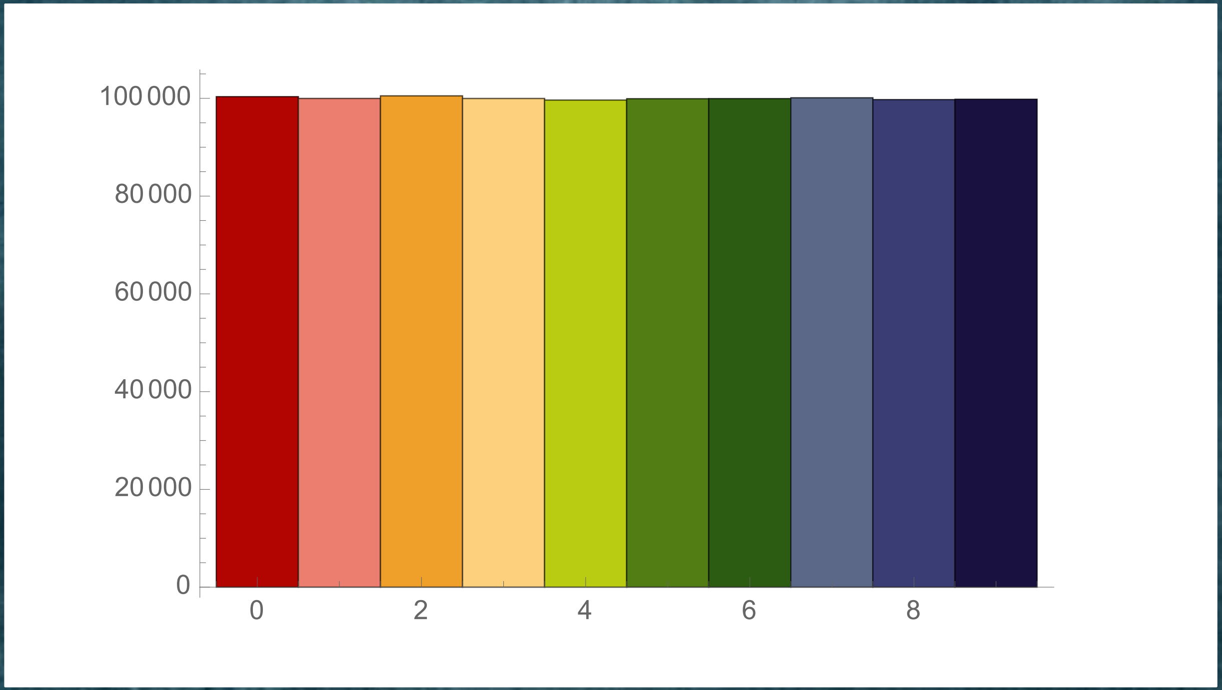
Law and (dis)order in π



First 60.000.000 digits

3.1415[redacted]26535[redacted]9793238462[redacted]4338[redacted]279502
884197169399375105[redacted]20974944592[redacted]0781
640628[redacted]2089986280348253421170679821
480865132823066470938446 09[redacted]50[redacted]82231
72[redacted]35940812848111745028410270193852
110555964462294895493038196442881[redacted]9
7566593344612[redacted]4756482337867831652 71
2019091456485669[redacted]3460348610454326 64
8213393607260249141273 72[redacted]5870066063
155[redacted]8174881520920962829254091715364
3678925903600113305305488204665[redacted]138
4146951941511[redacted]09433057270365759591 9
5309218611738193261179310511[redacted]548074
46237996274956735188575272489122793

Law and (dis)order in π



First 60.000.000 digits

Displacement function of the digits of π

$$\overline{d} = 4.5$$

Displacement function of the digits of π

$$S(n) = \sum_{i=1}^{n} (d_i - \overline{d})$$

$$\overline{d} = 4.5$$

How is the S(n) distributed?

20.000 digits

The strange realm of large numbers

- Number of distict prime divisors

$$n = p_1^{a_1} \cdot p_2^{a_2} \cdot p_3^{a_3} \cdots p_k^{a_k}$$

$$\omega(n) = k$$

What is the average of ω(n) and how ω(n) is distributed?

The strange realm of large numbers

$$n \sim 10^{60}$$

$$\omega = \{\ldots, 6, 5, 6, 5, 1, 6, 7, 2, 7, 5, 3, 3, 2, 6, 3, 6, 4, 4, 3, 2, 2, 5, 5, 5, 2, 5, 4, \ldots\}$$

The strange realm of large numbers

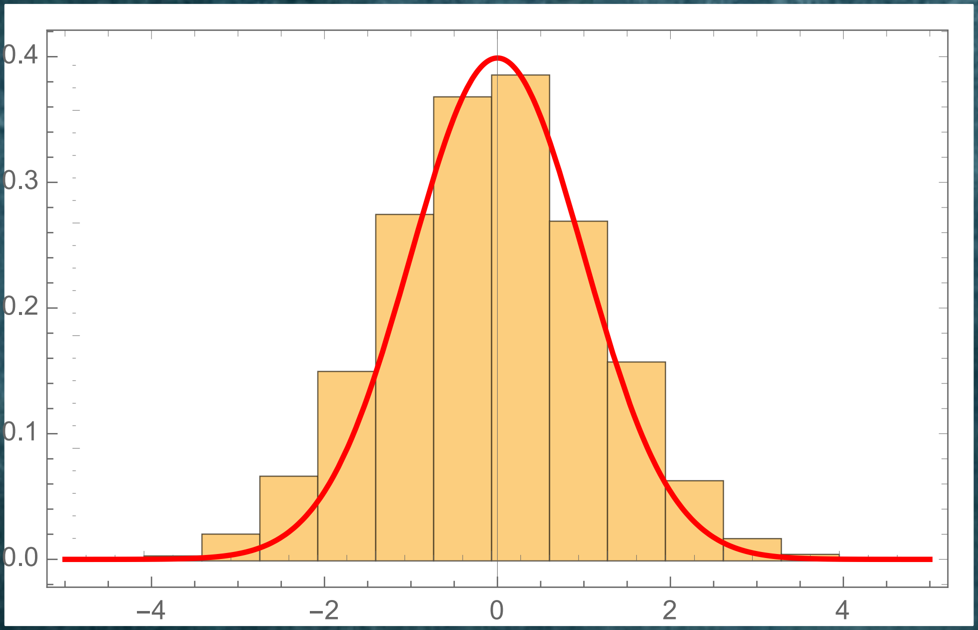
$$\overline{\omega} \simeq \log \log n$$

So, the factorization of integer numbers

of order 10^{1000}

have in average, only 7 different primes!!

The strange realm of large numbers

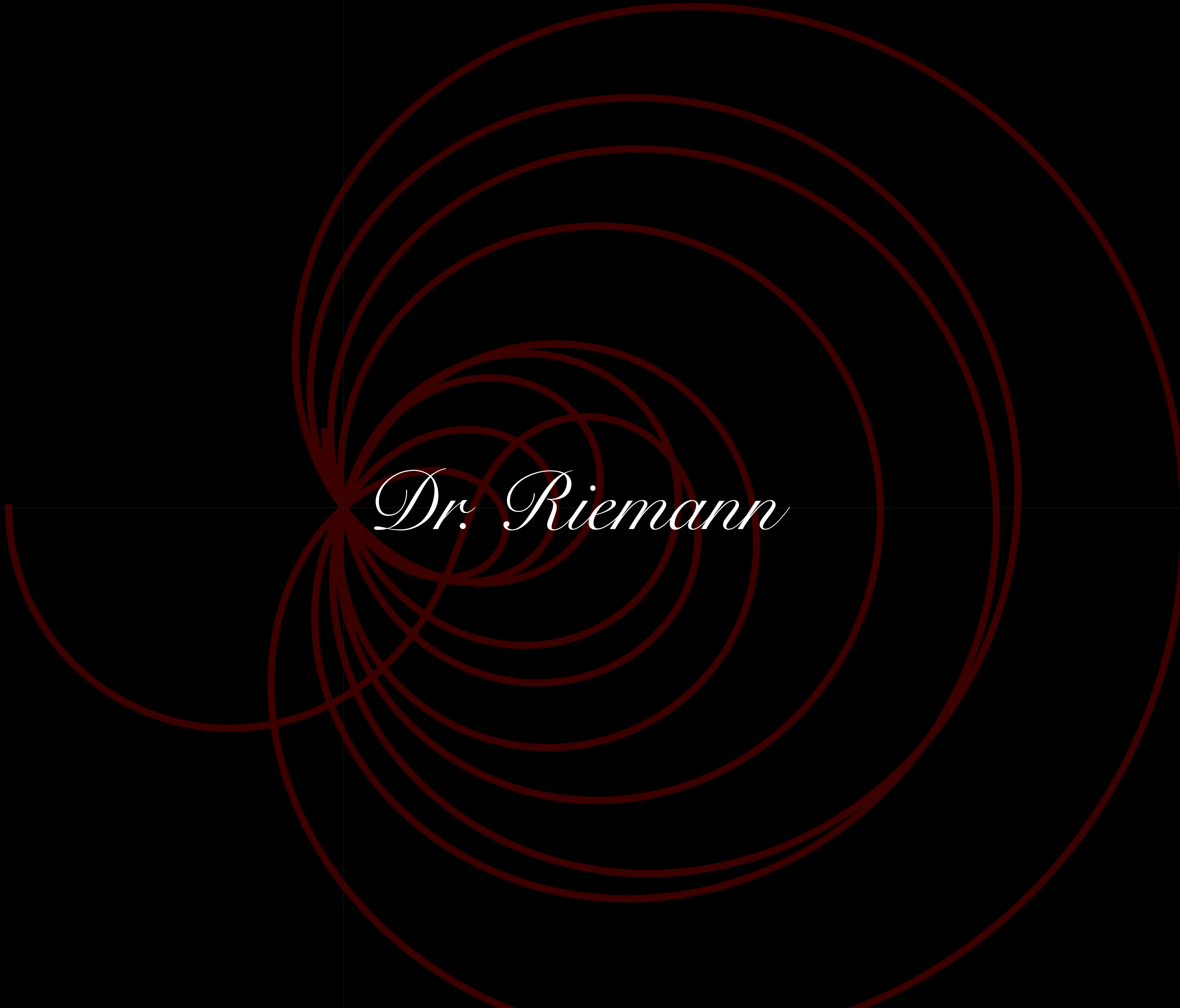


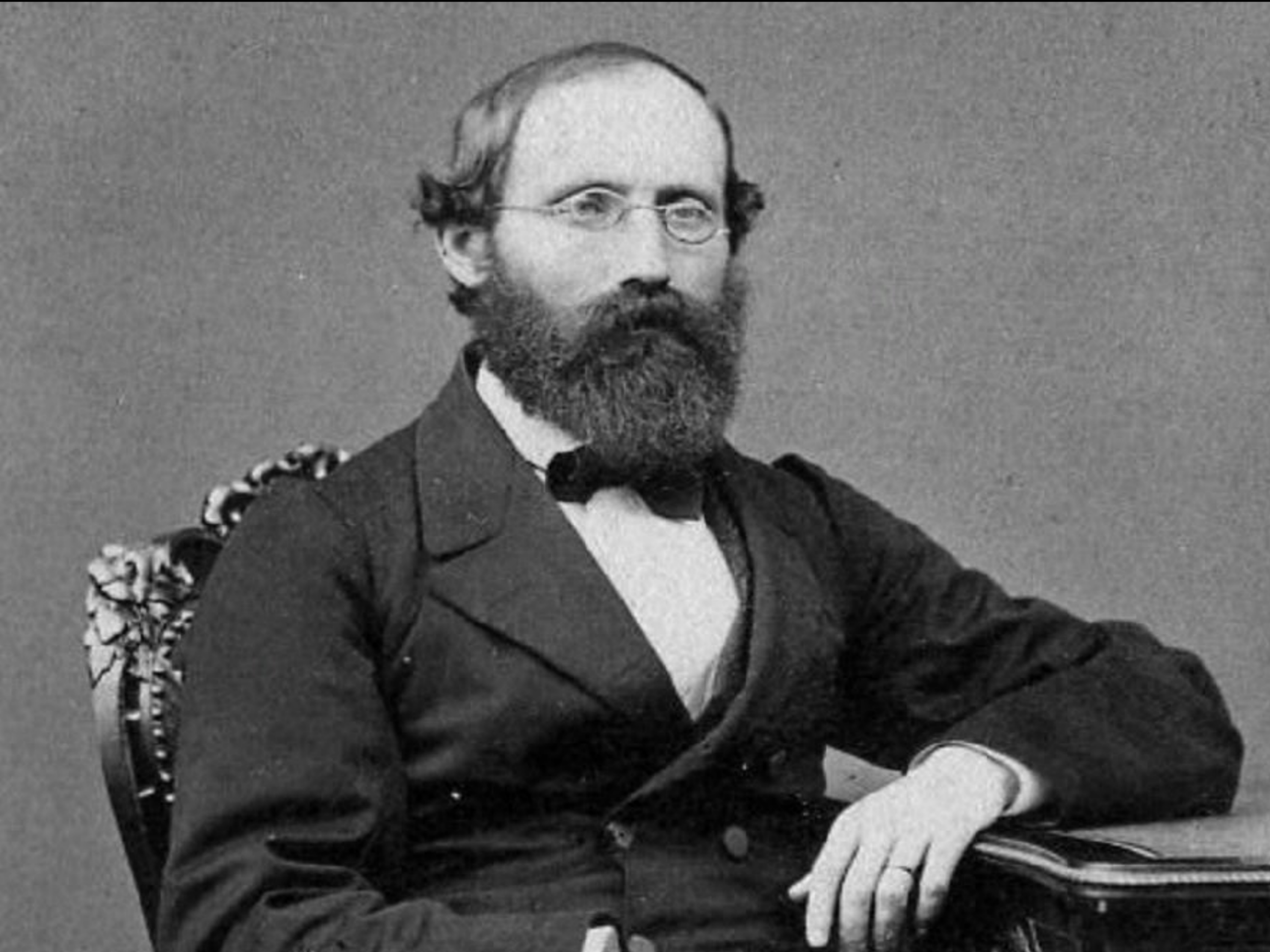
For large n, the variable

$$z = \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}}$$

is normal distributed!

(Erdos-Kac theorem)


Dr. Riemann



Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse.

(Berliner Monatsberichte, 1859. November)

Meinen Dank für die Auszeichnung, welche mir die Akademie durch die Aufnahme unter ihre Correspondenten hat zu Theil werden lassen, glaube ich am besten dadurch zu erkennen zu geben, dass ich von der hierdurch erhaltenen Erlaubniss baldigst Gebrauch mache durch Mittheilung einer Untersuchung über die Häufigkeit der Primzahlen; ein Gegenstand, welcher durch das Interesse, welches Gauss und Dirichlet demselben längere Zeit geschenkt haben, einer solchen Mittheilung vielleicht nicht ganz unwerth erscheint.

Bei dieser Untersuchung diente mir als Ausgangspunkt die von Euler gemachte Bemerkung, dass das Product

$$\prod \frac{1}{1-\frac{1}{p^s}} = \sum \frac{1}{n^s},$$

wenn für p alle Primzahlen, für n alle ganzen Zahlen gesetzt werden. Die Function der complexen Veränderlichen s, welche durch diese beiden Ausdrücke, so lange sie convergiren, dargestellt wird, bezeichne ich durch $\zeta(s)$. Beide convergiren nur, so lange der reelle Theil von s grösser als 1 ist; es lässt sich indess leicht ein immer gültig bleibender Ausdruck der Function finden. Durch Anwendung der Gleichung

$$\int_0^\infty e^{-nx} x^{s-1} dx = \frac{\Pi(s-1)}{n^s}$$

erhält man zunächst

$$\Pi(s-1)\cdot\zeta(s) = \int_0^\infty \frac{x^{s-1}dx}{e^x-1}.$$

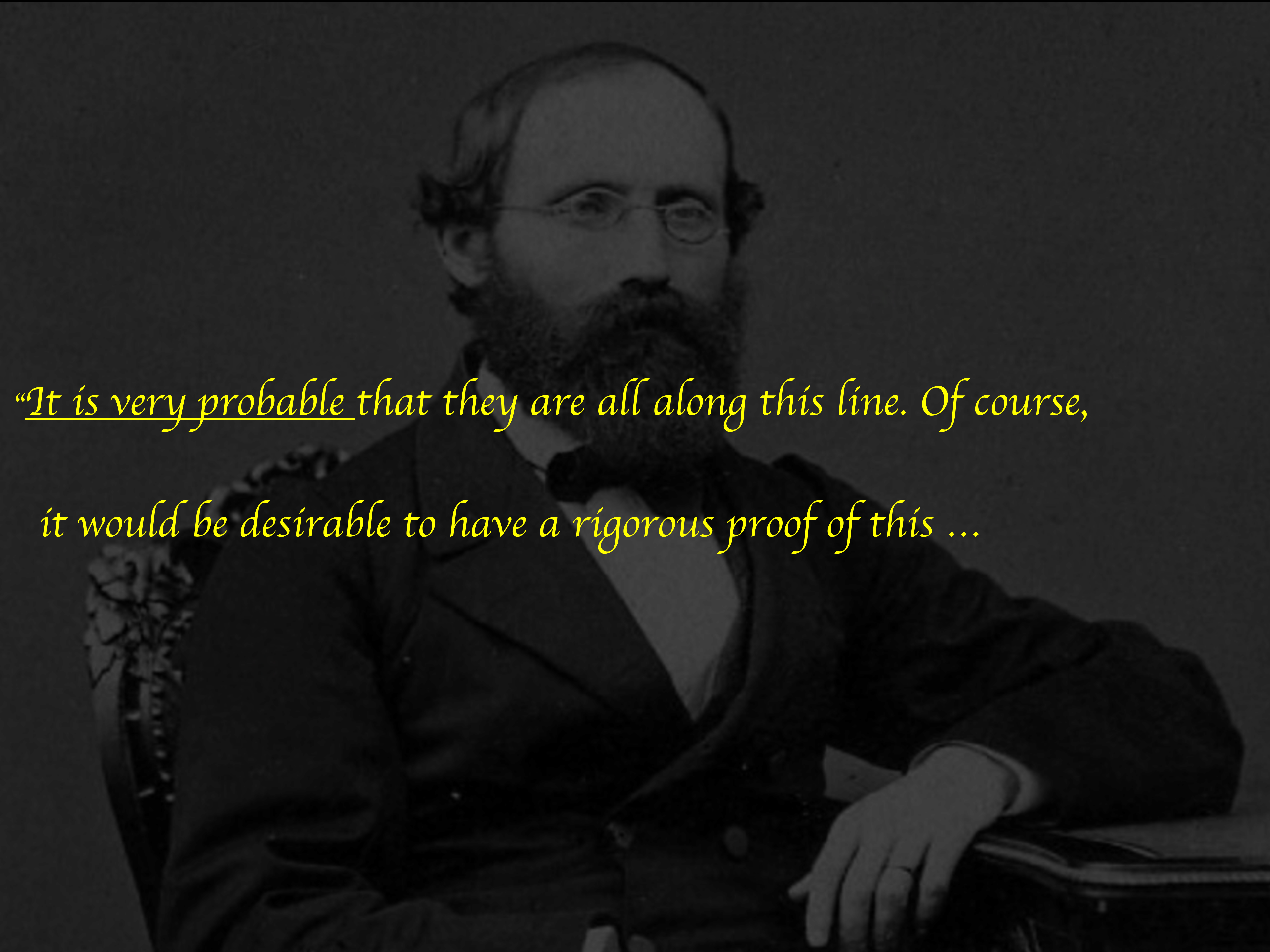
Betrachtet man nun das Integral

$$\int \frac{(-x)^{s-1}dx}{e^x-1}$$

von $+\infty$ bis $+\infty$ positiv um ein Grössengebiet erstreckt, welches den Werth 0, aber keinen andern Unstetigkeitswerth der Function unter dem Integralzeichen im Innern enthält, so ergiebt sich dieses leicht gleich

$$(e^{-\pi si} - e^{\pi si})\int_0^\infty \frac{x^{s-1}dx}{e^x-1},$$

vorausgesetzt, dass in der vieldeutigen Function $(-x)^{s-1} = e^{(s-1)\log(-x)}$ der Logarithmus von $-x$ so bestimmt worden ist, dass er für ein negatives x reell wird. Man

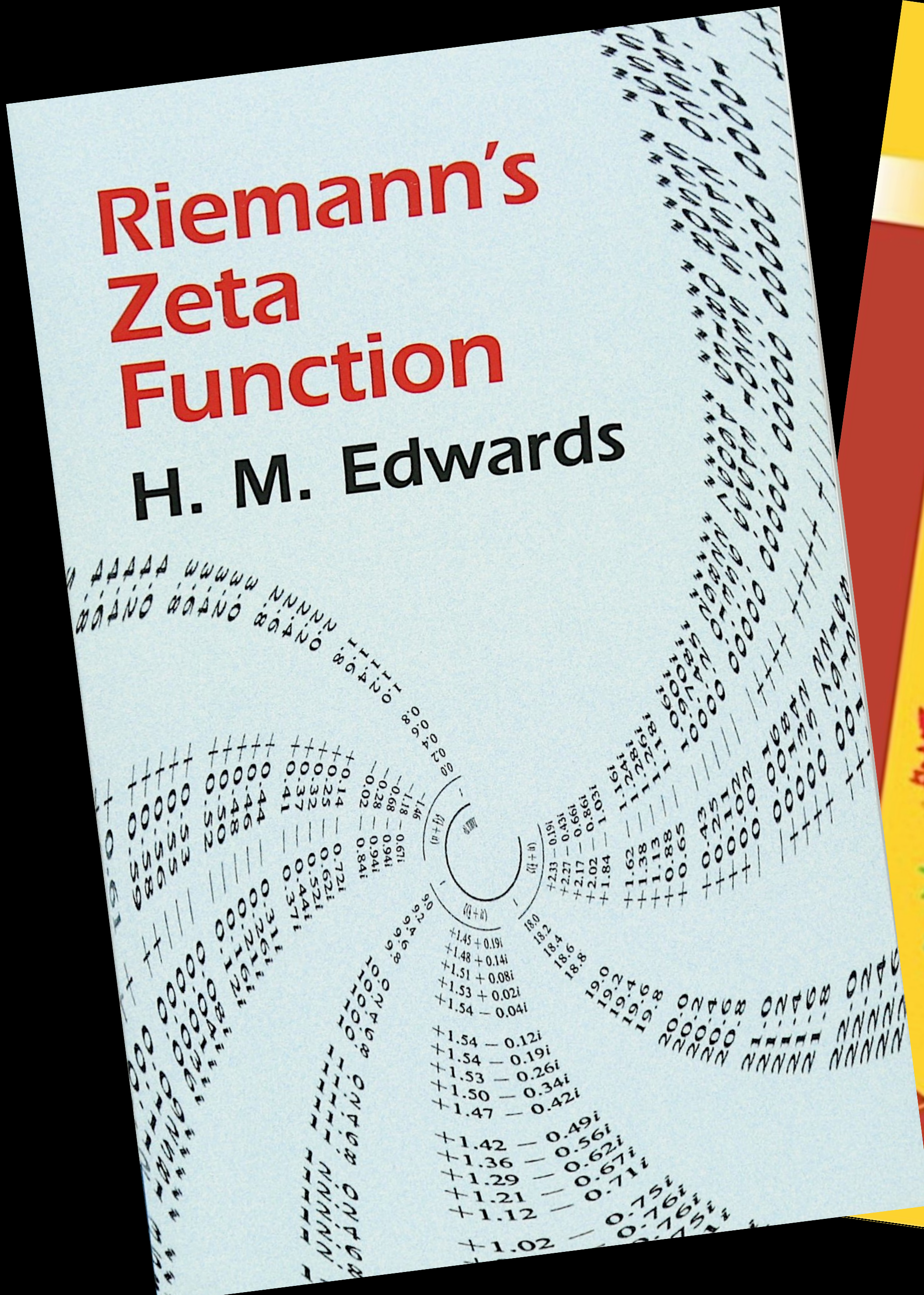


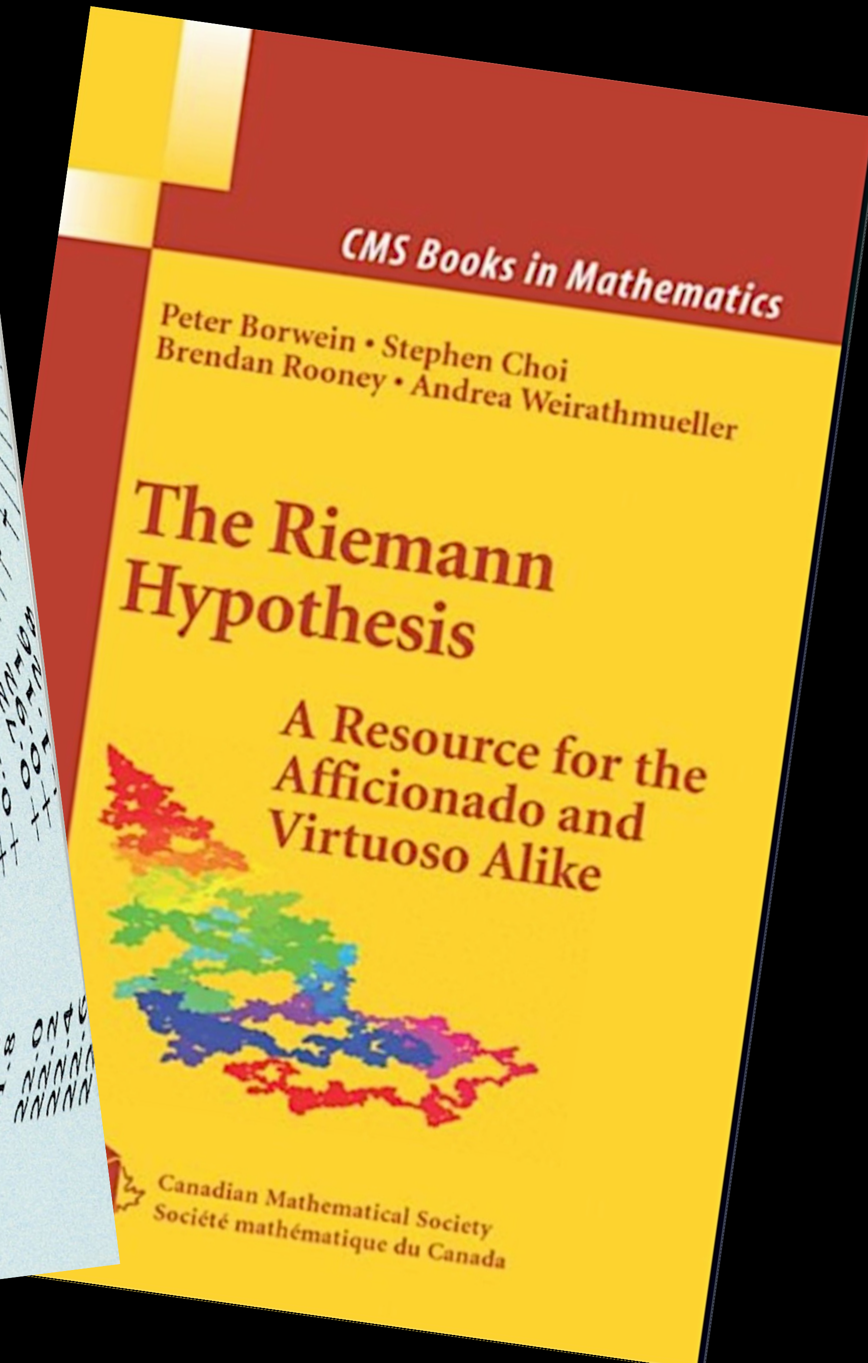
"It is very probable that they are all along this line. Of course,

it would be desirable to have a rigorous proof of this …

Innumerable consequences of the Riemann Hypothesis

- Distribution of the primes (large gap, etc.)
- Growth of some arithmetic functions
- Zoo of zeta functions
- Random Matrices and Quantum Chaos
- General Theory of Phase Transitions (Fisher’s zeros, etc.)


Riemann's
Zeta
Function
H. M. Edwards

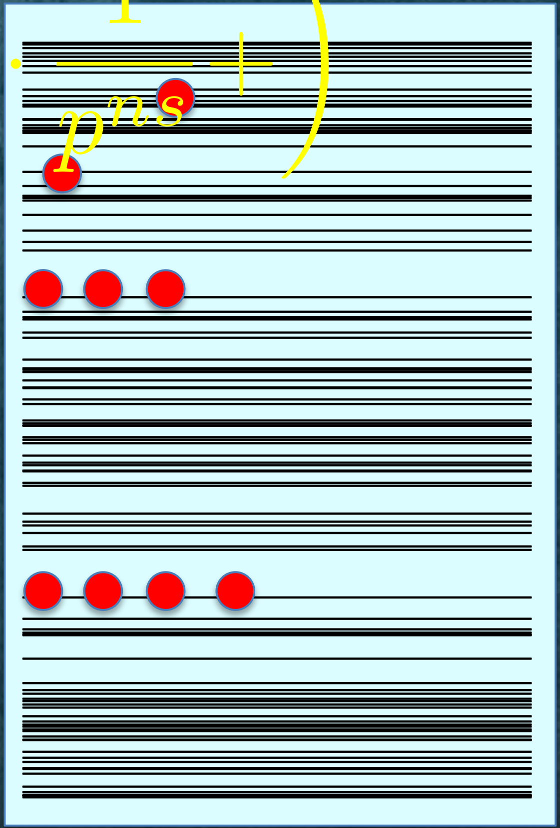

CMS Books in Mathematics
Peter Borwein • Stephen Choi
Brendan Rooney • Andrea Weirathmueller
The Riemann
Hypothesis
A Resource for the
Afficionado and
Virtuoso Alike
Canadian Mathematical Society
Société mathématique du Canada

The Riemann zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}} \qquad Re(s) > 1$$

$$= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots \frac{1}{p^{ns}} + \right)$$

Free bosonic system



$$E_p = \log p$$

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

The Riemann zeta Function

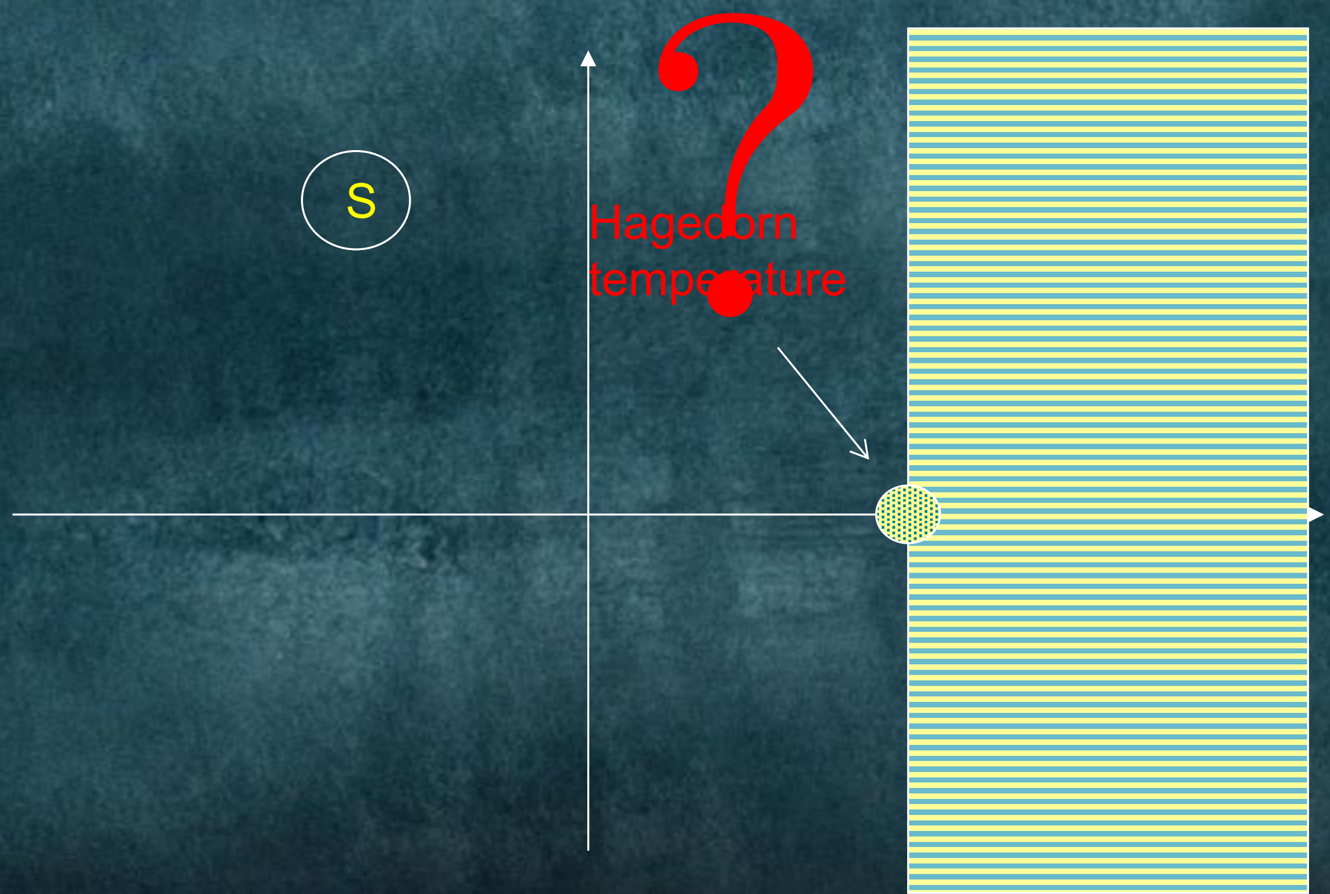
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

Grand canonical Ensemble

Micro canonical Ensemble

The Riemann zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

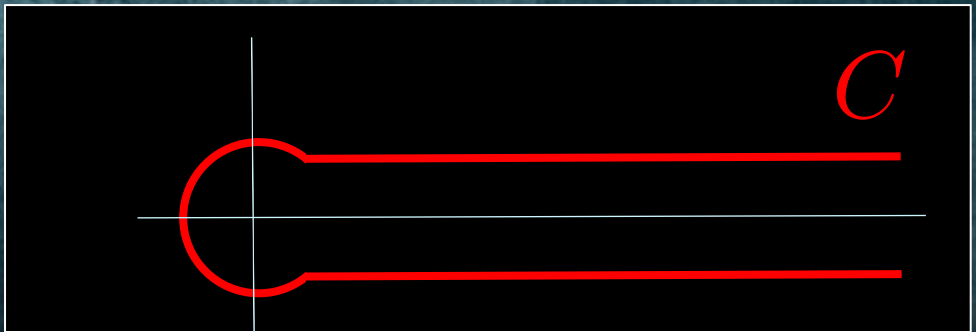


The Riemann zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$

Analytic continuation

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi i} \oint_C \frac{(-x)^{s-1}}{e^x - 1}$$



The Riemann zeta Function

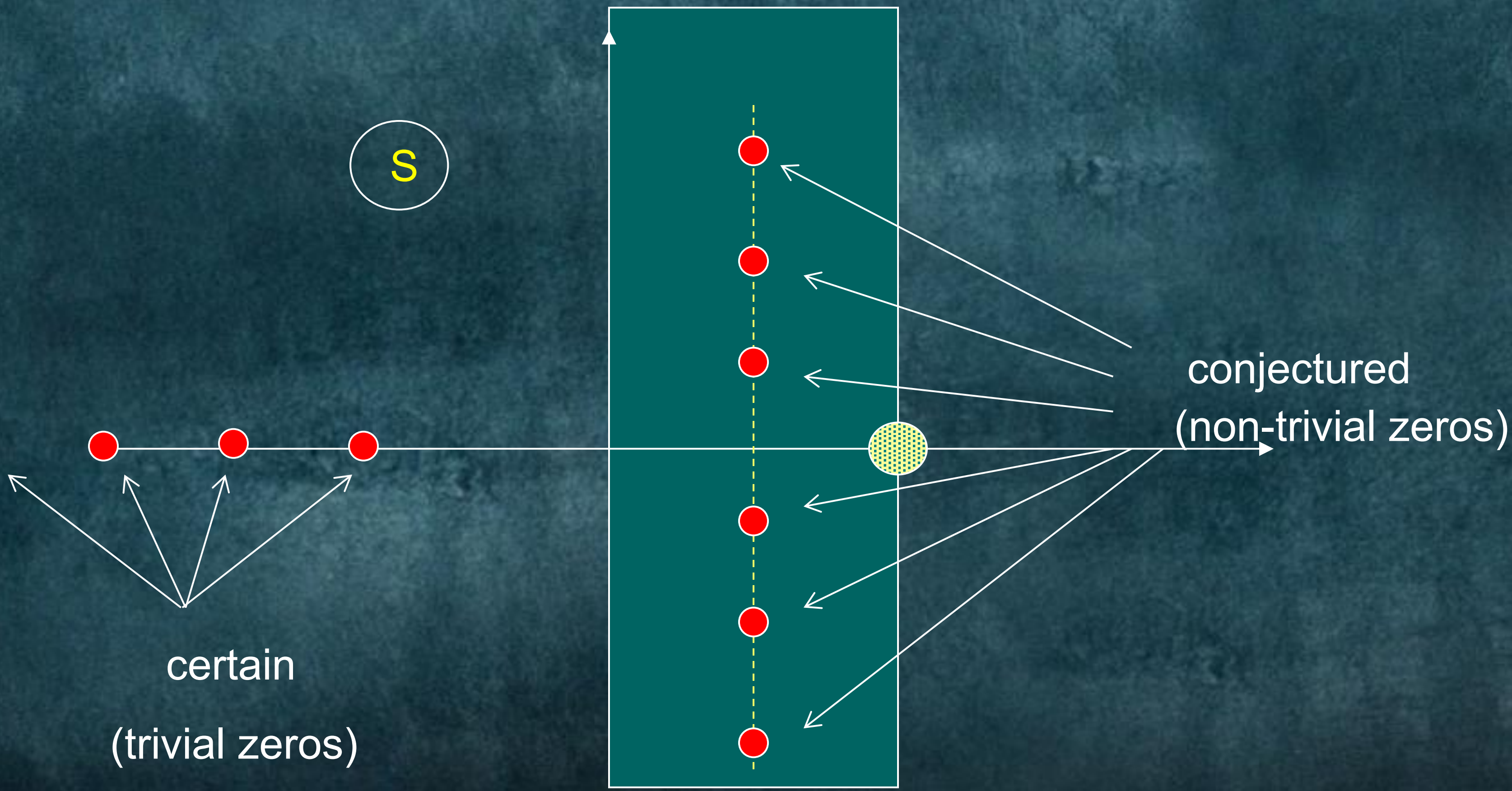
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}}$$

Functional equation

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

The Riemann zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}}$$



The Riemann zeta Function

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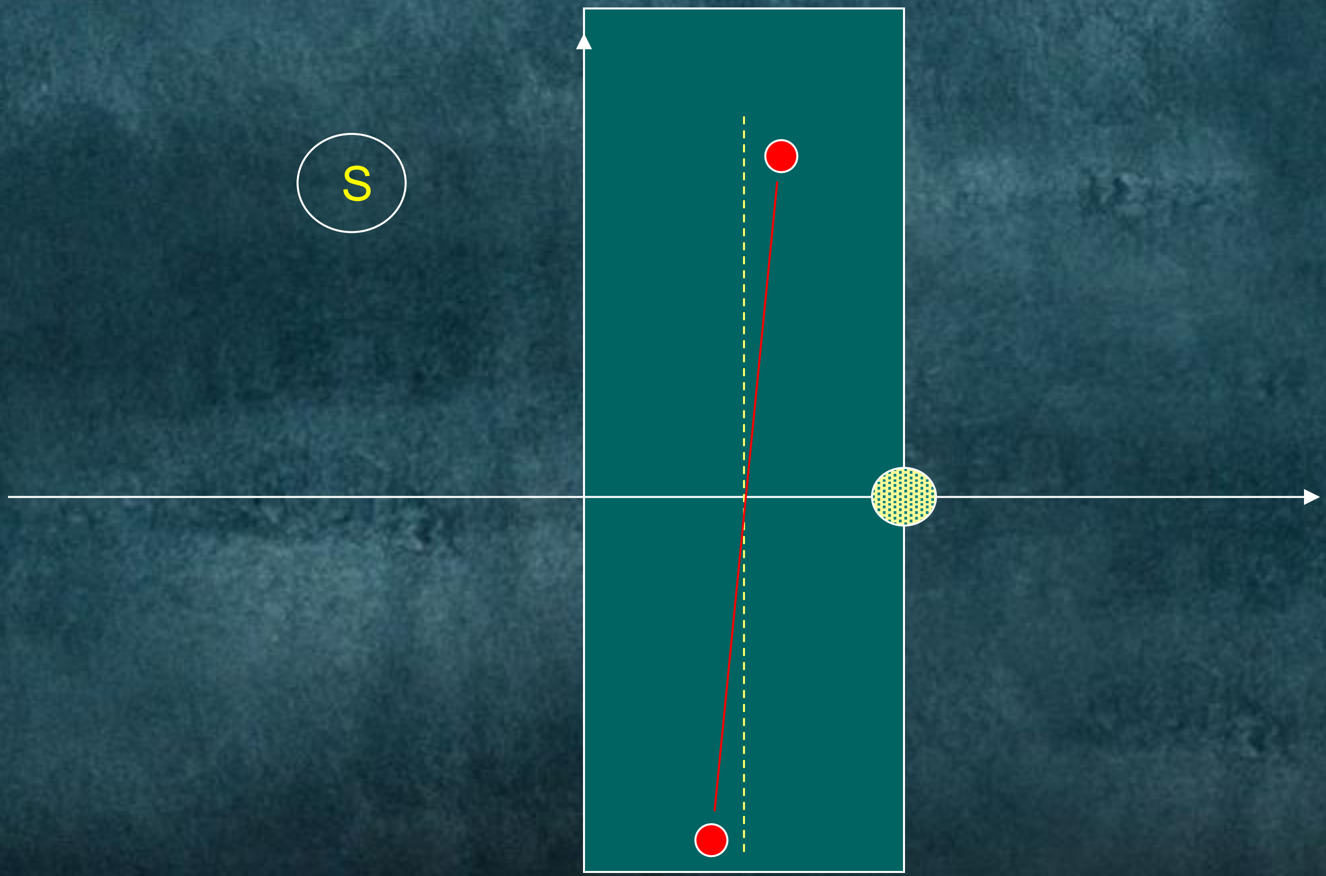

Number of zeros in the critical strip

(Riemann, von Mangoldt)

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + \frac{7}{8} + \mathcal{O}\left(\frac{1}{T}\right)$$

Number of zeros in the critical strip

(Riemann, von Mangoldt)

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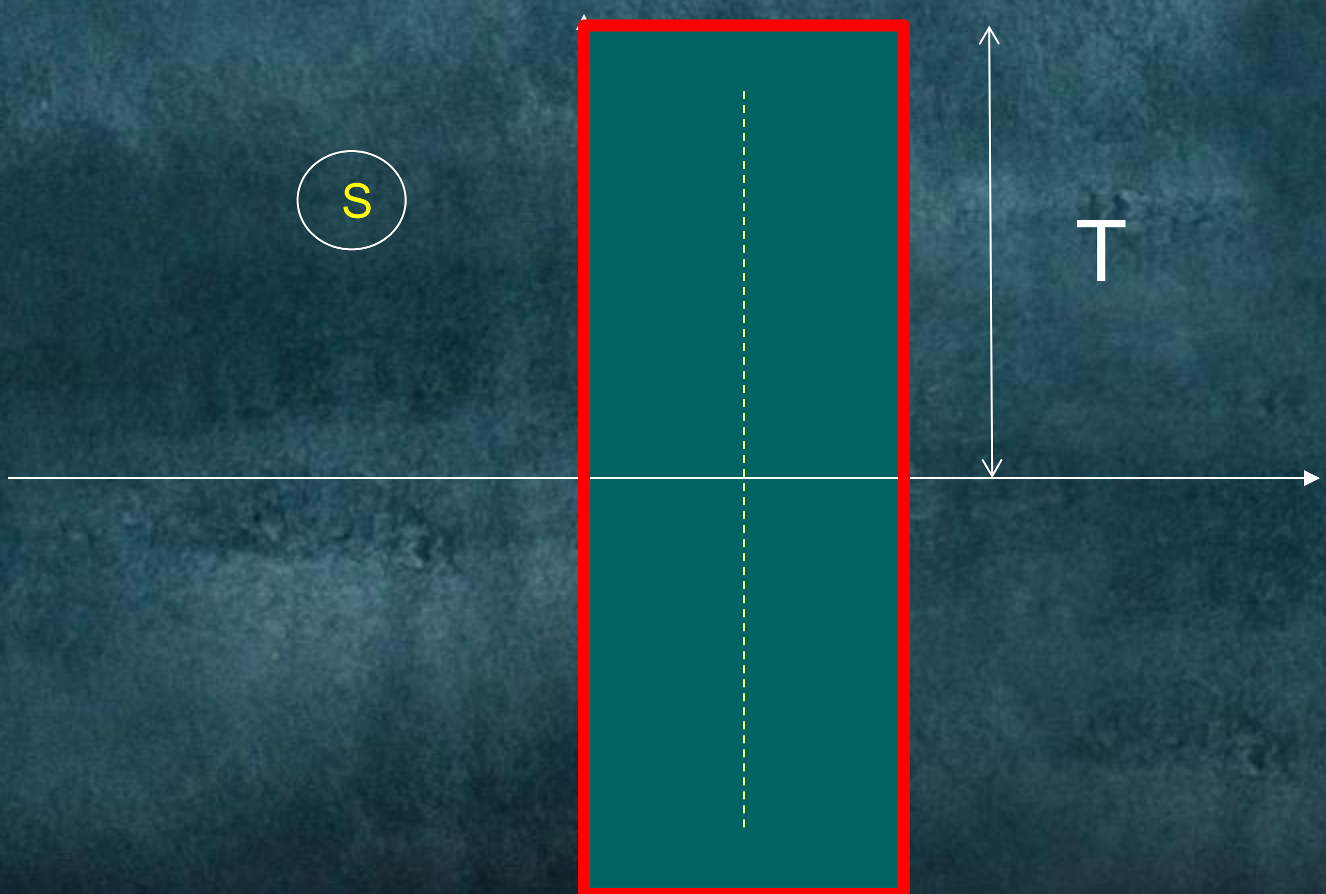
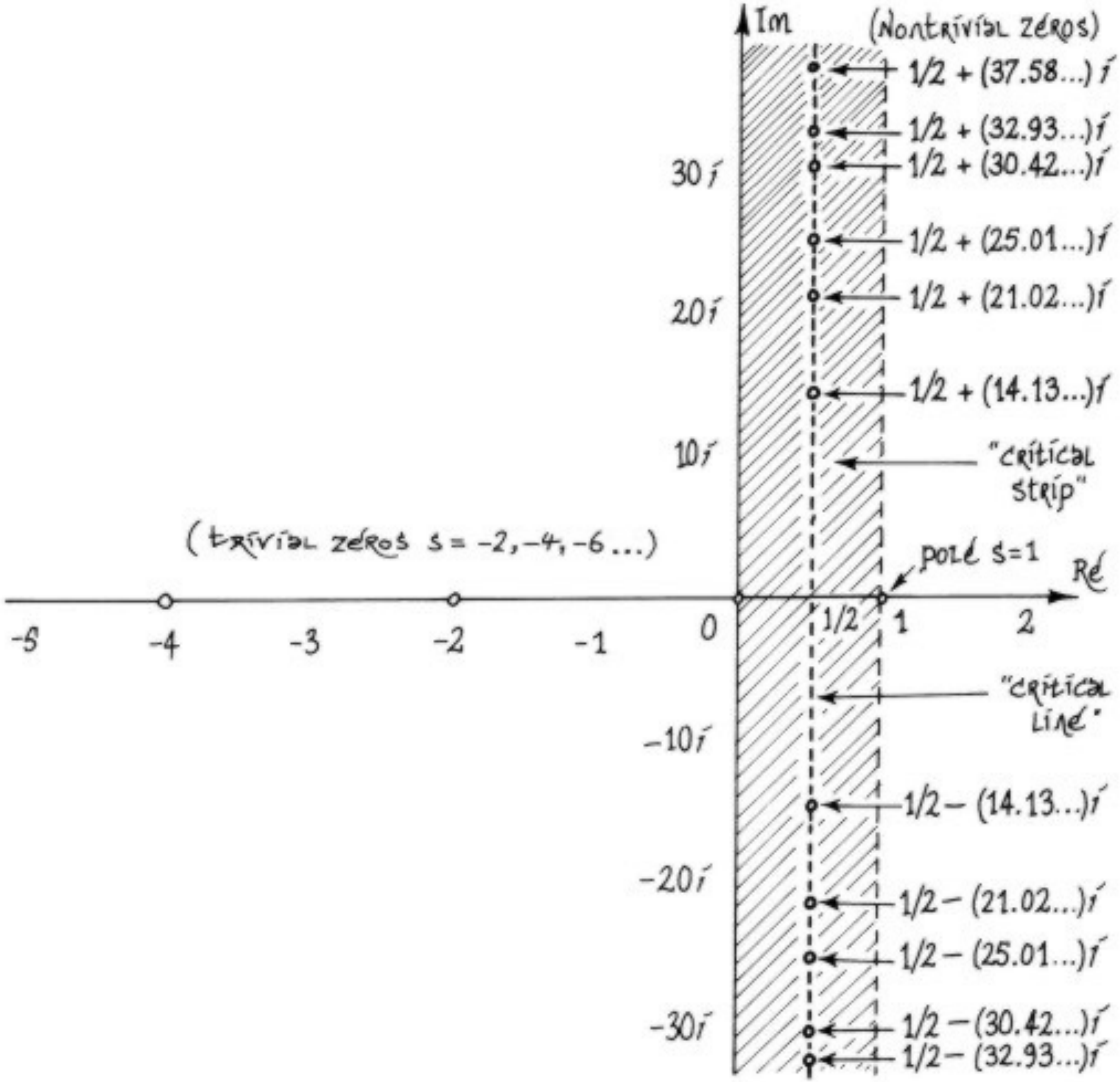


Year	Number of zeros	Computed by
1859 (approx.)	1	B. Riemann
1903	15	J. P. Gram
1914	79	R. J. Backlund
1925	138	J. I. Hutchinson
1935	1,041	E. C. Titchmarsh
1953	1,104	A. M. Turing
1956	15,000	D. H. Lehmer
1956	25,000	D. H. Lehmer
1958	35,337	N. A. Meller
1966	250,000	R. S. Lehman
1968	3,500,000	J. B. Rosser, et al.
1977	40,000,000	R. P. Brent
1979	81,000,001	R. P. Brent
1982	200,000,001	R. P. Brent, et al.
1983	300,000,001	J. van de Lune, H. J. J. te Riele
1986	1,500,000,001	J. van de Lune, et al.
2001	10,000,000,000	J. van de Lune (unpublished)
2004	900,000,000,000	S. Wedeniwski
2004	10,000,000,000,000	X. Gourdon



Im
(Nontrivial zeros)
1/2 + (37.58...) i
1/2 + (32.93...) i
1/2 + (30.42...) i
1/2 + (25.01...) i
1/2 + (21.02...) i
1/2 + (14.13...) i
30 i
20 i
10 i
"critical strip"
(trivial zeros s = −2, −4, −6 ...)
pole s=1
Re
−5
−4
−3
−2
−1
0
1/2
1
2
"critical line"
−10 i
−20 i
−30 i
1/2 − (14.13...) i
1/2 − (21.02...) i
1/2 − (25.01...) i
1/2 − (30.42...) i
1/2 − (32.93...) i

A Glimpse on "Randomness" of the Riemann Zeta Function

Inverse of the Riemann zeta function

$$\frac{1}{\zeta(s)} = \prod_p \left(1 - \frac{1}{p^s}\right) = \left(1 - \frac{1}{2^s}\right)\left(1 - \frac{1}{3^s}\right)\left(1 - \frac{1}{5^s}\right)\cdots$$

$$= 1 - \frac{1}{2^s} - \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{(2\cdot 3)^s} - \frac{1}{7^s} + \frac{1}{(2\cdot 5)^s} + \cdots$$

$$\frac{(-1)^k}{(p_1 p_2 \cdots p_k)^s}$$

Inverse of the Riemann zeta function

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Square-free numbers

$$f_n = p_1 \cdot p_2 \cdots p_k$$

$$\hat{\mu}(n) = (-1)^k$$

The beauty of the Mellin transform

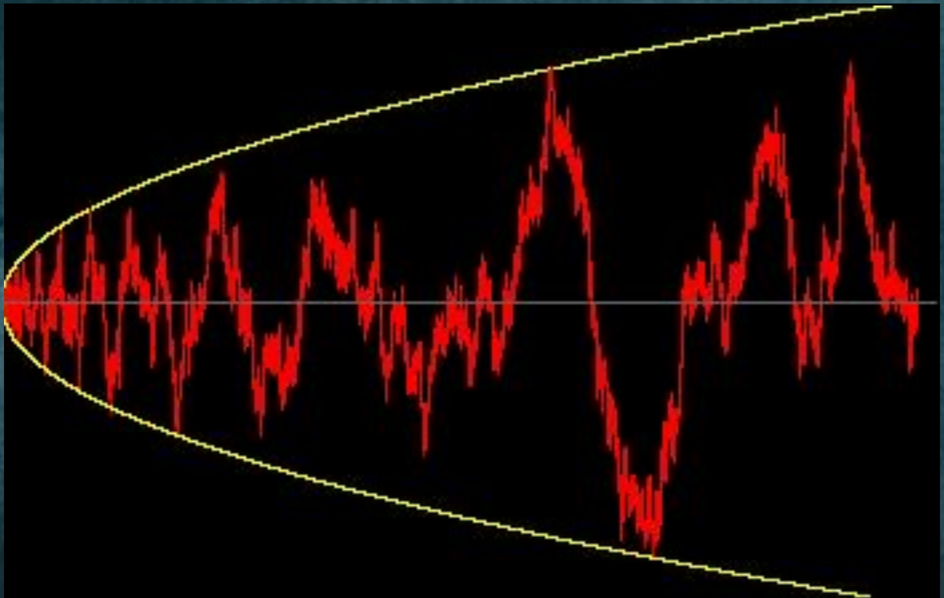
$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\hat{\mu}(n)}{f_n^s} = s \int_1^{\infty} \frac{\hat{M}(x)}{x^{s+1}}\, dx$$

$$\hat{M}(x) = \sum_{n=1}^{[x]} \hat{\mu}(n)$$

Brownian motion

$$\hat{\mu}(n) = \{-,-,-,+,-,+,-,-,+,+,-,-,+,+,-,\cdots\}$$

$$\hat{M}(x) = \sum_{n=1}^{[x]} \hat{\mu}(n)$$



Location of the first singularity

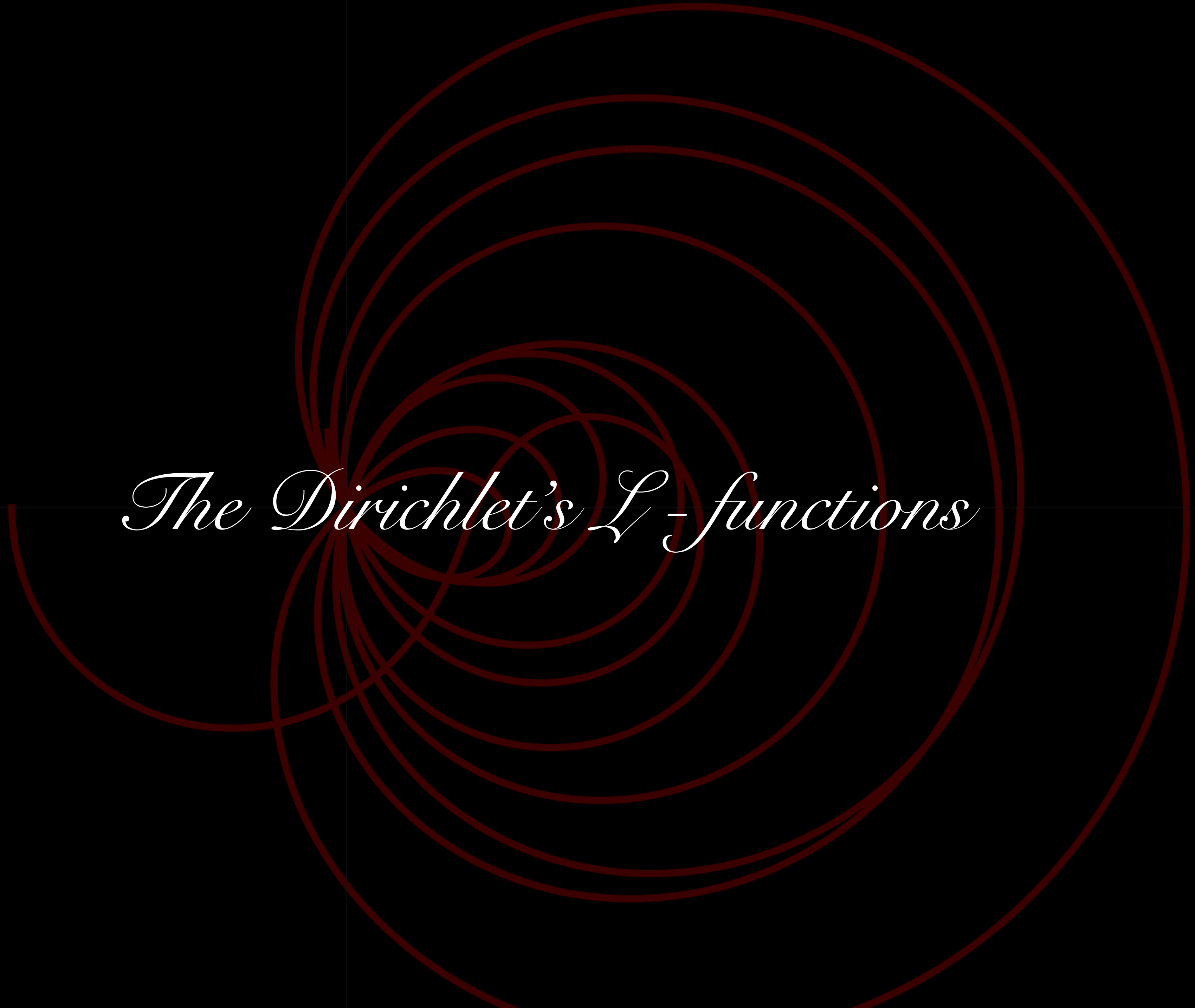
$$\frac{1}{\zeta(s)} = s \int_1^\infty \frac{\hat{M}(x)}{x^{s+1}}\, dx$$

If $\hat{M}(x) \simeq x^{1/2}$, the integral diverges at Re(s) = ½

$\Downarrow$

$\frac{1}{\zeta(s)}$ has its first singularities at Re(s) = ½!!

The Dirichlet's $\mathcal{L}$-functions



Arithmetic Progressions

$$S_n = qn + h$$

$$q, h \in \mathbb{N}$$

q = modulus h = residue

Dirichlet question: Under which conditions on q and h, the sequence contains infinite number of primes?

Dirichlet theorem: q and h must be coprime! Necessary but also sufficient condition

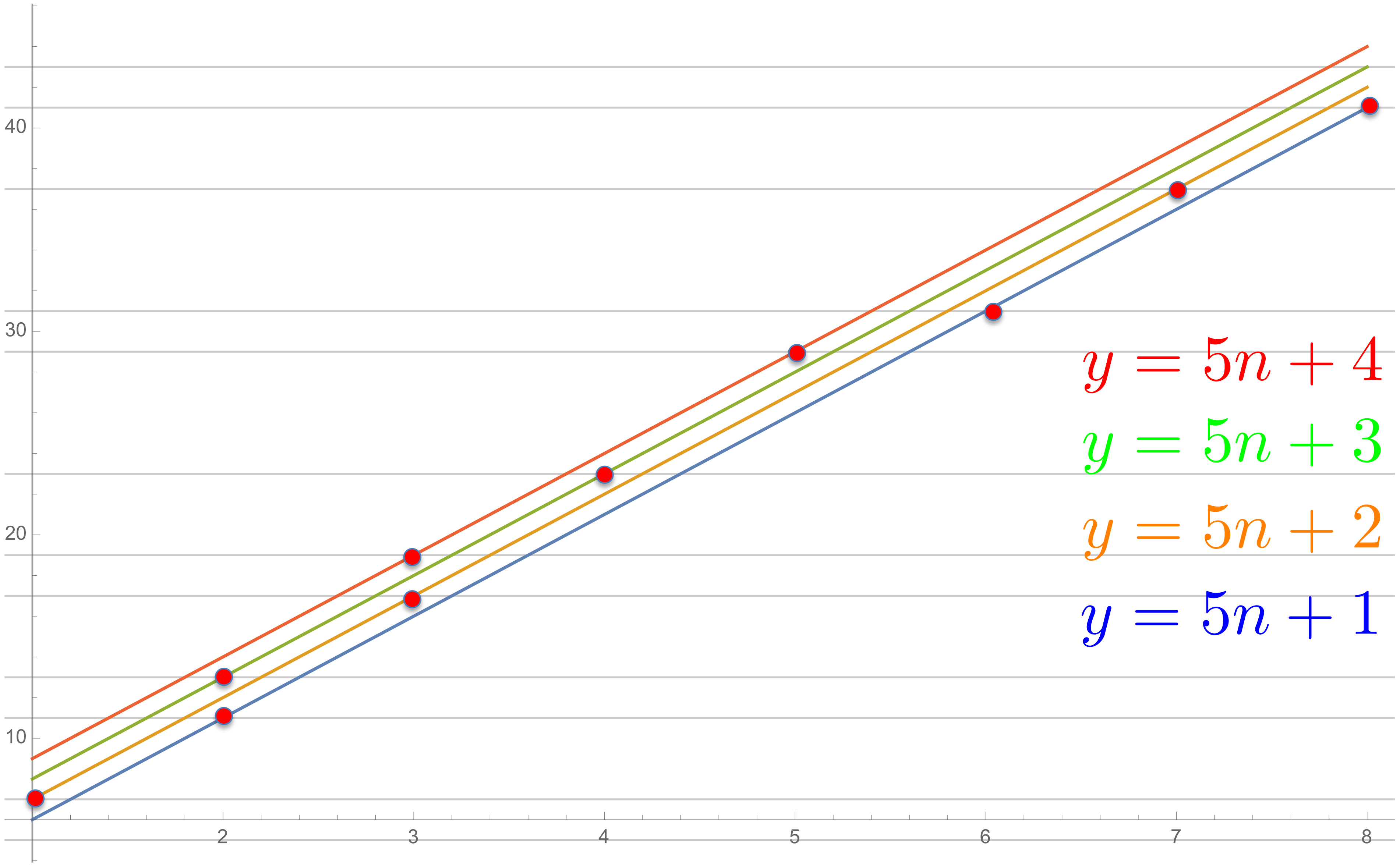
Examples

$$S_n = 5n + 3$$

S_n = {8, 13, 18, 23, 28, 33, 38, 43, 48, 53, 58, 63, 68, 73, 78, 83, 88, 93, 98, 103,…}

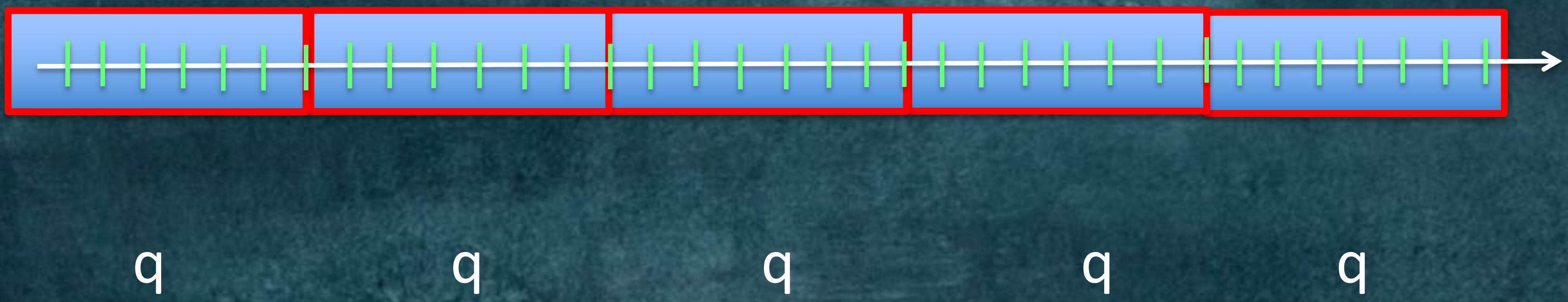
$$S_n = 5n + 2$$

S_n = {7, 12, 17, 22, 27, 32, 37, 42, 47,52, 57, 62, 67, 72, 77, 82, 87, 92, 97, 102, 107, 117, 122, 127, 132,…}



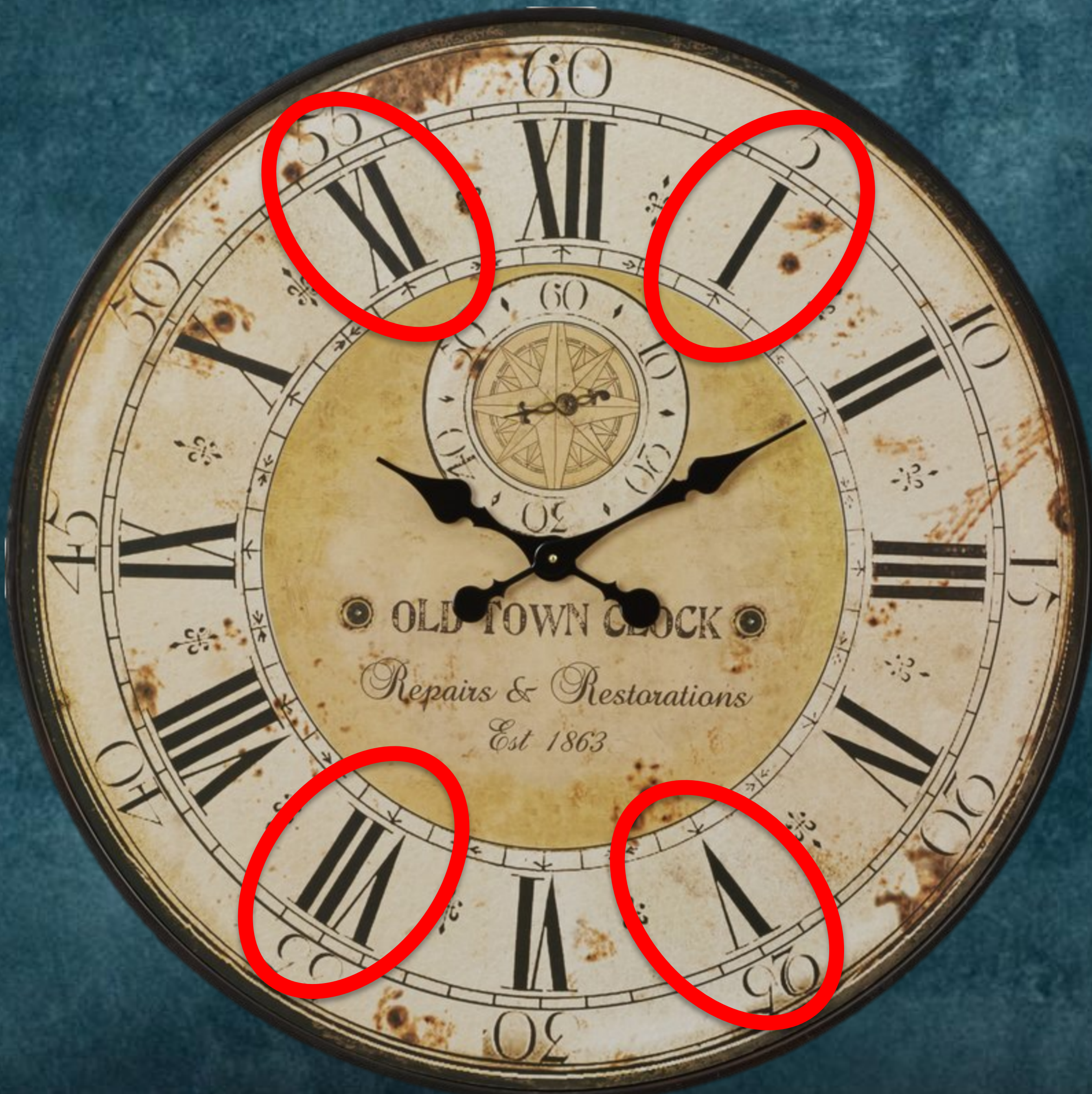
40
30
20
10
2
3
4
5
6
7
8
$y = 5n + 4$
$y = 5n + 3$
$y = 5n + 2$
$y = 5n + 1$

Lattices on integers



Modular Arithmetic

Modular Arithmetic



Group Multiplication Table

1	5	7	11
5	1	11	7
7	11	1	5
11	7	5	1

$$5 \times 5 = 25 = 24 + 1$$

$$5 \times 7 = 35 = 24 + 11$$

$$5 \times 11 = 55 = 48 + 7$$

$$2 \times 2 = 4$$

$$2 \times 8 = 16 = 12 + 4$$

The abelian group of prime residue classes modulo q

$(\mathbb{Z}/q\mathbb{Z})^*$: = { a mod q : gcd(a,q) = 1 }

$$\#\,\text{elements} = \varphi(q) = q\prod_{p|q}\left(1-\frac{1}{p}\right)$$

Examples

q=7 $(\mathbb{Z}/7\mathbb{Z})^* = \{1,2,3,4,5,6\}$ (q-1)

q=20 $(\mathbb{Z}/20\mathbb{Z})^* = \{1,3,7,9,11,13,17,19\}$

Characters

- These are 1-d representation of the residue class group
- There are as many characters as many elements, i.e. φ(q)

$$\chi_a(n) = e^{i\theta_a(n)} \qquad a = 1, 2, \ldots \varphi(q)$$

- These arithmetic functions satisfy a series of properties

Characters

- $\chi(n+q) = \chi(n).$

- $\chi(nm) = \chi(n)\,\chi(m).$

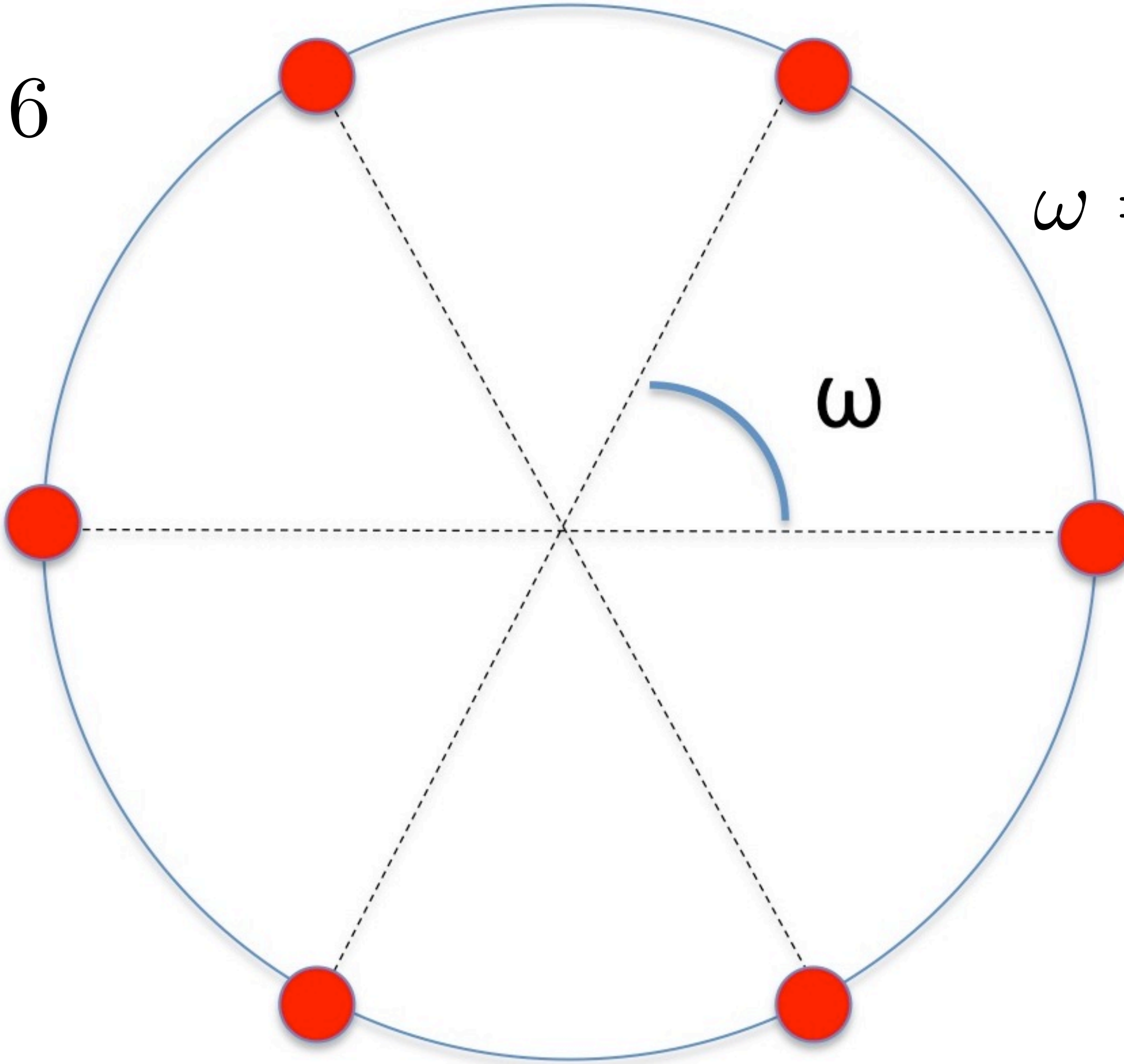
- $\chi(n) \neq 0$ if $(n,q) = 1.$

- $(\chi(n))^{\varphi(q)} = 1$,

namely $\chi(n)$ have to be $\varphi(q)$-roots of unity.

q=7

$$\varphi(7) = 6$$



$$\omega = e^{i2\pi/6} = e^{i\pi/3}$$

Character Table for q=7

Principal character

n	1	2	3	4	5	6	7
$\chi_1(n)$	1	1	1	1	1	1	0
$\chi_2(n)$	1	ω^2	ω	$-\omega$	$-\omega^2$	-1	0
$\chi_3(n)$	1	$-\omega$	ω^2	ω^2	$-\omega$	1	0
$\chi_4(n)$	1	1	-1	1	-1	-1	0
$\chi_5(n)$	1	ω^2	$-\omega$	$-\omega$	ω^2	1	0
$\chi_6(n)$	1	$-\omega$	$-\omega^2$	ω^2	ω	-1	0

Dirichlet L-functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^s}\right)^{-1}$$

$$\chi_a(n) = e^{i\theta_a(n)} \qquad a = 1, 2, \ldots \varphi(q)$$

Partition functions of free bosons with an extra $Z_{\varphi(q)}$ abelian charge

Dirichlet L-functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^s}\right)^{-1}$$

The Riemann function is a very special case

of a Dirichlet function (q=1)

Dirichlet L-functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^s}\right)^{-1}$$

Functional equation

$$L(1-s,\chi) = \mathcal{A}\, L(s,\overline{\chi})$$

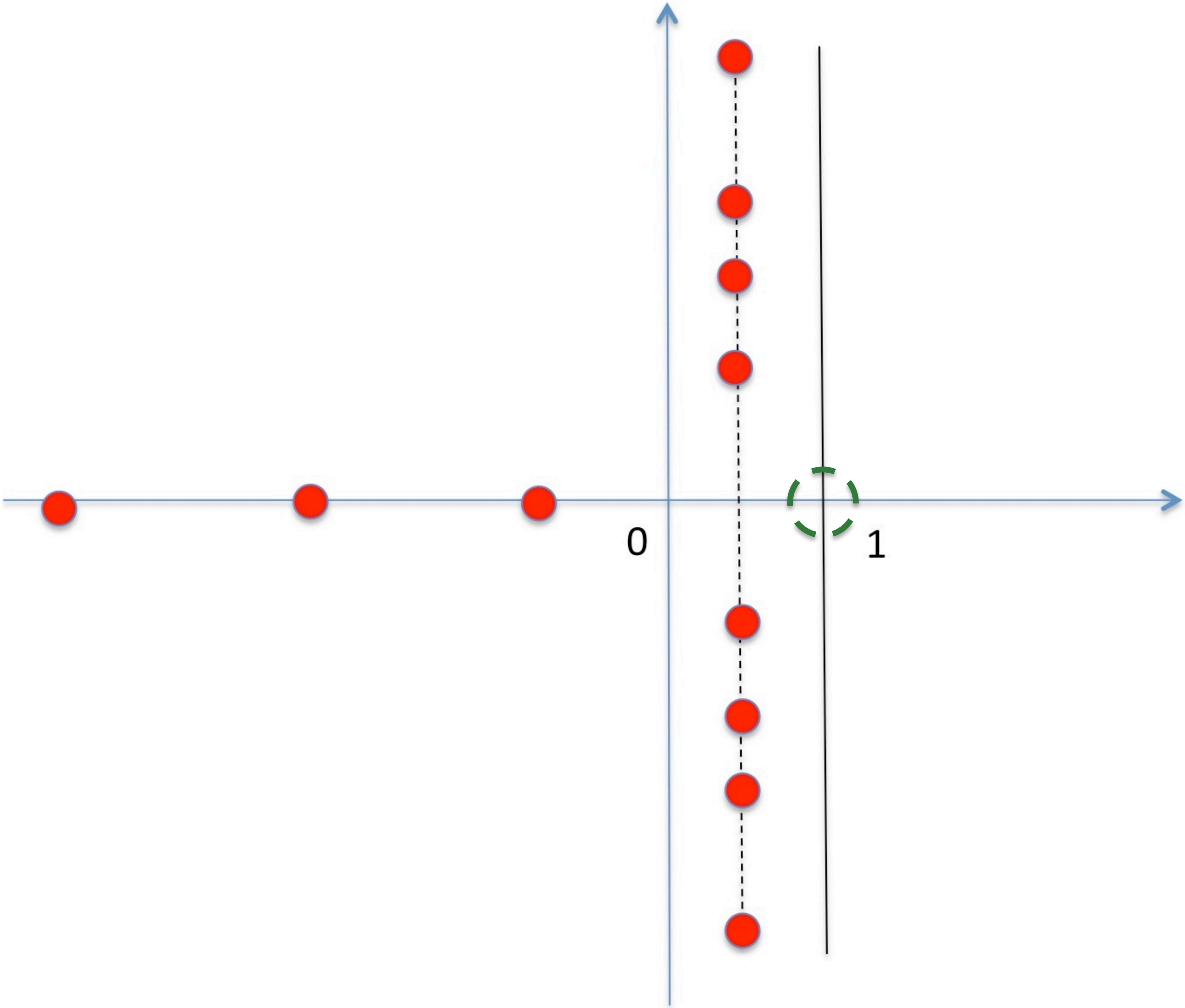
$$\mathcal{A} = \frac{q^{s-1}\Gamma(s)}{(2\pi)^s} \cos(\pi s/2)$$

Dirichlet L-functions

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_{n=1}^{\infty} \left(1 - \frac{\chi(p_n)}{p_n^s}\right)^{-1}$$

Residue at the pole s = 1

$$\mathrm{Res}\, L(s,\chi) = \frac{1}{q} \sum_{r=0}^{q} \chi(r) = \begin{cases} \frac{\varphi(q)}{q} & \text{if } \chi = \chi_1 \\ 0 & \text{if } \chi \neq \chi_1 \end{cases}.$$


0
1


Urbi et Orbi

Generalised Riemann Hypothesis

(Piltz 1884)

The non-trivial zeros of ALL Dirichlet functions

(i.e. for any modulus q and any character)

are ALL along the critical line placed at

$$\mathbb{R}e\, s = \frac{1}{2}$$

Today, in this talk, we discuss the

Generalised Riemann Hypothesis

for non-principal characters

Randomness of Möbius coefficents and brownian motion: growth of the Mertens function and the Riemann Hypothesis

Giuseppe Mussardo[1] and André LeClair[2]

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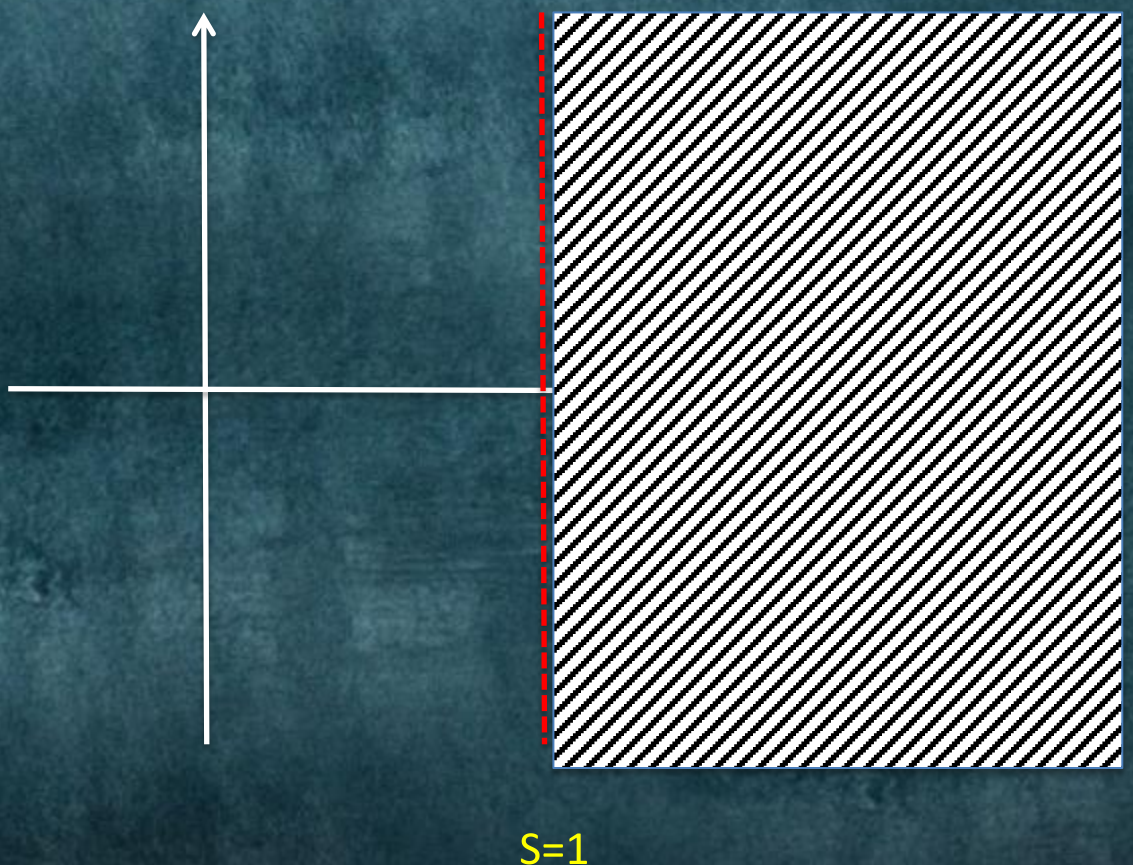
[2]Cornell University, Physics Department, Ithaca, NY 14850

The validity of the Riemann Hypothesis (RH) on the location of the non-trivial zeros of the Riemann ζ-function is directly related to the growth of the Mertens function $M(x) = \sum_{k=1}^{x} \mu(k)$, where $\mu(k)$ is the Möbius coefficient of the integer k: the RH is indeed true if the Mertens function goes asymptotically as $M(x) \simeq x^{1/2+\epsilon}$. We show that this behavior can be established on the basis of a new probabilistic approach based on the *global* properties of Mertens function. To this aim, we focus the attention on the square-free numbers and we derive a series of probabilistic results concerning the prime number distribution along the series of square-free numbers, the average number of prime divisors, the Erdős-Kac theorem for square-free numbers, etc. These results lead us to the conclusion that the Mertens function is subject to a normal distribution as much as any other random walk, therefore with an asymptotic behaviour given by $x^{1/2+\epsilon}$. This represents a theoretical advance in the field. We also argue how the Riemann Hypothesis implies the Generalised Riemann Hypothesis for the Dirichlet L-functions. Next we study the *local* properties of the Mertens function dictated by the Möbius coefficients restricted to the square-free numbers. Motivated by the natural curiosity to see how close to a purely random walk is any sub-sequence extracted by the sequence of the Möbius coefficients for the square-free numbers, we perform a massive statistical analysis on these coefficients, applying to them a series of randomness tests of increasing precision and complexity: together with several frequency tests within a block, the list of our tests include those for the longest run of ones in a block, the binary matrix rank test, the Discrete Fourier Transform test, the non-overlapping template matching test, the entropy test, the cumulative sum test, the random excursion tests, etc. The successful outputs of all these tests (with a level of confidence of 99% that all the sub-sequences analyzed are indeed random) can be seen as impressive "experimental" confirmations of the brownian nature of the restricted Möbius coefficients and the probabilistic normal law distribution of the Mertens function analytically established earlier. In view of the theoretical probabilistic argument and the large battery of statistical tests, we can conclude that while a violation of the RH is strictly speaking not impossible, it is however ridiculously improbable.

The main idea

- We will look at enlarging the domain of convergence of the Mellin transform of the L-functions



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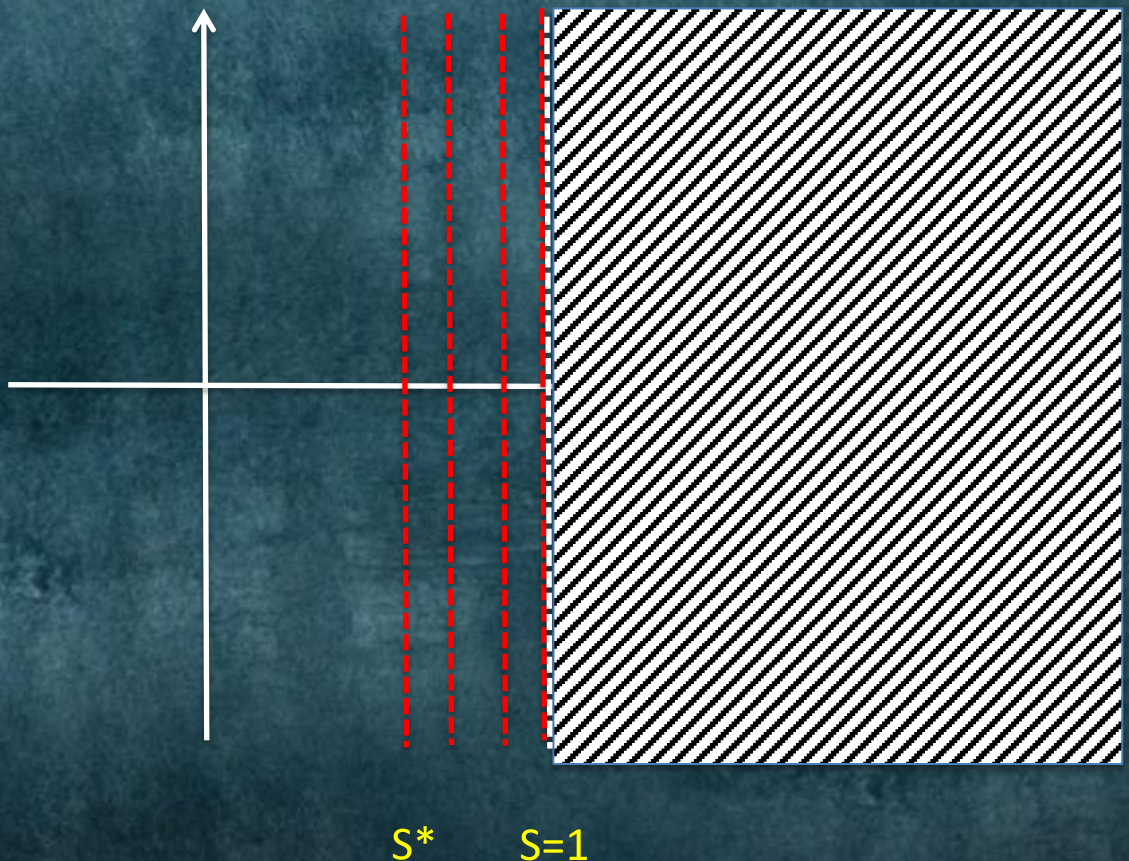


The main idea

- We will look at enlarging the domain of convergence of the Mellin transform of the L-functions

$$P(s,\chi) = \log L(s,\chi)$$

$$\log \prod_{n=1}^{\infty}\left(1-\frac{\chi(p_n)}{p_n^s}\right)^{-1}$$

- The singularities of this quantity depend ONLY on the zeros of the L-functions of non-principal characters

$$s = \sigma + it$$

The main idea

$$\log P(\sigma) = \sum_{n=1} \frac{\cos\theta(p_n)}{p_n^\sigma} = \sigma \int_2^\infty \frac{B(u)}{u^{\sigma+1}} du$$

$$B(x) = \sum_{n=1}^{[x]} \cos\theta(p_n)$$

The angles $\theta(p_n)$ are computed along the sequence of the primes p_n

The main idea of our argument

$$\log\ P(\sigma)\ = \sum_{n=1} \frac{\cos\theta(p_n)}{p_n^\sigma} \quad = \sigma \int_2^\infty \frac{B(u)}{u^{\sigma+1}} du$$

$$B(N) \simeq N^\alpha$$

If α=1/2, i.e.

$$B \simeq \sqrt{N}$$

then all zeros are on the critical line!

Corrections by logarithms to this scaling law do not spoil the conclusion

$$B(N) \simeq \sqrt{N}\,\log^a N$$

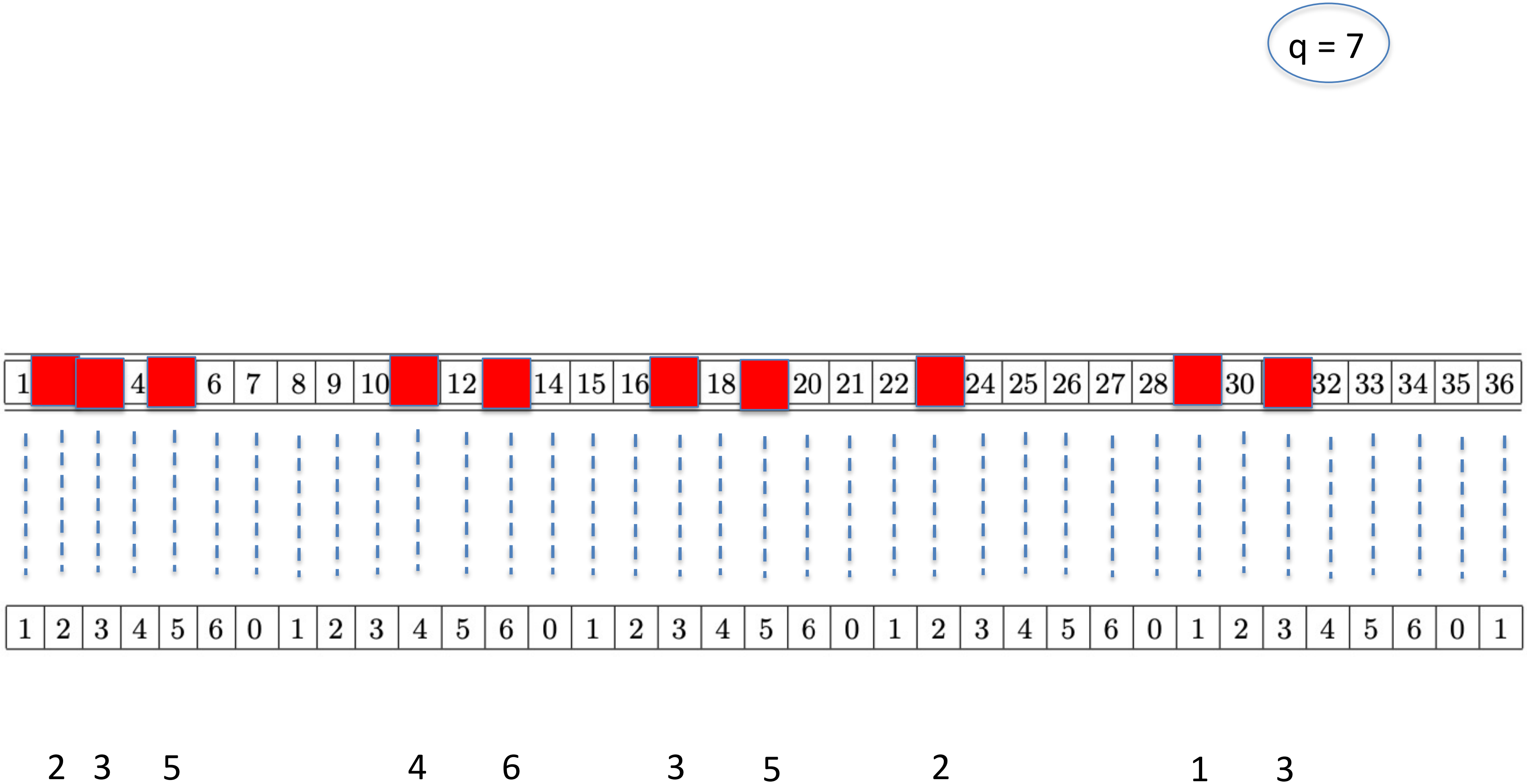
$$\log P(s) = \int_2^\infty \frac{B(u)}{u^{\sigma+1}} = \int_2^\infty \frac{\sqrt{u}\,\log^a u}{u^{\sigma+1}}$$

$$= \frac{\Gamma(a+1)}{(\sigma - 1/2)^{a+1}}$$

The large asymptotic behaviour of the series B(N)

$$B(N) \simeq N^{\alpha}$$

is an ideal diagnosis for the real part of the zeros



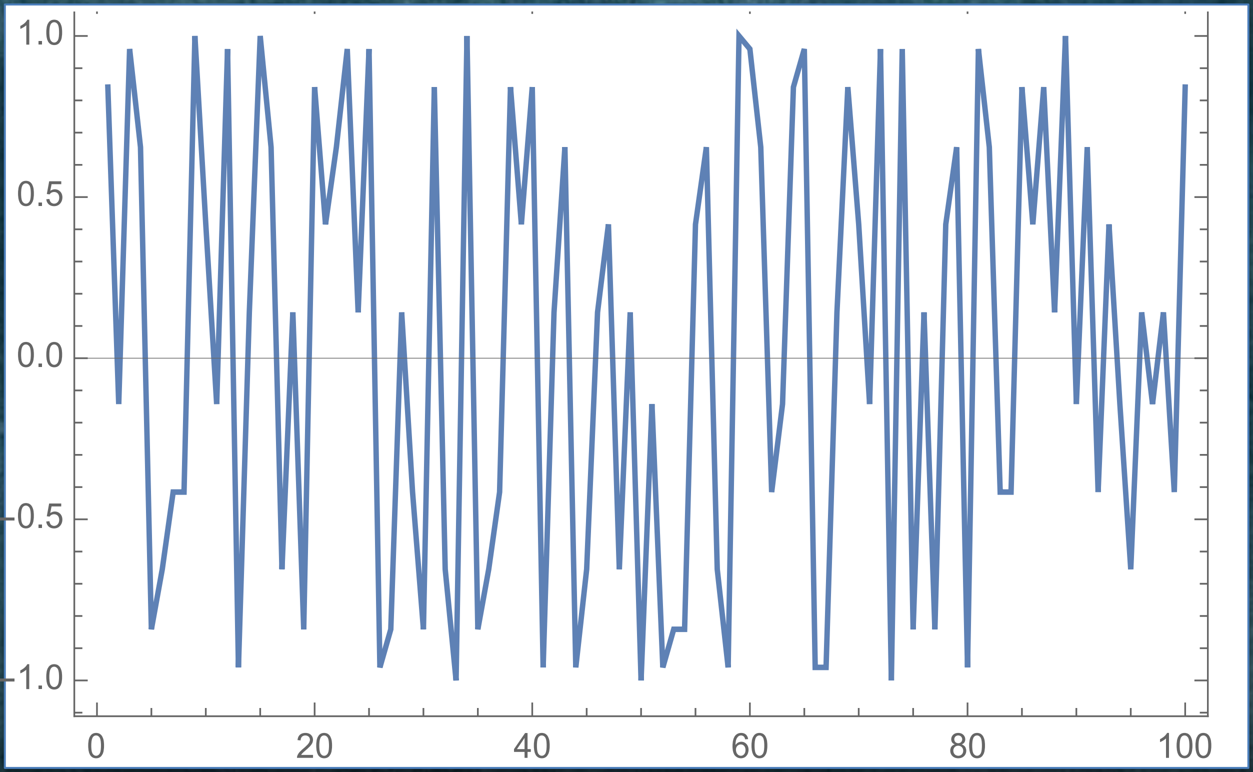
q = 7
1 4 6 7 8 9 10 12 14 15 16 18 20 21 22 24 25 26 27 28 30 32 33 34 35 36
1 2 3 4 5 6 0 1 2 3 4 5 6 0 1 2 3 4 5 6 0 1 2 3 4 5 6 0 1 2 3 4 5 6 0 1
2 3 5 4 6 3 5 2 1 3

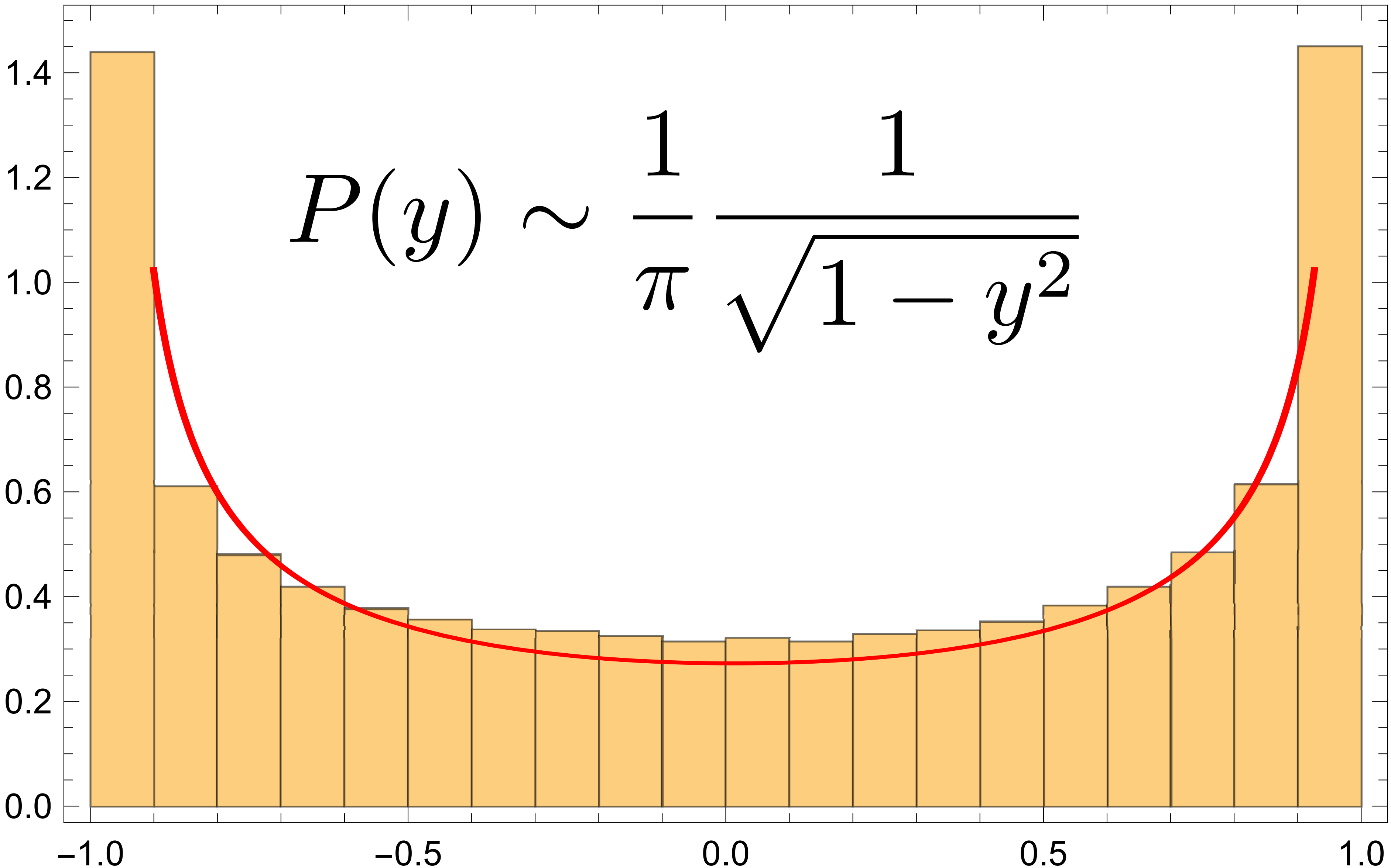
4, 6, 3, 5, 2, 1, 3, 2, 6, 1, 5, 4, 3, 5, 4, 1, 3, 2, 6, 5, 6, 3, 5, 2, 4, 1, 1, 5, 4, 6, 2, 4, 3, 2, 6, 5, 4, 6, 2, 4, 1

Properties of the series B(N)

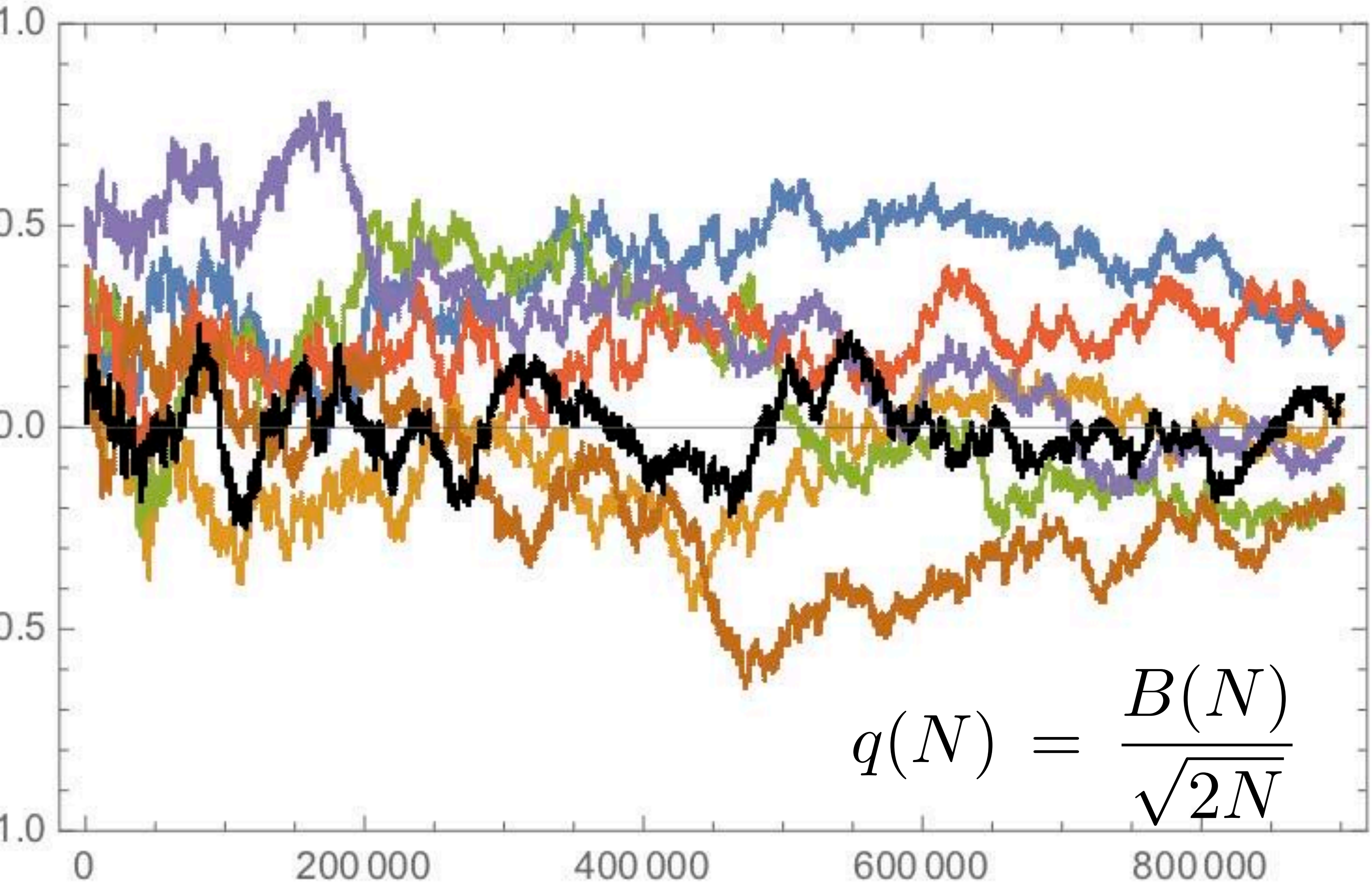
$$B(N) = \sum_{n=1}^{N} \cos\theta(p_n)$$

$\cos\theta(p_n)$





$P(y) \sim \frac{1}{\pi}\frac{1}{\sqrt{1-y^2}}$
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−1.0
−0.5
0.0
0.5
1.0



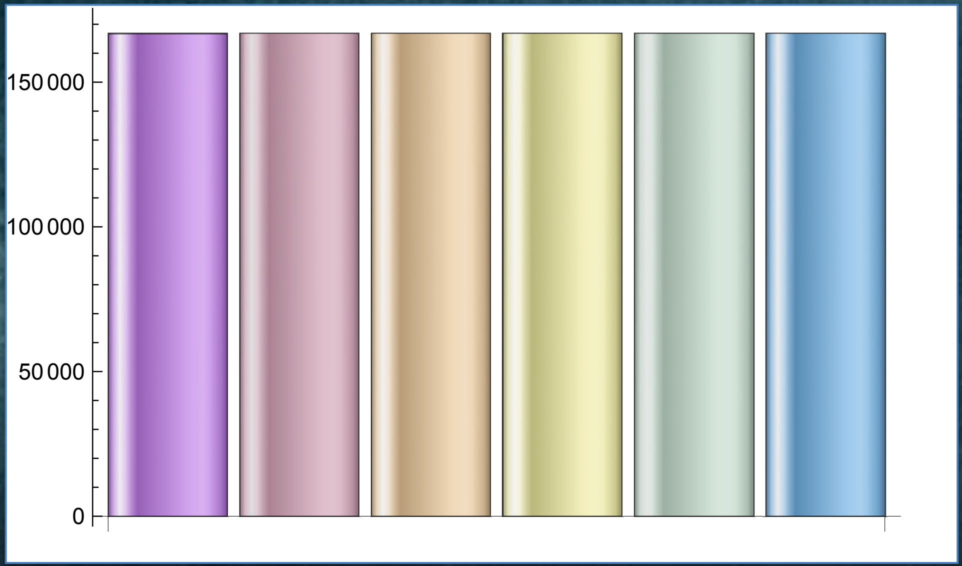
1.0
0.5
0.0
0.5
1.0
0
200 000
400 000
600 000
800 000
$q(N) = \frac{B(N)}{\sqrt{2N}}$

Properties of the series B(N)

$$B(N) = \sum_{n=1}^{N} \cos\theta(p_n) \equiv \sum_{n=1}^{N} c_n$$

- All terms are of the same order (order 1)

- All the angles are equiprobable (Dirichlet theorem)



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- As consequence, the mean of the series B(N) vanishes

$$\langle B(N) \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n = 0$$

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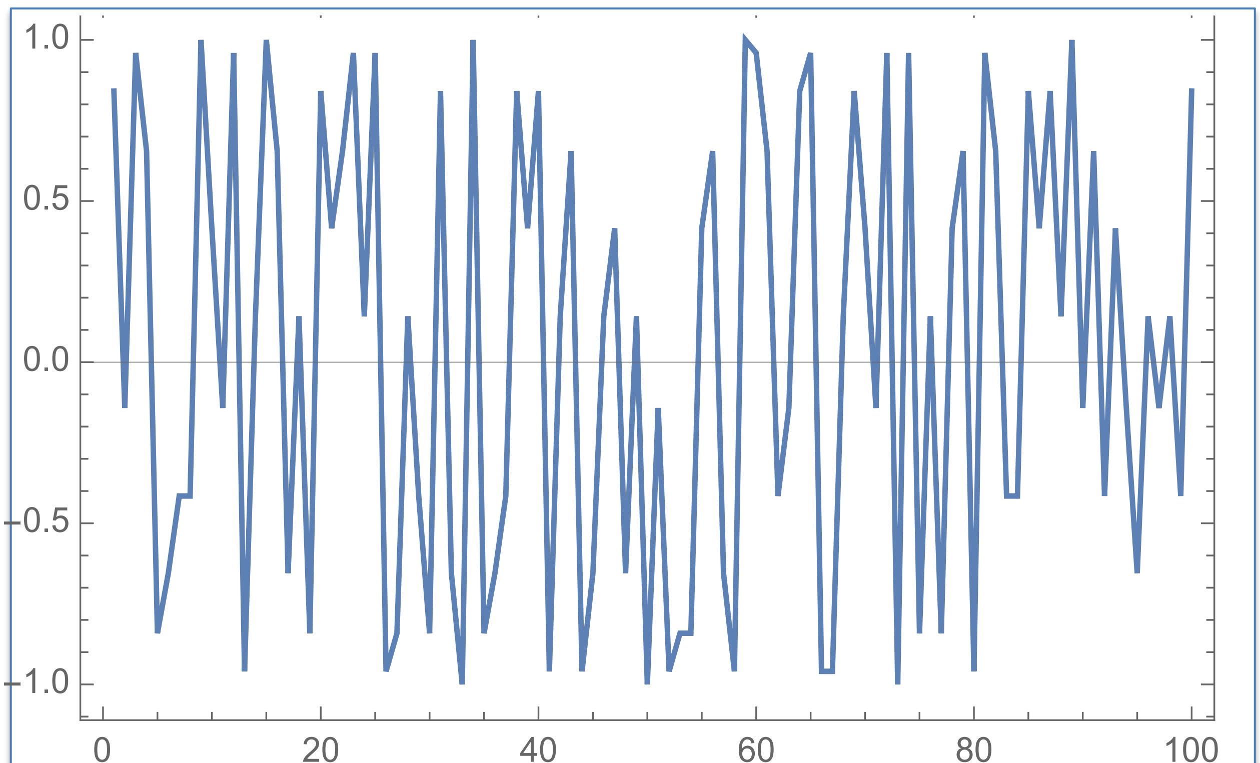
$$\langle B(N) \rangle = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} c_n = 0$$

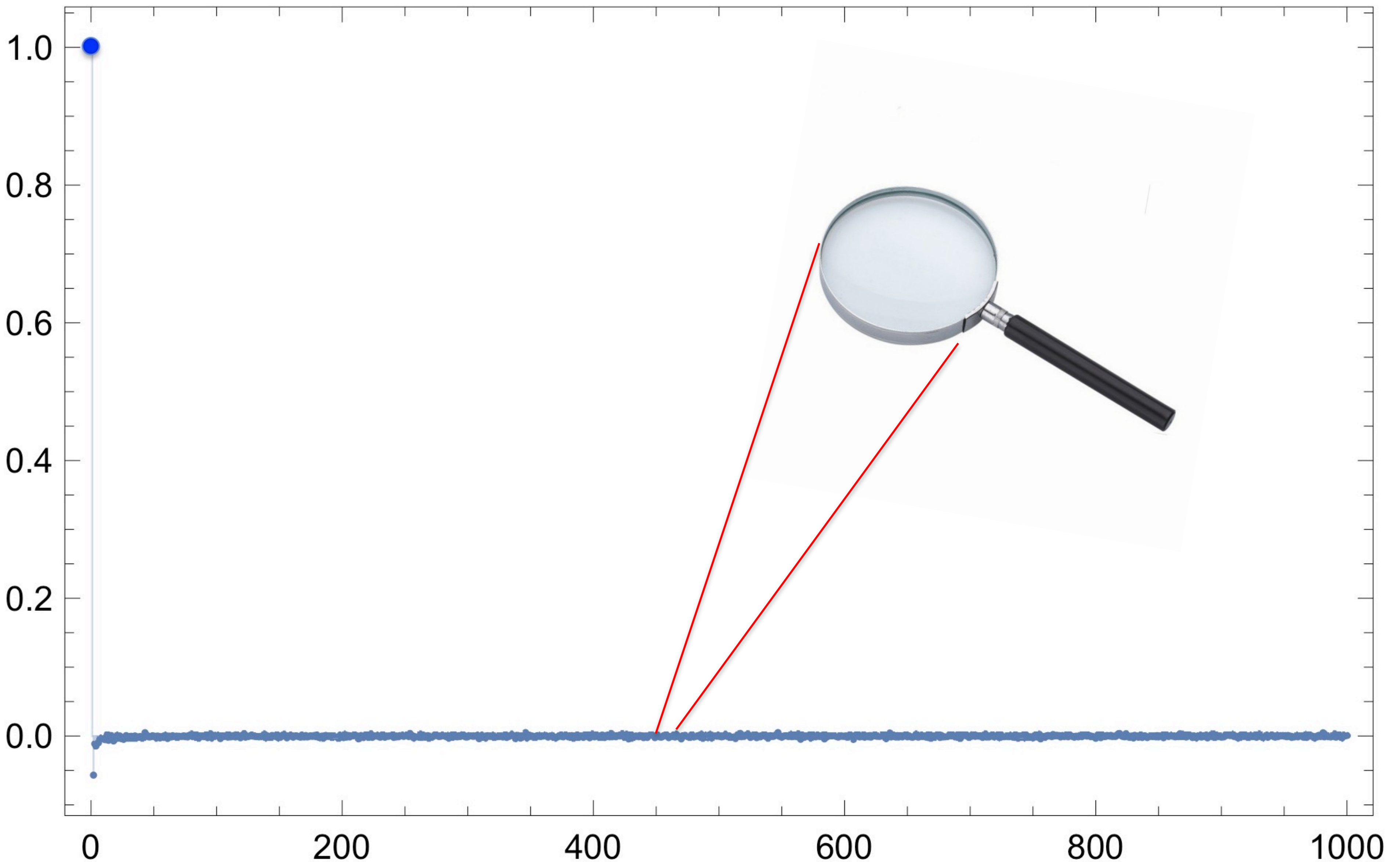
- The variable c_n are very weakly correlated

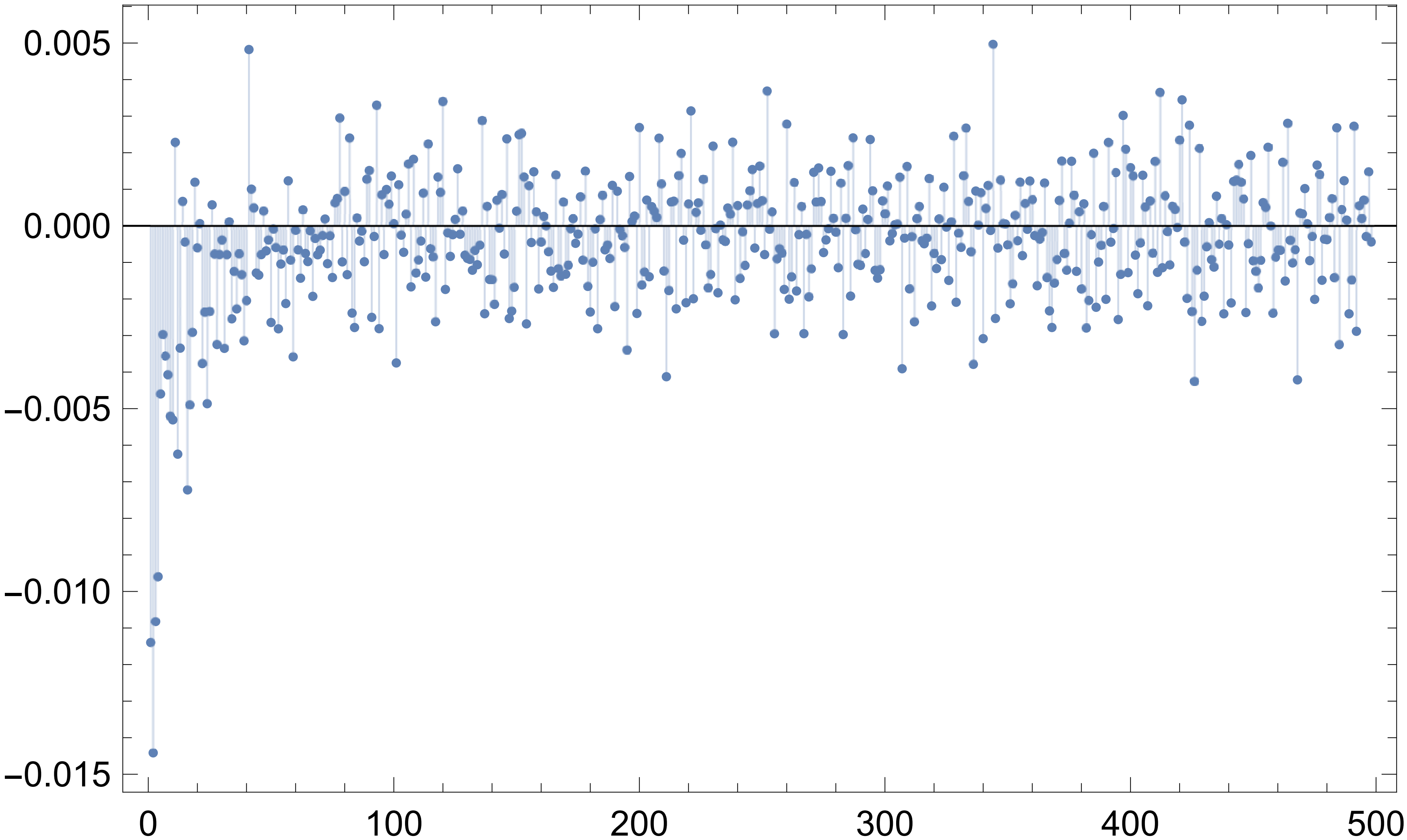
Correlation Function of lag j

$$G(j) = \frac{\sum_{n=1}^{N-1}(c_n - \mu)(c_{n+j} - \mu)}{\sum_{n=1}^{N}(c_n - \mu)^2}$$



Correlation Function of lag j





0.005
0.000
−0.005
−0.010
−0.015
0
100
200
300
400
500

As a matter of fact, at finite size we have a very clear idea of all their correlations!

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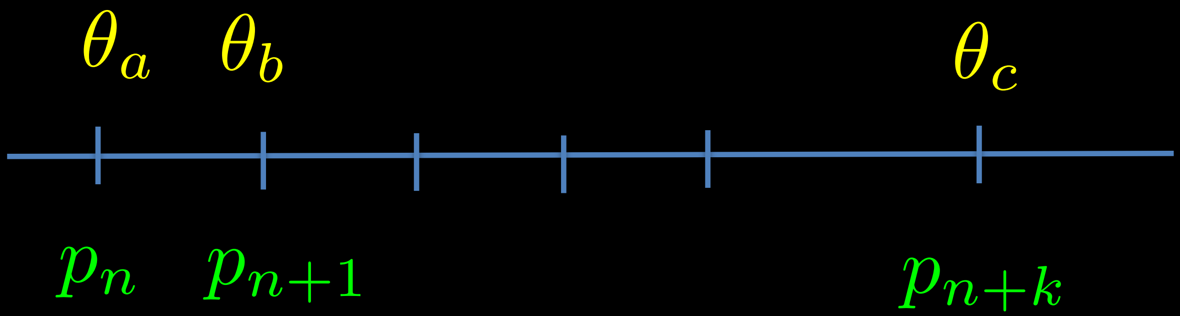
(Lemke Oliver-Soundararajan)



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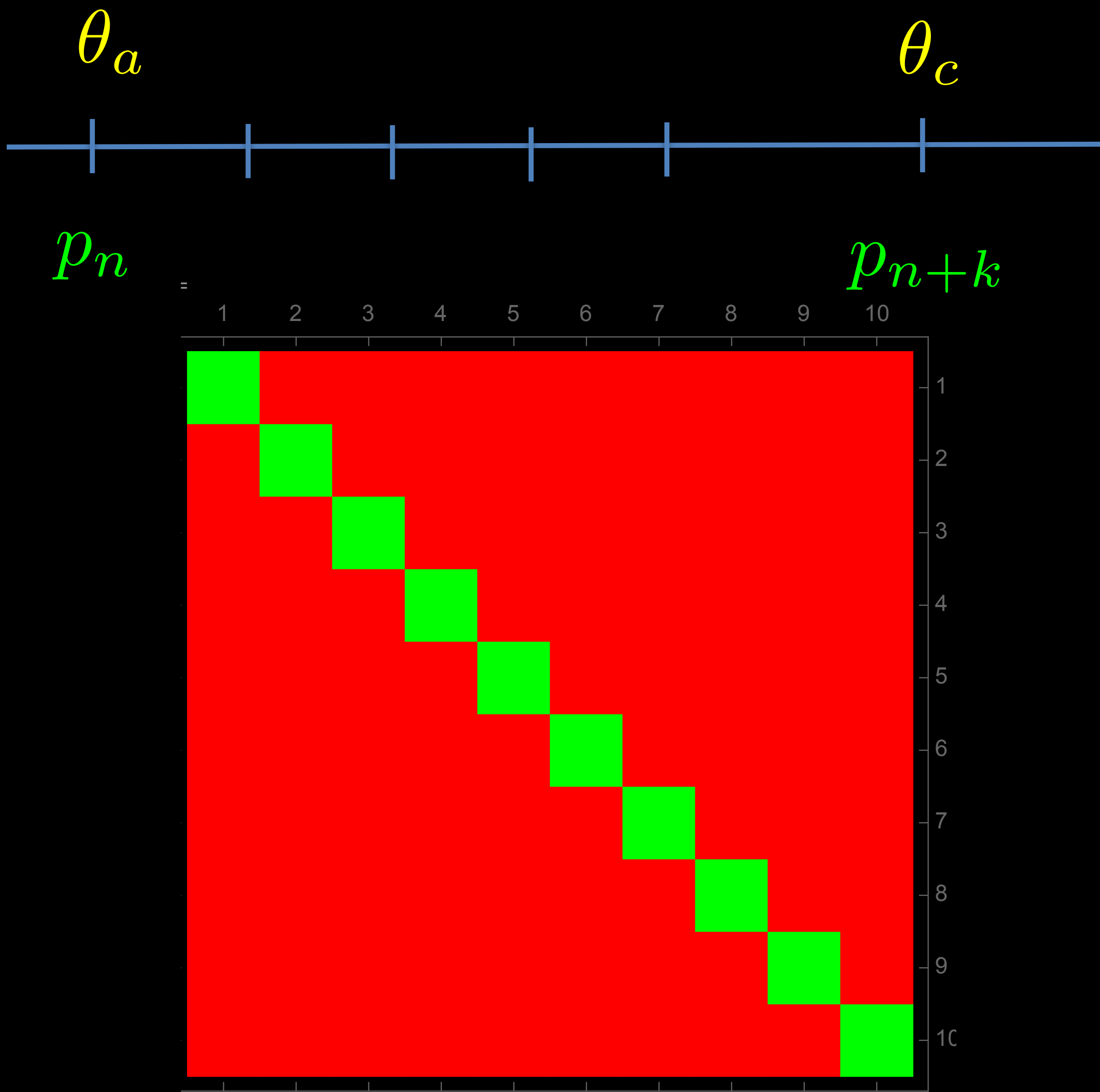

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(Lemke Oliver-Soundararajan)

$$f_{ab}(N,q,k) = \frac{\#\{p_n \le p_N : p_n \equiv h_a \pmod q \ , \ p_{n+k} \equiv h_b \pmod q\}}{N}$$

$$f_{aa}(N,q,k) = \frac{1}{(\varphi(q))^2}\left(1 - \frac{(\varphi(q)-1)}{2(k-1)}\frac{1}{\log N} + O\left(\frac{1}{(\log N)^{7/4}}\right)\right)$$

$$f_{ab}(N,q,k) = \frac{1}{(\varphi(q))^2}\left(1 + \frac{(1)}{2(k-1)}\frac{1}{\log N} + O\left(\frac{1}{(\log N)^{7/4}}\right)\right)$$

As a matter of fact, at finite size we have a very clear idea of all their correlations!

(Lemke Oliver-Soundararajan)

$$f_{ab}(N,q,k) = \frac{\#\{p_n \le p_N : p_n \equiv h_a \pmod q \;,\; p_{n+k} \equiv h_b \pmod q\}}{N}$$

$$f_{aa}(N,q,k) = \frac{1}{(\varphi(q))^2}$$

$$f_{ab}(N,q,k) = \frac{1}{(\varphi(q))^2} \qquad N \to \infty$$

FOR LARGE N,

THE RESIDUES ARE UNCORRELATED !!!

Time Series

$$B(N) = \sum_{n=1}^{N} c_n \qquad c_n = \cos\theta_{p_n}$$

B(N) grows as a brownian motion !

- $$\langle B(N) \rangle = 0$$
- $$\langle B^2(N) \rangle = b^2 \, N$$
- $$\langle B^{2k}(N) \rangle = b^{2k} \, N^k$$

Law of iterated logarithms

(Khinchin –Kolmogorov)

$$B(N) = \sum_{n=1}^{N} c_n$$

For a brownian motion we have to control the tails

$$\lim_{N\to\infty} \sup \frac{|B(N)|}{\sqrt{2N \log\log N}} = 1$$

Experimental Mathematics

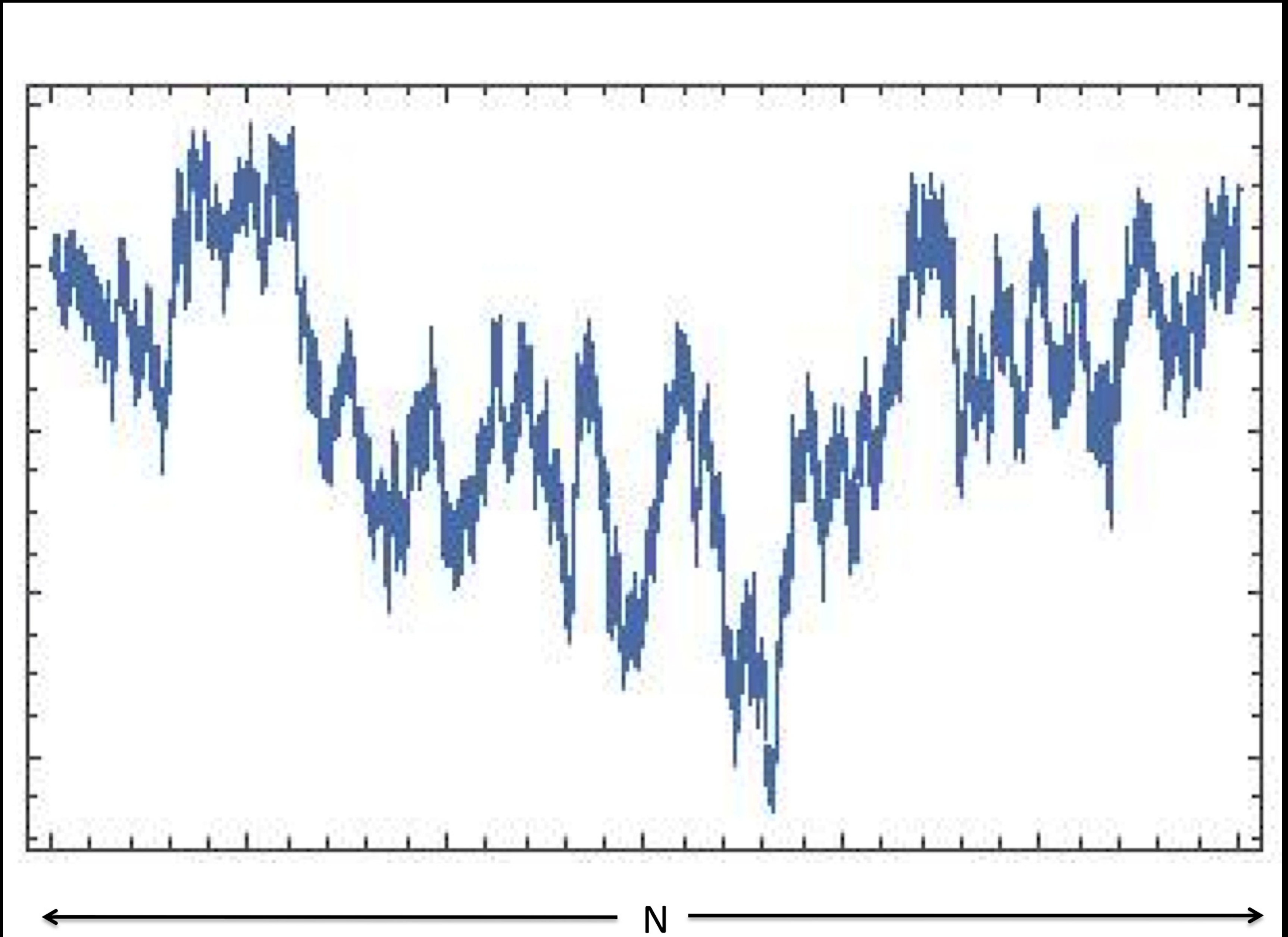
All previous considerations lead to conclude

$$B(N) \sim N^{1/2+\epsilon}$$

But, can we actually check or disprove it ?

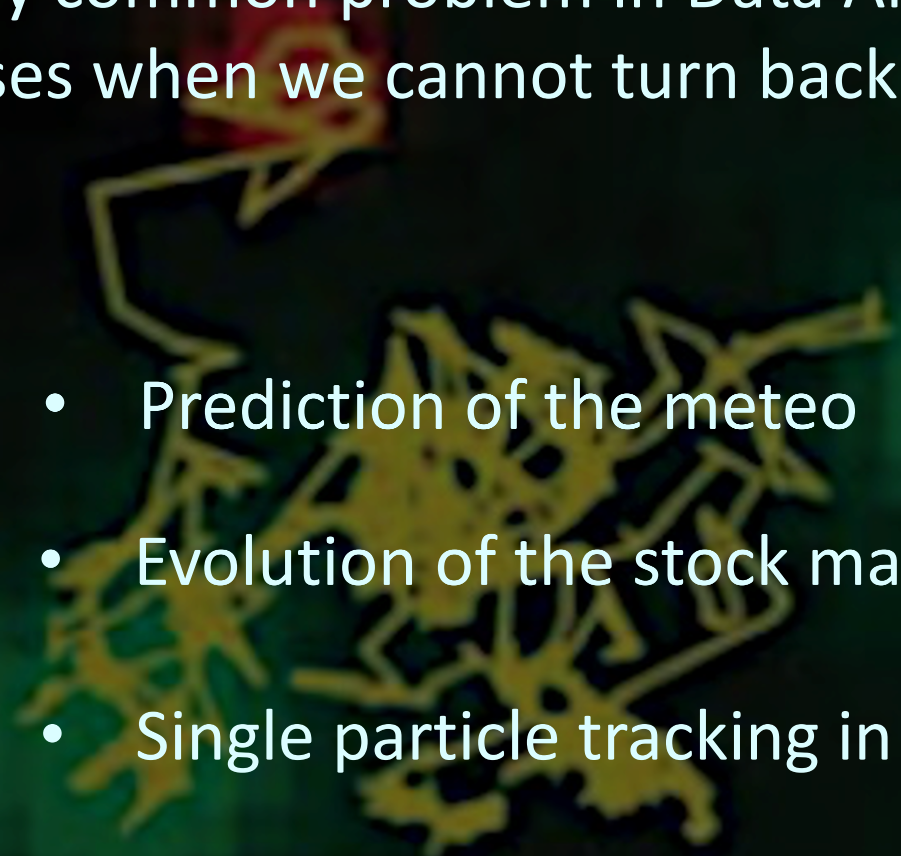
Can we falsify such a conclusion ?

Problem of single Brownian trajectory

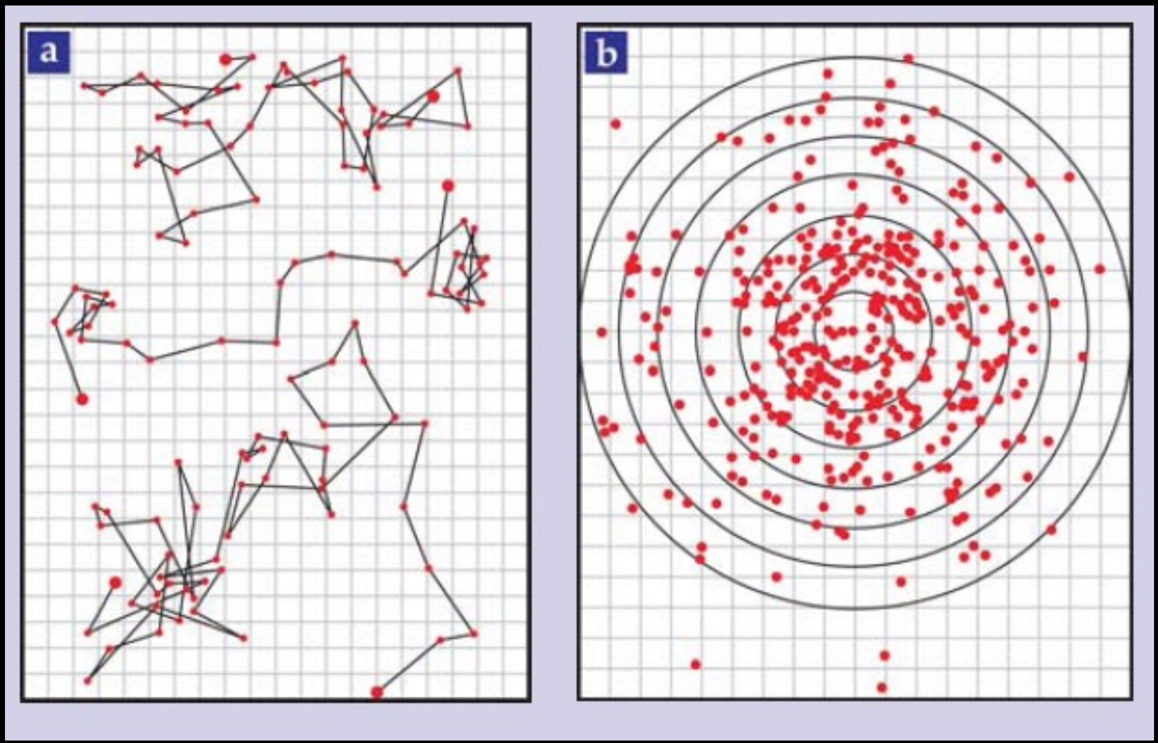


Problem of single Brownian trajectory

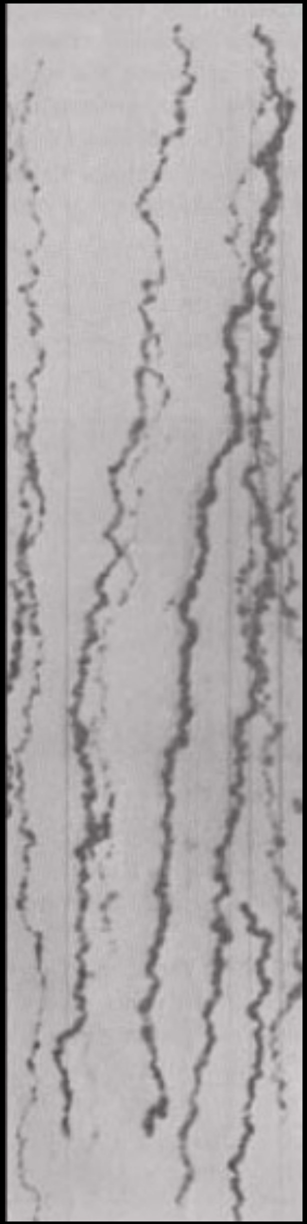
Very common problem in Data Analysis which arises when we cannot turn back time…



- Prediction of the meteo
- Evolution of the stock market
- Single particle tracking in living cell



Jean Perrin (1908)

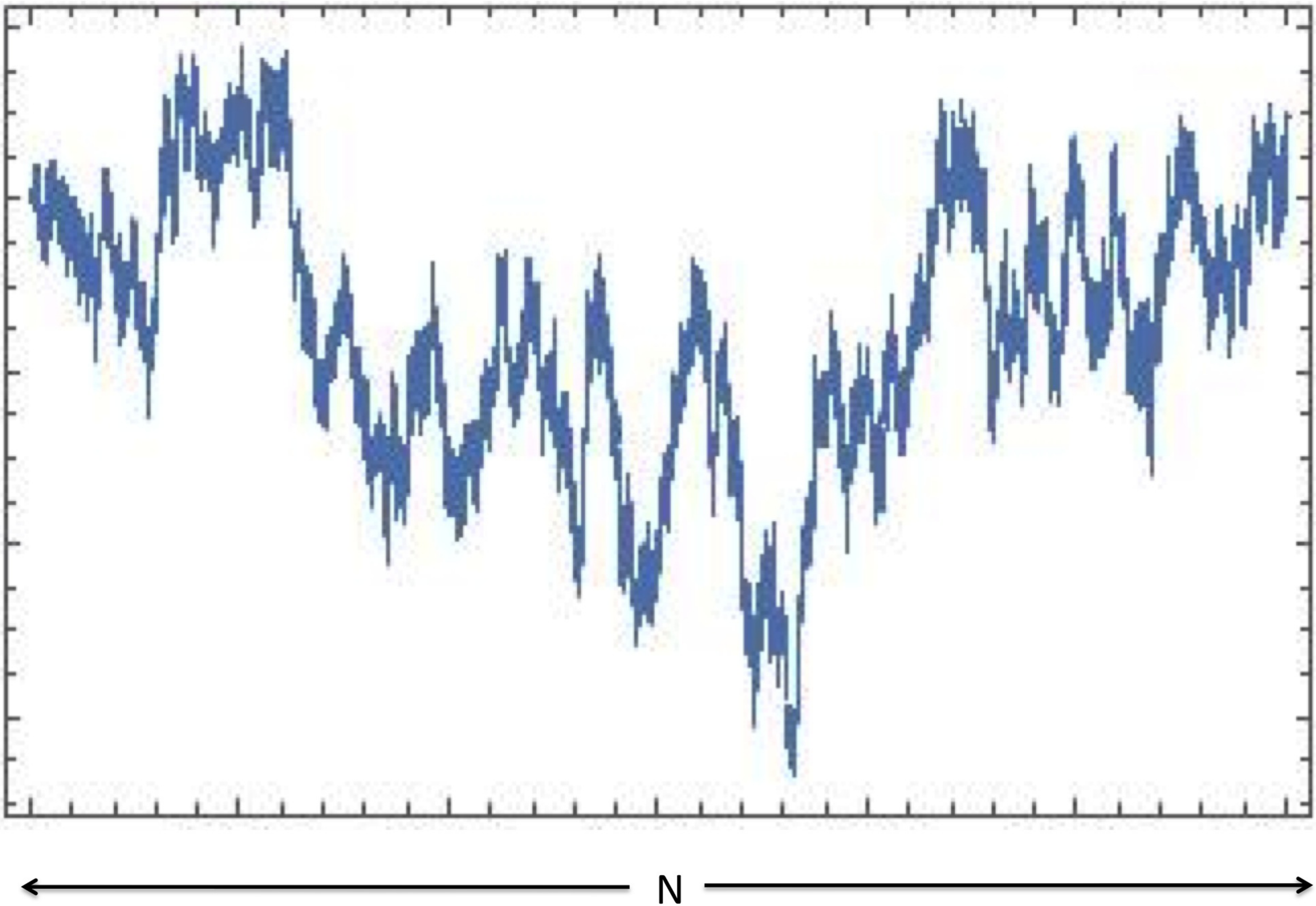


Ivar Nordlund (1914)

Problem of single Brownian trajectory

We can use parts of the infinite time series to generate the statistical ensembles!

With an abuse of language, the time series gives rise to its own "*thermal bath*"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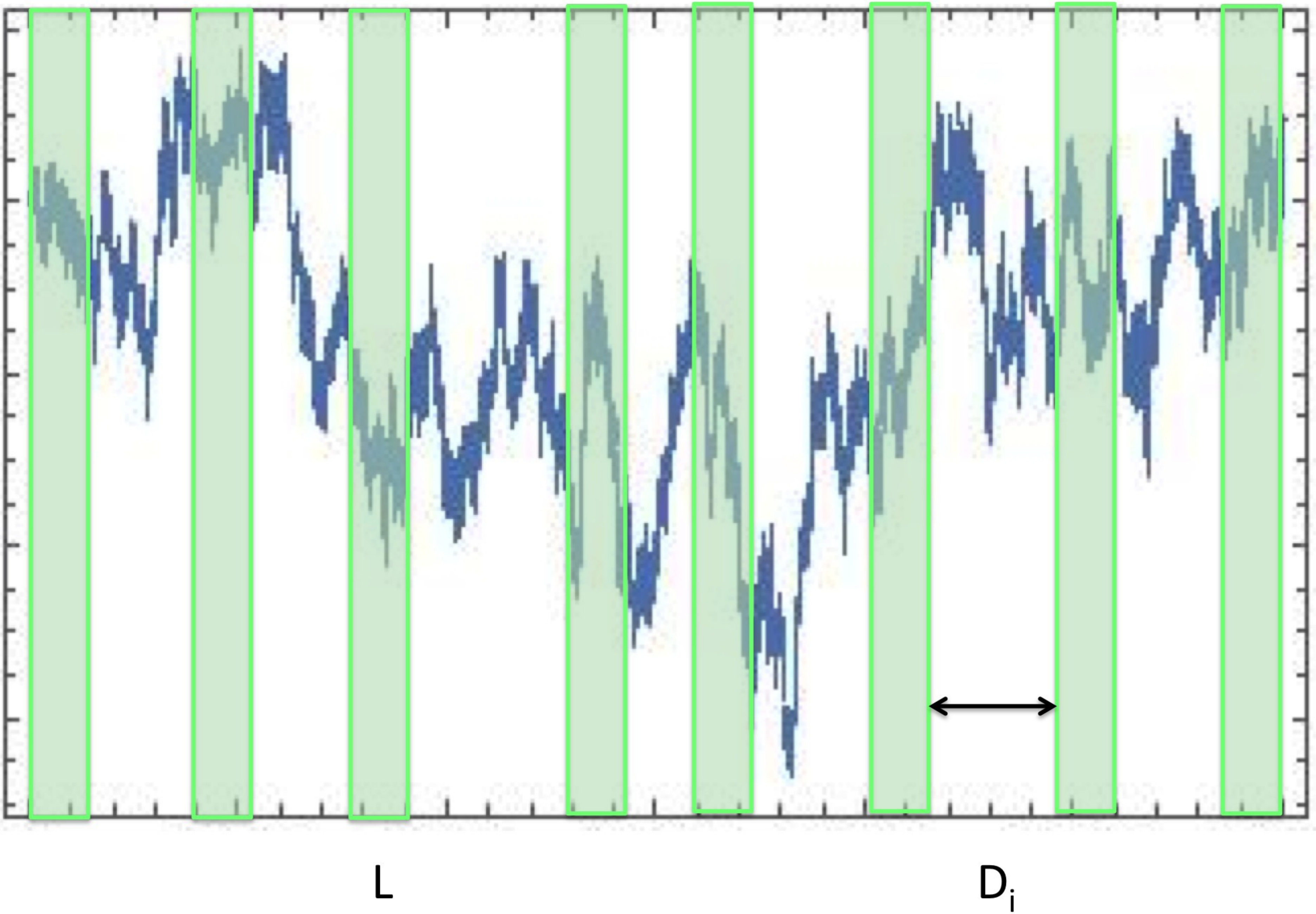
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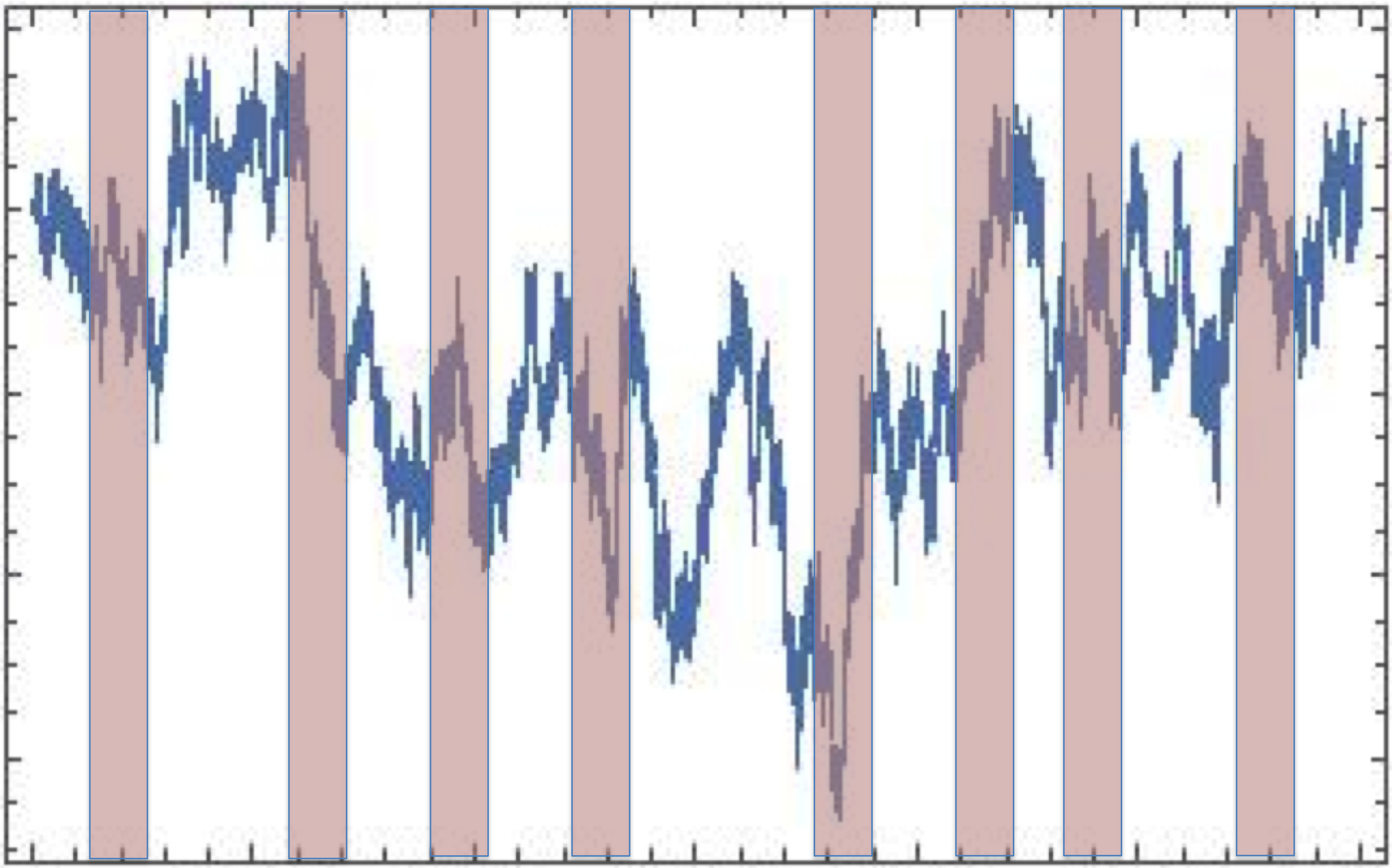
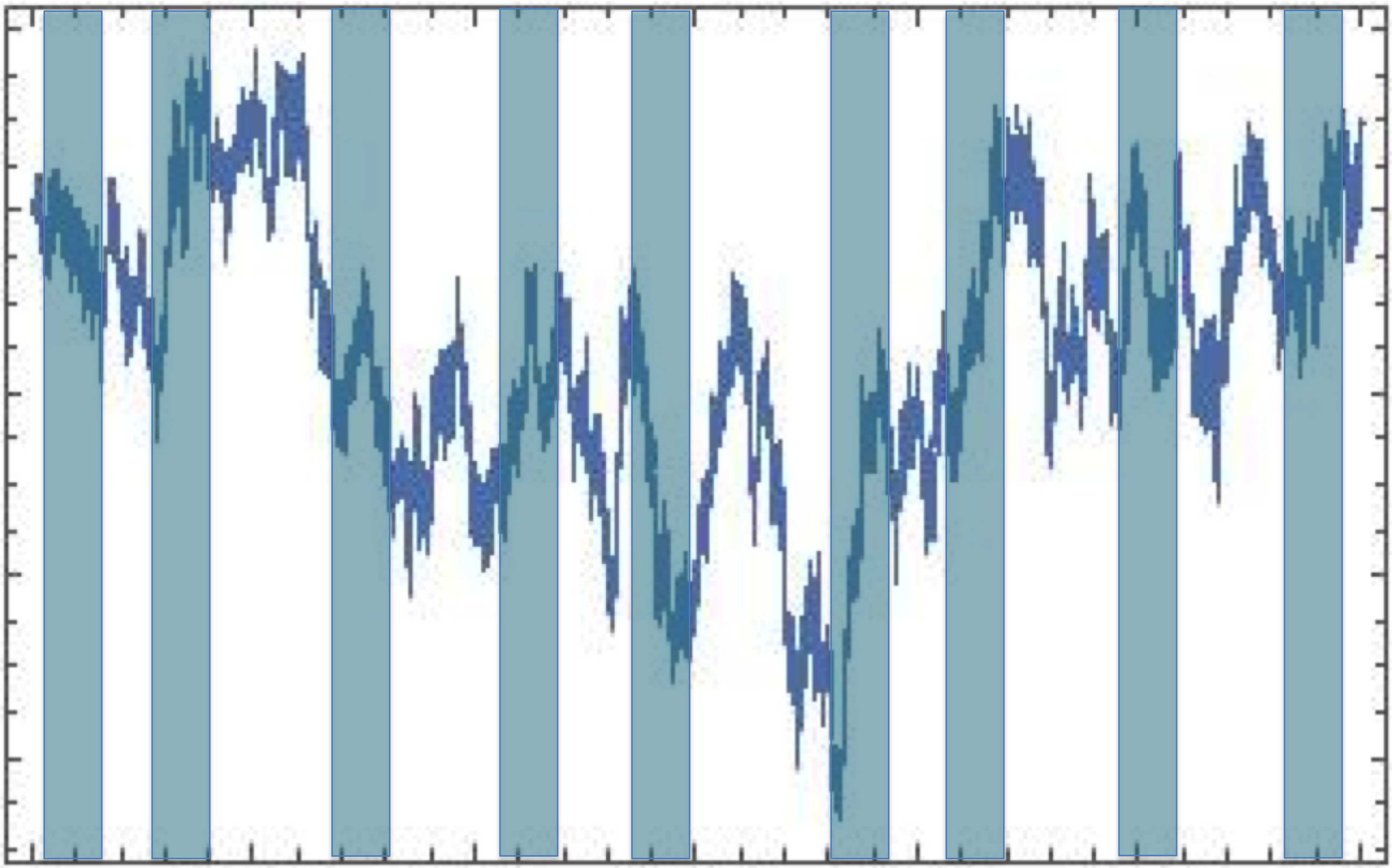
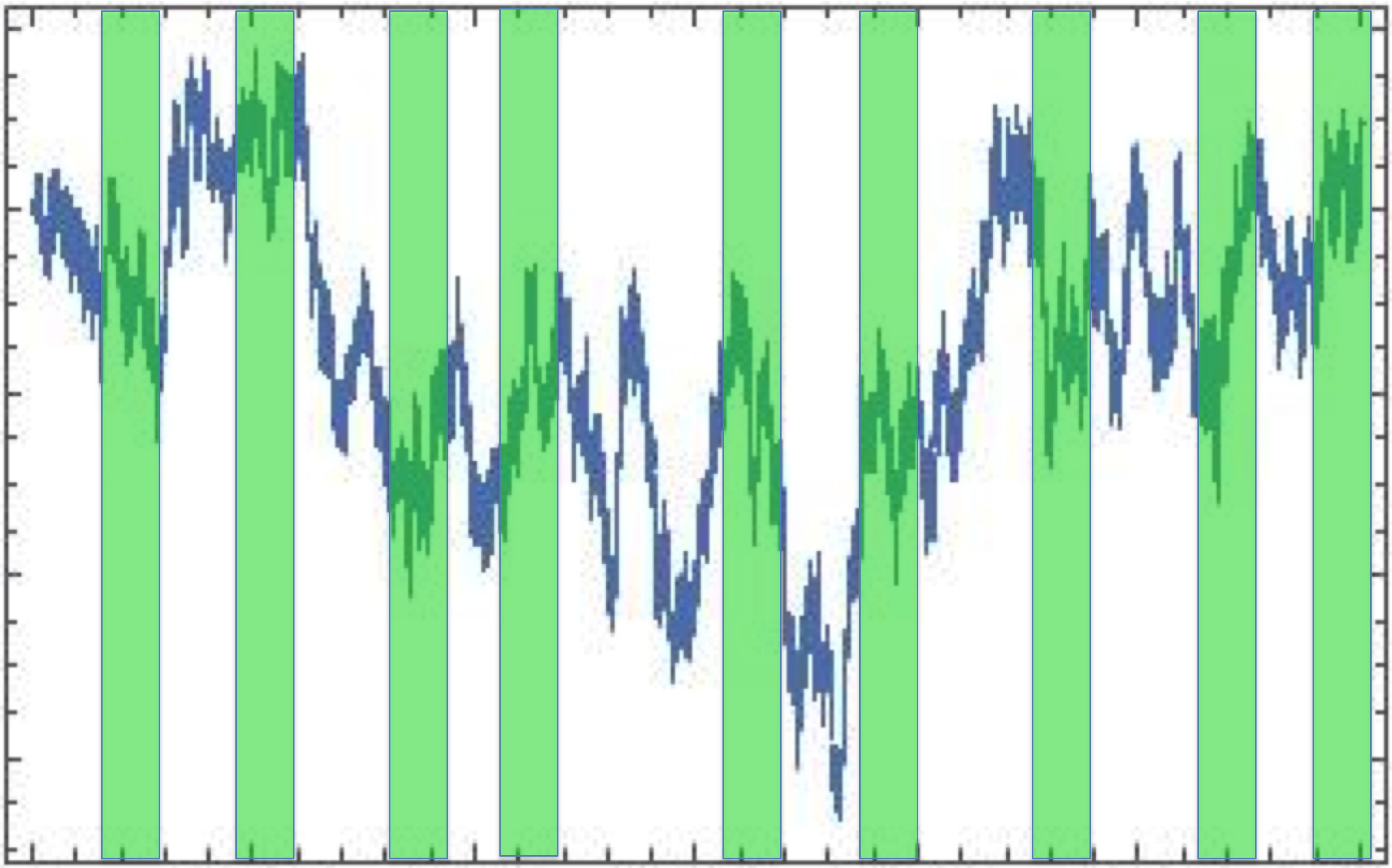
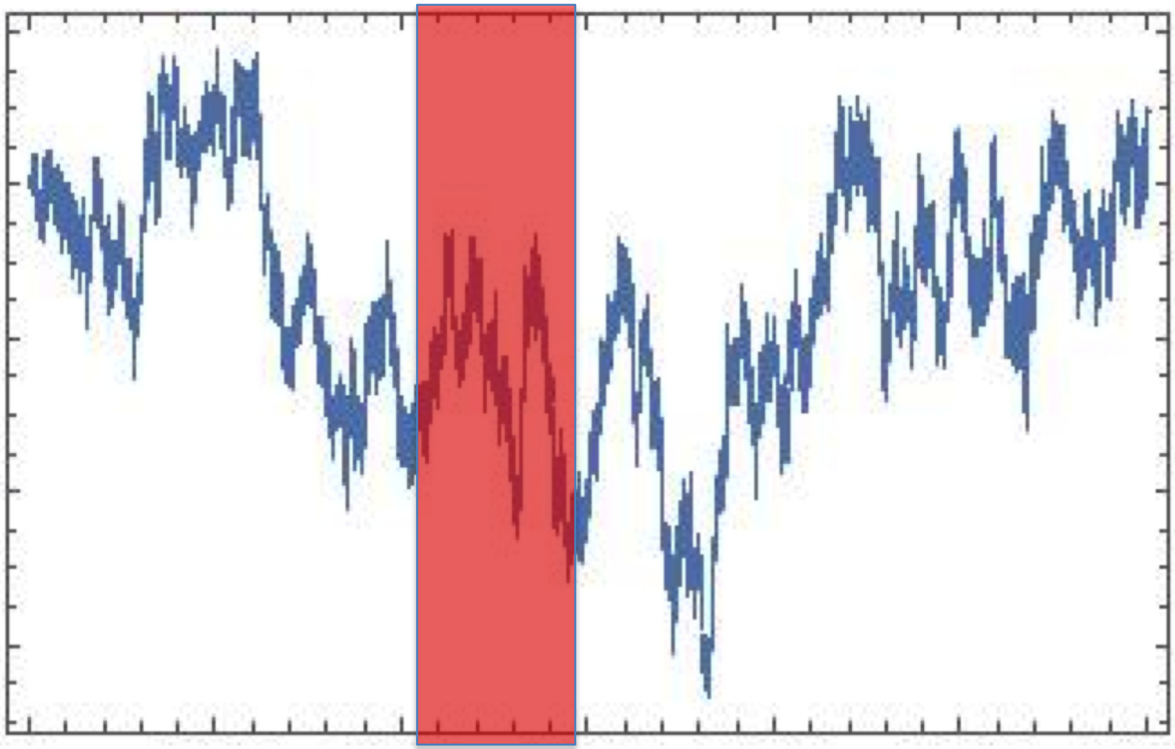
D_i







Block Variables



$$X_a(L) = \sum_{n=1}^{L} c_{a+n}$$

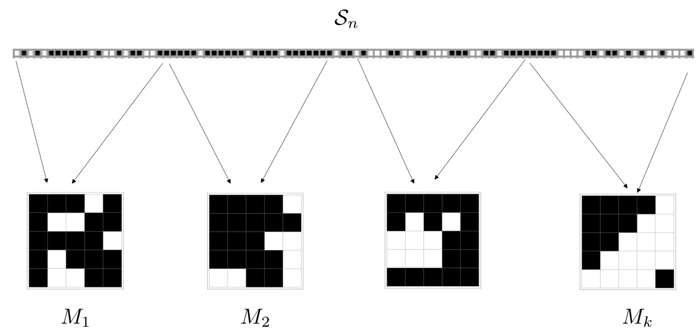
If the original series goes as

$$B_N \simeq \sqrt{N}$$

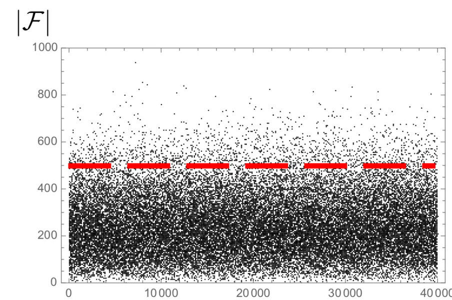
the same should happen for any of its subsets!

$$B'_L \simeq \sqrt{L}$$

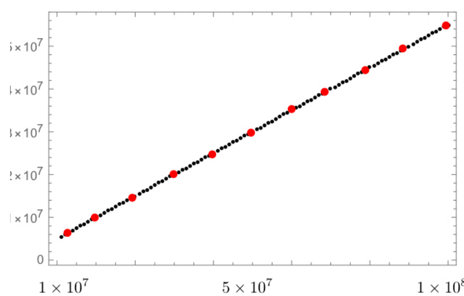
Huge battery of highly sophisticate statistical tests

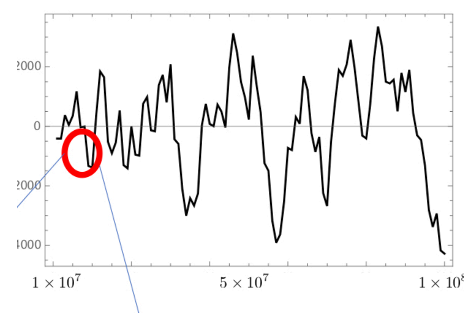


- Longest run of positive values
- Matrix rank test
- Discrete Fourier Transform distribution

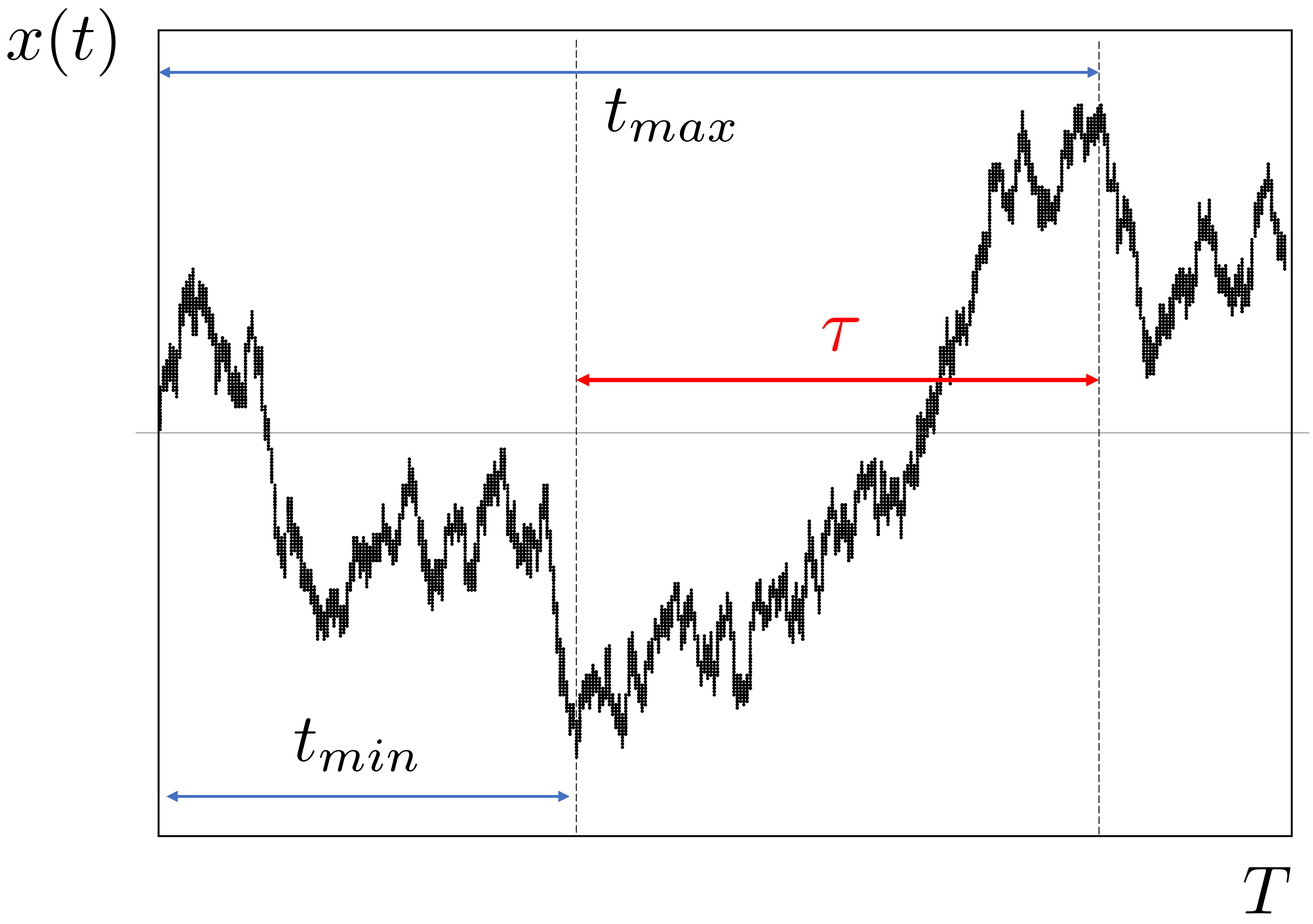


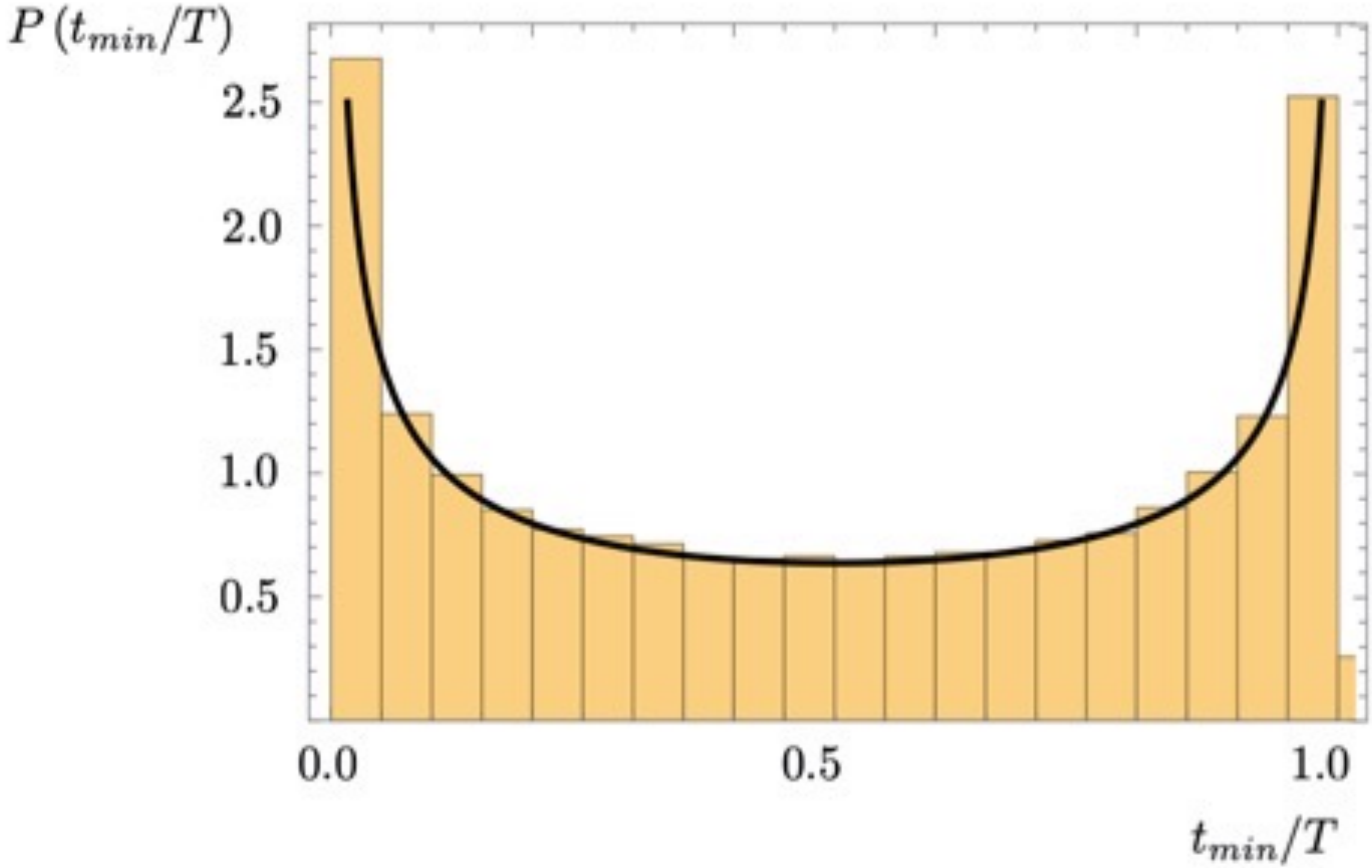
- Min-max time distribution
- Entropy test
- Cumulative sum test
- Random excursion test
- … … …





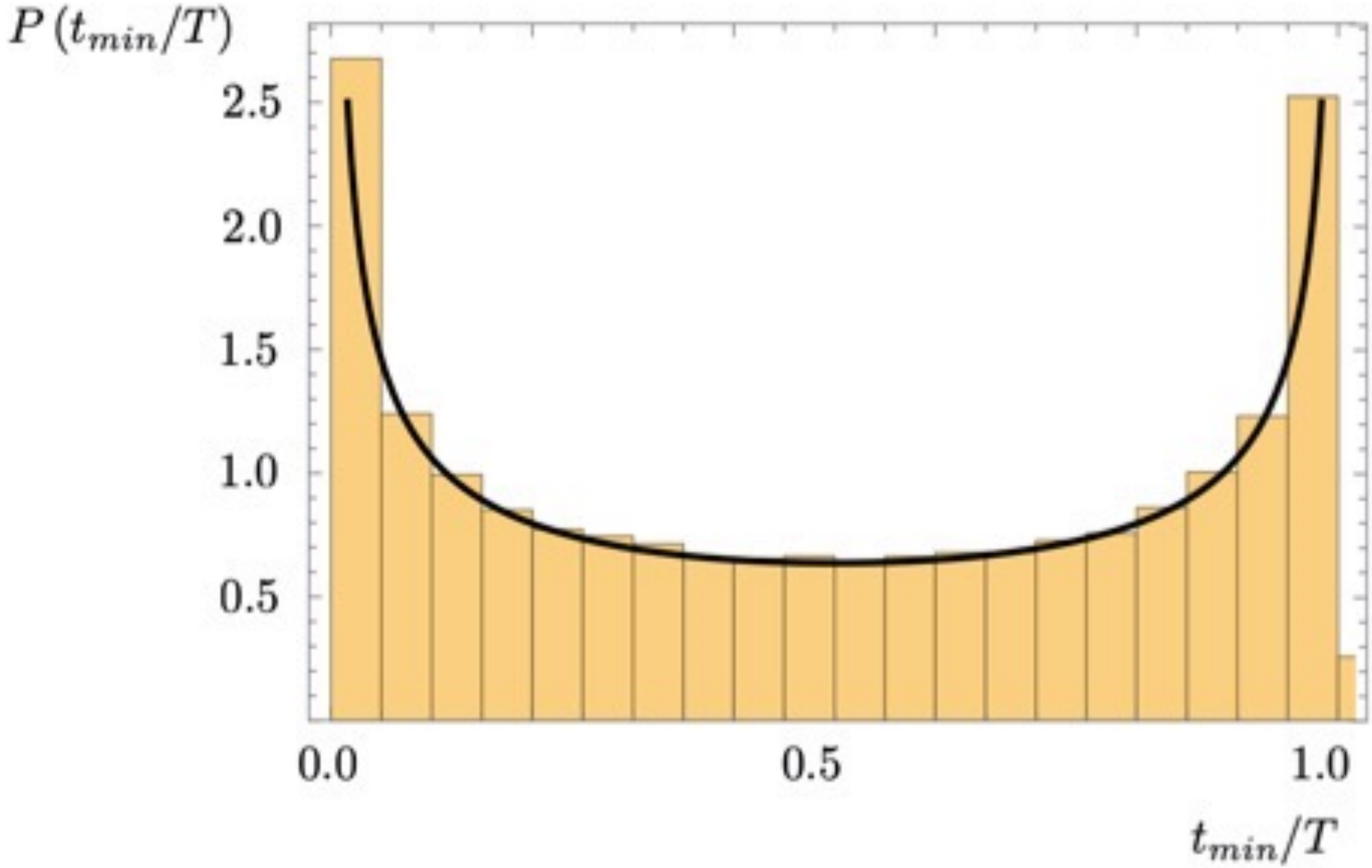
Min-max distribution





$$P(x|T) = \frac{T}{\pi} \frac{1}{\sqrt{x(1-x)}}$$

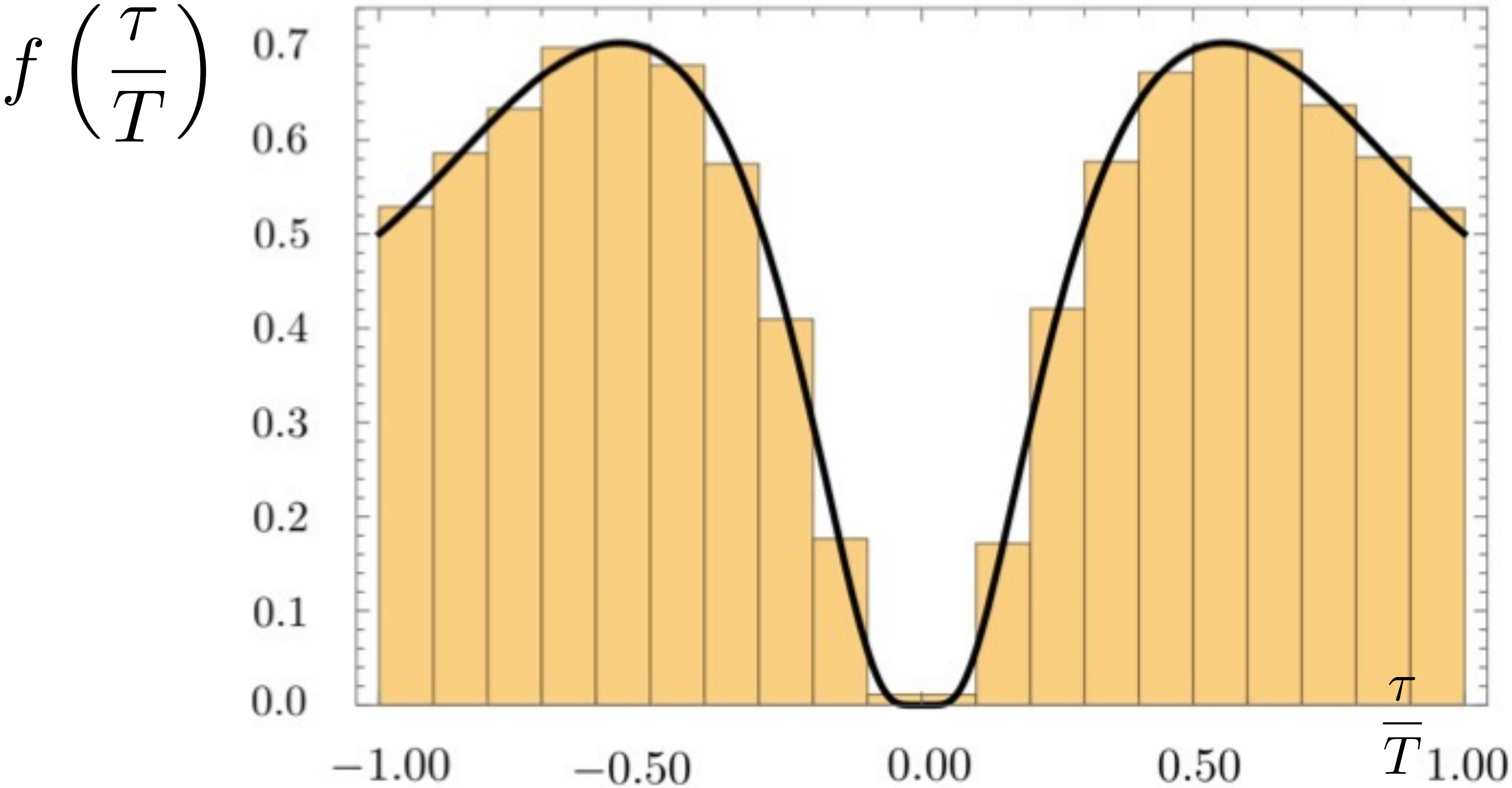
(E.T. Jaynes, The Theory of Probability)



- Pearson χ^2 = 33.35 with d = 35 degrees of freedom

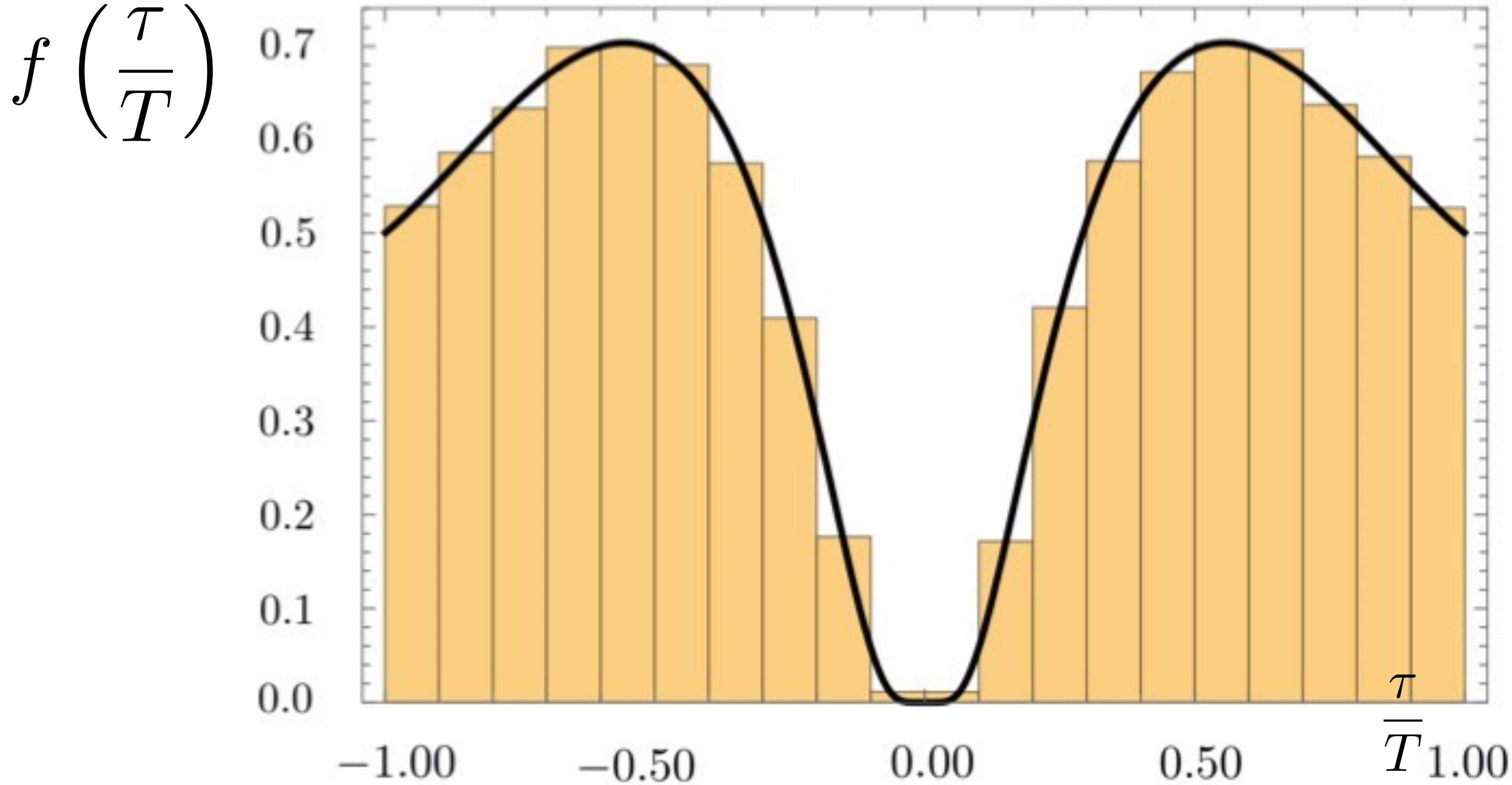
  P-value = 0.66

- Kolmogorov-Smirnov test ➔ 99.99%



$$f(x) = \frac{2(1-|x|)}{x^2} \sum_{m=-\infty}^{\infty} \frac{2m+1}{\sinh\left((2m+1)\pi\pi\sqrt{\frac{1-|x|}{|x|}}\right)}$$

(Mori, Majumdar, Schehr, 2019)



- Pearson χ^2 = 31.12 with d = 35 degrees of freedom

P-value = 0.77

- Kolmogorov-Smirnov test ➔ 99.99%

Variance of the block variables

$$\sigma_L^2 = L\,\lambda(L)$$

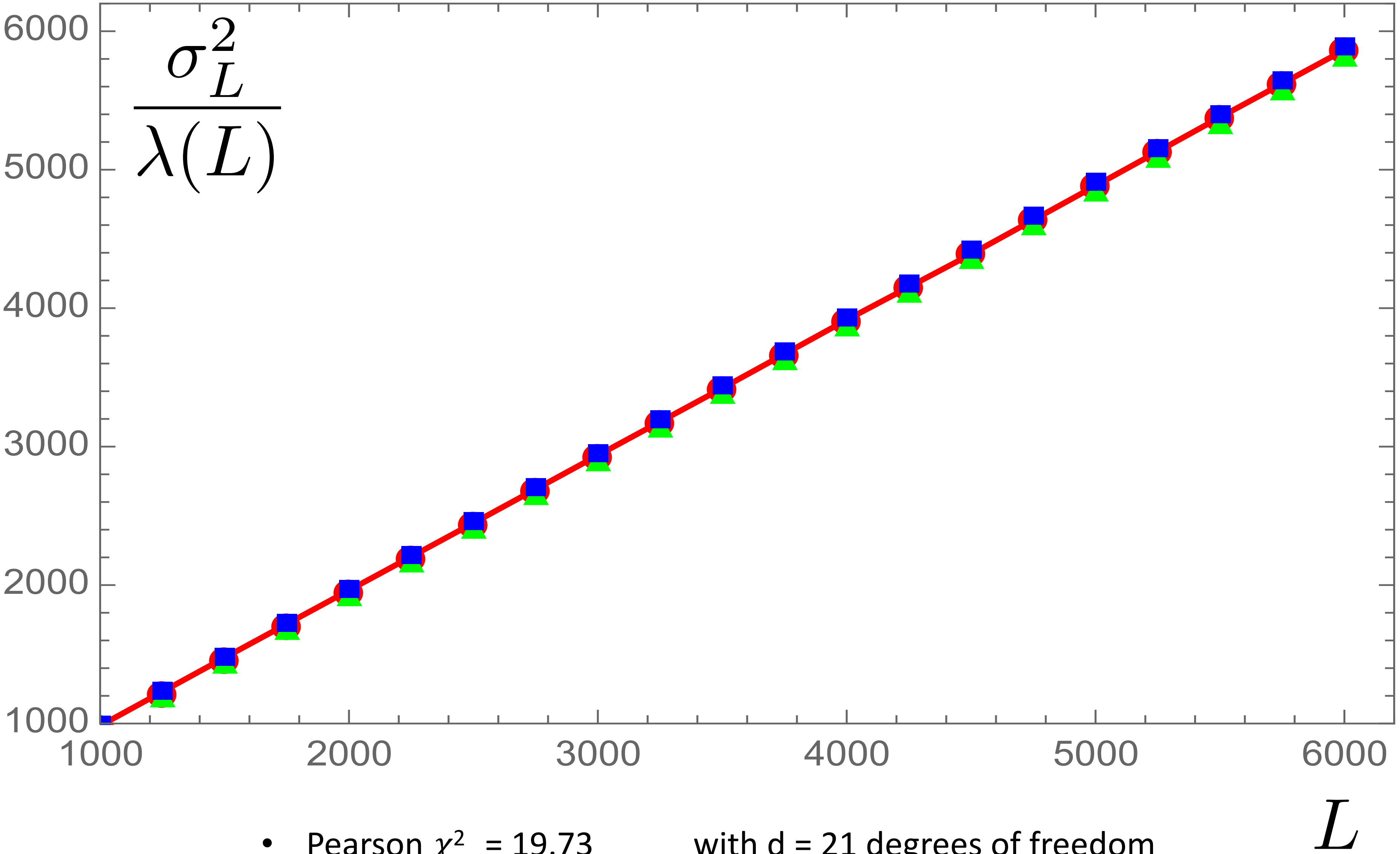
$$\lambda(L) = b^2 \left[1 - \frac{1}{\log N}(\log L - \gamma_E - 1) + \mathcal{O}\left(\frac{1}{\log^{7/4} N}\right)\right]$$

$$b^2 \equiv \frac{1}{r}\sum_{k=1}^{r}\cos^2\phi_k = \begin{cases} 1 & ; \quad \text{if } \chi = \chi^* \\ 1/2 & ; \quad \text{if } \chi \neq \chi^* \end{cases}$$

Hence, for **ANY** ensemble made of sub-interval of length L

$$\frac{\sigma_L^2}{\lambda(L)} = L$$

indipendent of the **modulus q**



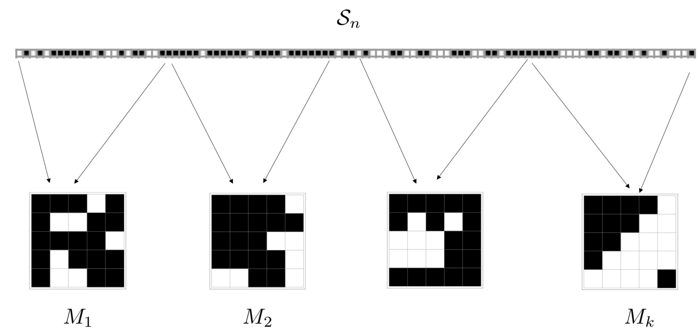
- Pearson χ^2 = 19.73 with d = 21 degrees of freedom

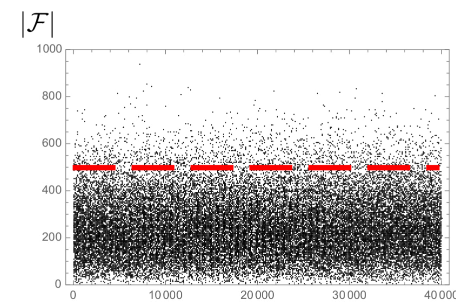
P-value = 0.65

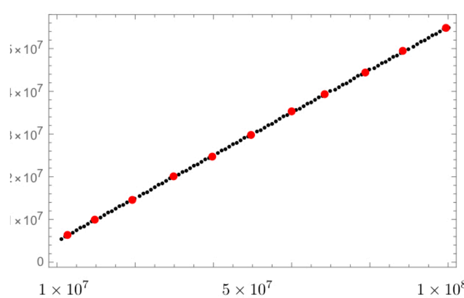
- Kolmogorov-Smirnov test ➔ 99.99%

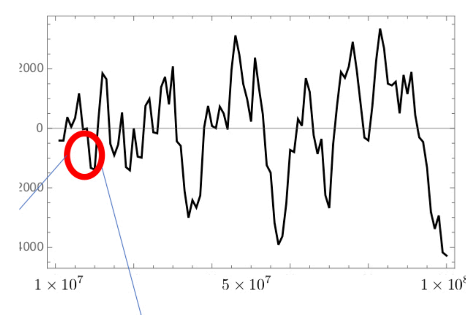
Huge battery of highly sophisticate statistical tests

- Longest run of positive values
- Matrix rank test
- Discrete Fourier Transform distribution
- Min-max time distribution
- Entropy test
- Cumulative sum test
- Random excursion test
- … … …

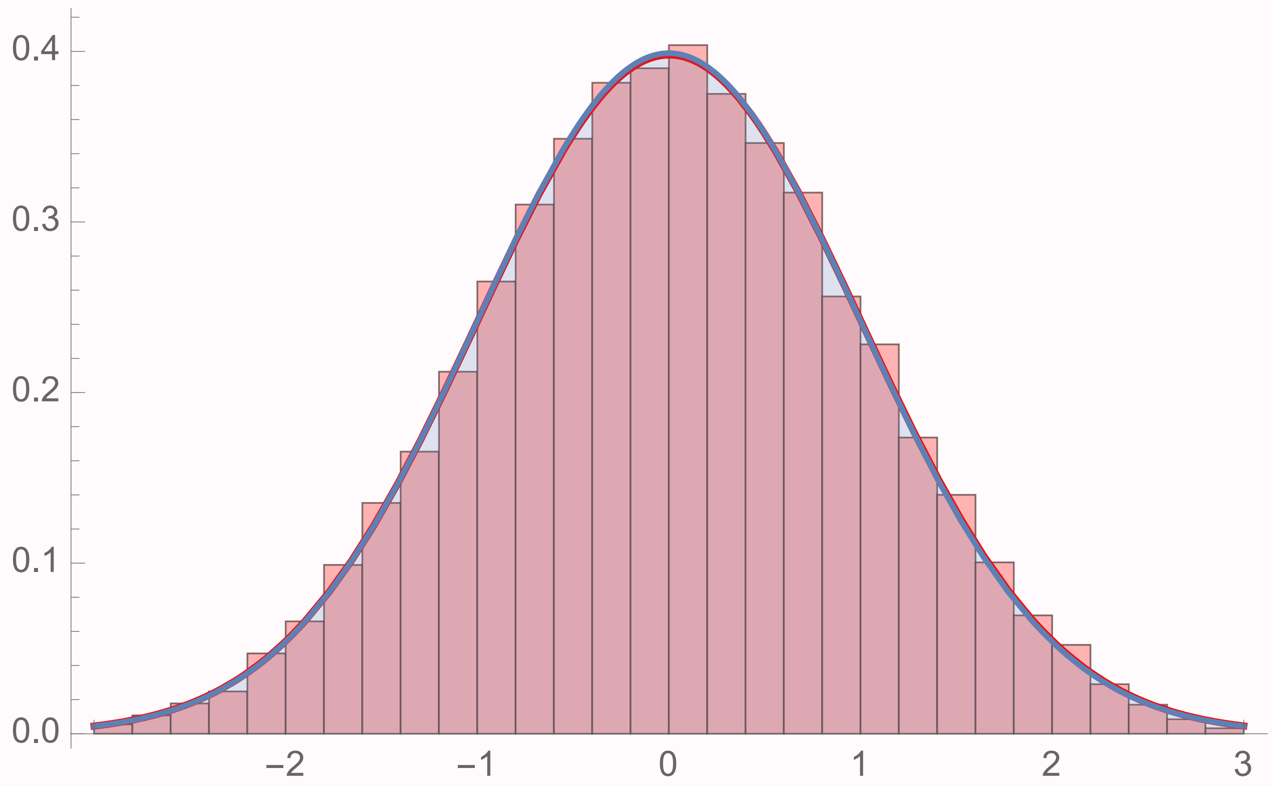








To Whom the Bell Tolls: Universal distribution of the B(N) Function



	Statistic	P-Value
Anderson-Darling	0.260545	0.726021
Baringhaus-Henze	0.487976	0.465461
Cramér-von Mises	0.0461625	0.576565
Jarque-Bera ALM	0.107547	0.946907
Mardia Combined	0.107547	0.946907
Mardia Kurtosis	0.287978	0.773363
Mardia Skewness	0.000700638	0.978883
Pearson χ^2	30.4856	0.492333
Shapiro-Wilk	0.99919	0.923721

Singularity of the Mellin Transform of L-functions

$$\log P(s) \quad = \sigma \int_2^\infty \frac{B(u)}{u^{\sigma+1}} du$$

$$B(N) = \sum_{n=1}^{N} \cos\theta_{p_n} \simeq N^{1/2+\epsilon}$$

The integral is then singular at σ =1/2

Namely, all zeros are on the critical line!

Conclusions

- There are infinitely many functions which, as the Riemann zeta function, has all their zeros along Re(s) = 1/2
- This properties can be traced back to some random properties of the primes
- The validity of the GRH relies on probability theorems involving the residues and the absence of their correlations
- Hence, the (Generalised) Riemann Hypothesis is just

“Random Walk theorem” !!!