GGI School on
"Quantum Computation and Sensing"

Quantum Algorithms and Protocols

LECTURE 3

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Theory and Phenomenology of Fundamental Interactions
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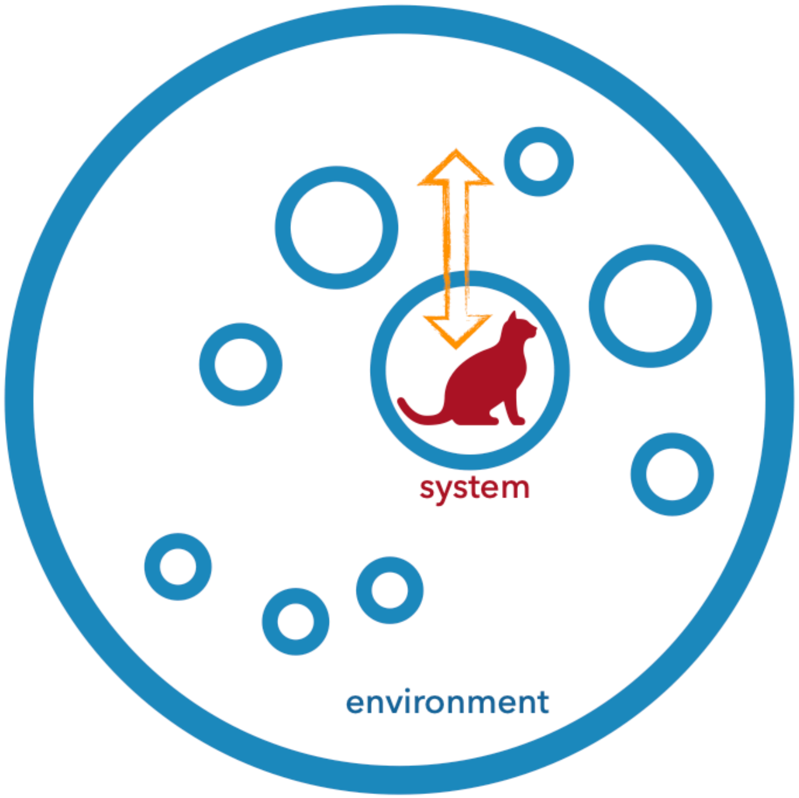
- We want to introduce now interactions with the *environment* that can “disturb” the system



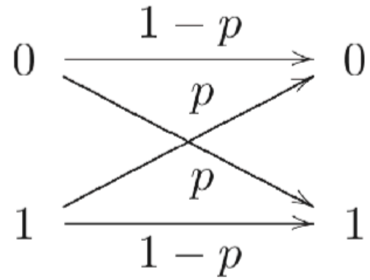
The system is open:

it can exchange energy, matter, … with the environment

The universe (system+environment) is closed,

i.e. completely isolated

- System=classical bit:



the environment can interact with it, resulting in a flipping of the bit with a probability p

Let's discretize time, and denote with

$p_0(0), p_1(0)$ the probability that the bit is in 0/1 at time $t = 0$

$p_0(1), p_1(1)$ the probability that the bit is in 0/1 at time $t = 1$

then

$$\begin{pmatrix} p_0(1) \\ p_1(1) \end{pmatrix} = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix} \begin{pmatrix} p_0(0) \\ p_1(0) \end{pmatrix}$$

We can repeat this for successive times steps, assuming each of them is independent from the previous one

- For a more general system described by the probability vector $\overrightarrow{p}(t)$, we assume its evolution to be described by a stochastic Markov process, for which
  - each time step is independent form the previous ones
  - the evolution is given by the law of conditional probability $p(Y=y) = \sum_x p(Y=y\,|\,X=x)\,p(X=x)$

    where X, Y represent the possible states of the system at time $t, t+1$ respectively.

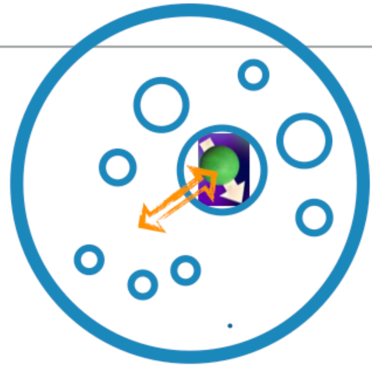
- In other words:

$$\overrightarrow{p}(t+1) = \mathscr{E}\,\overrightarrow{p}(t)$$

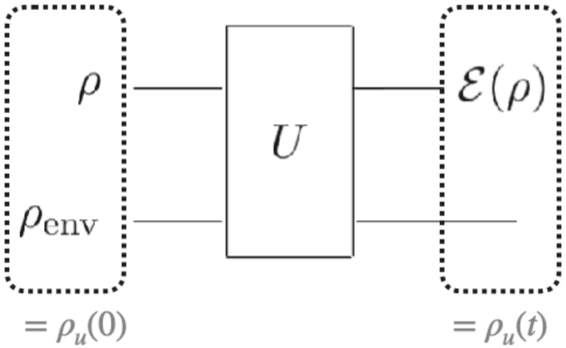
$\mathscr{E}$ is the Evolution matrix , whose coefficients are the transition probabilities $p(Y=y\,|\,X=x)$, satisfying:

positivity = all coefficients are positive

completeness = entires in each column sum to 1



- For our universe we assume that at initial time, the density matrix is a product of a state of the system and a state of the environment
- The system and the environment interact: the total system is closed and hence undergoes a unitary evolution, that might produce entanglement between the two parts.
- At the final time, the system does no longer interact with the environment and its density matrix is obtained by taking the partial trace over the degrees of freedom of the environment



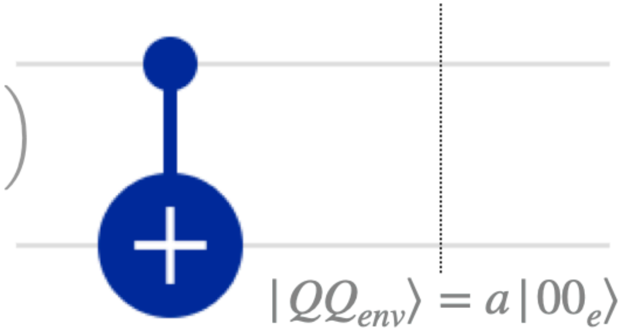
$$\rho_u(0) = \rho \otimes \rho_{env} \mapsto U(t)\rho_u(0) = \rho_u(t) \mapsto Tr_{env}[\rho_u(t)] \equiv \mathscr{E}(\rho)(t)$$

- Theorem: the state of the system is mixed iff the finale state of the universe is entangled

Example: system=qubit environment= qubit unitary evolution=CNOT

$|Q\rangle = a|0\rangle + b|1\rangle\,,\;\; \rho_Q = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix}$

$|Q_{env}\rangle = |0_e\rangle\,, \rho_{env} = |0_e\rangle\langle 0_e|$



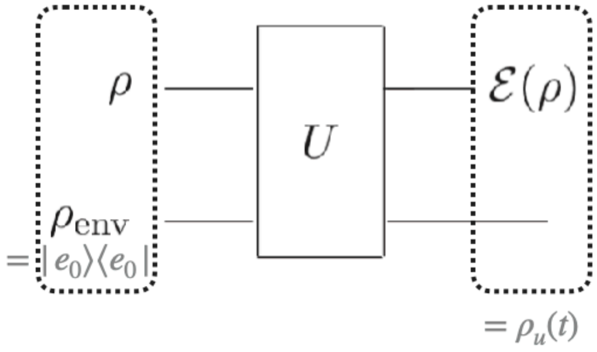
$|QQ_{env}\rangle = a|00_e\rangle + b|11_e\rangle$

$$\mathscr{E}(\rho) = |a|^2|0\rangle\langle 0| + |b|^2|1\rangle\langle 1| = \begin{pmatrix} |a|^2 & 0 \\ 0 & |b|^2 \end{pmatrix}$$

Remark 1. The off-diagonal terms have been lost: passing from a quantum to a classical mixture

Remark 2. A measure of the environment gives the coefficients/probability of the final mixed state of the system

- Let $\{|e_k\rangle\}_k$ be an (orthonormal) set of states of the environment and suppose that at the initial time the environment is in the state $|e_0\rangle$:



$$\mathscr{E}(\rho) = \sum_k E_k \rho E_k^\dagger$$

$$E_k = \langle e_k | U | e_0 \rangle$$

$$\sum_k E_k E_k^\dagger = \mathbb{1}$$

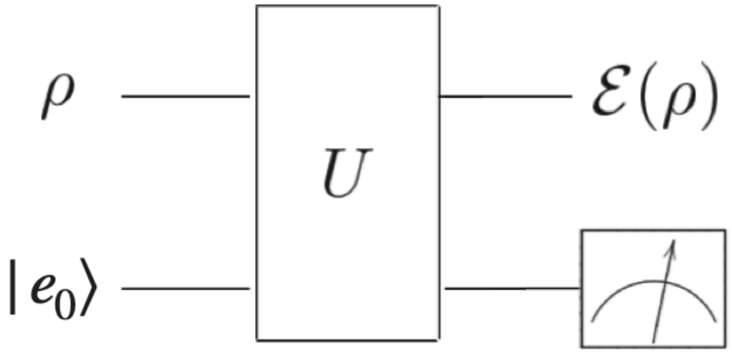
- Notice that $\mathscr{E}(\rho)$, as mixed density matrix, can be written as:

$$\mathscr{E}(\rho) = \sum_k E_k \rho E_k^\dagger = \sum_k p_k \rho_k$$

$$\rho_k = \frac{E_k \rho E_k^\dagger}{Tr[E_k \rho E_k^\dagger]}$$

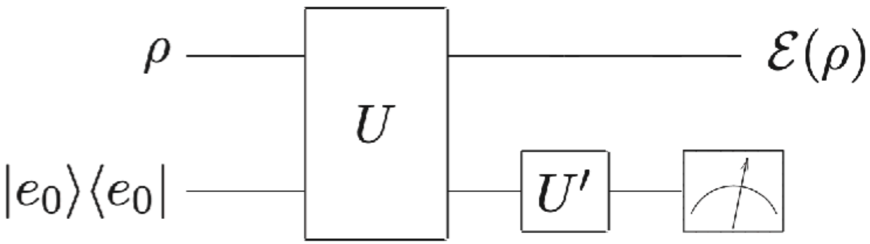
$$p_k = Tr[E_k \rho E_k^\dagger]$$

Thus, making a measurement of the environment in the basis $\{|e_k\rangle\}_k$, gives us the probabilities p_k.



More general definition (that does not refer to the environment):

- The (super)operator $\mathscr{E}$, on the space of density matrices defines a quantum operation iff:
  - $0 \leq Tr[\mathscr{E}(\rho)] \leq 1 \quad \forall \rho$ (if $Tr[\mathscr{E}(\rho)] = 1 \quad \forall \rho$, we say $\mathscr{E}$ is trace-preserving)
  - it is a convex linear map, $\mathscr{E}(\sum_j p_j \rho_j) = \sum_j p_j \mathscr{E}(\rho_j)$
  - it is completely positive (i.e. the operator $(\mathbb{I} \otimes \mathscr{E})(A)$ is positive for any positive operator A for any extension)
- Such a map is of the kind: $\mathscr{E}(\rho) = \sum_k E_k \rho E_k^\dagger$, with $\sum_k E_k E_k^\dagger \leq \mathbb{I}$
- Two sets $E = \{E_1, \cdots, E_m\}$, $F = \{F_1, \cdots, F_m\}$ of operators define the same super-operator $\mathscr{E}$ iff $E_j = \sum_k u_{jk} F_k$ where $U' = [u_{jk}]$ is a unitary transformation



$$\mathscr{E}(\rho) = \sum_k E_k \rho E_k^\dagger \quad , \quad \sum_k E_k E_K^\dagger = \mathbb{I} \qquad \text{with} \qquad E_k = \alpha_k \mathbb{I} + \sum_j \alpha_{kj} \sigma_j$$

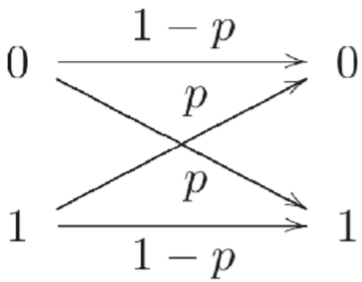
Since $\rho = \frac{\mathbb{I} + \vec{n} \cdot \vec{\sigma}}{2}$ with $|\vec{n}|^2 \leq 1$, one can represent $\mathscr{E}$ via the affine map on the bloch sphere:

$$\mathscr{M} : \vec{n} \mapsto \vec{n}' = M\vec{n} + \vec{c}$$

where

$$M_{jk} = \sum_l \left[a_{lj} a_{lk}^* + a_{lj}^* a_{lk} + \left(|\alpha_l|^2 - \sum_p a_{lp} a_{lp}^* \right) \delta_{jk} + i \sum_p \epsilon_{jkp} (\alpha_l a_{lp}^* - \alpha_l^* a_{lp}) \right]$$

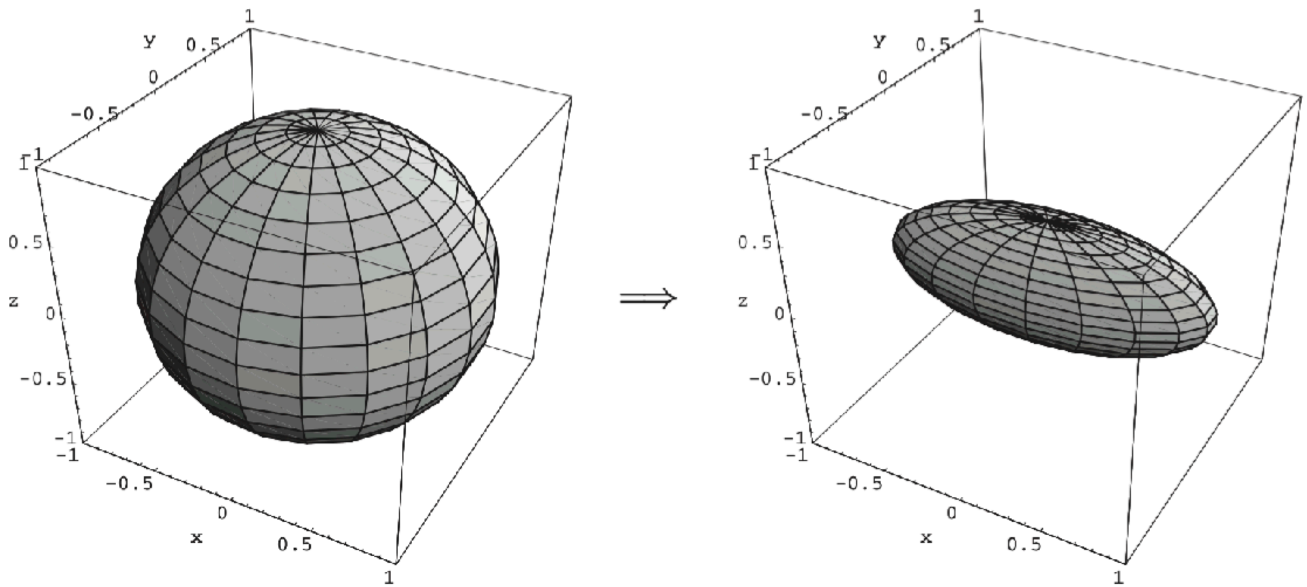
$$c_k = 2i \sum_l \sum_{jp} \epsilon_{jpk} a_{lj} a_{lp}^*,$$



$$E_0 = \sqrt{p}\,\mathbb{I} \;,\; E_1 = \sqrt{1-p}\,X$$

giving the only non-zero coefficients: $\alpha_0 = \sqrt{p}$, $\alpha_{11} = \sqrt{1-p}$ which yield

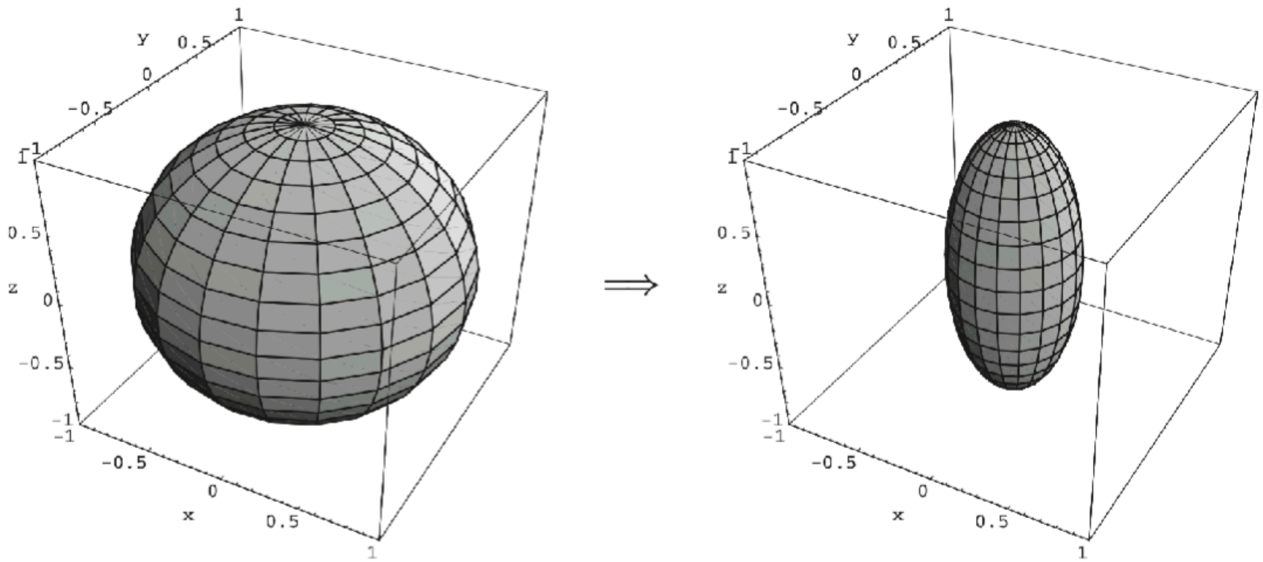
$$M = diag(1, 1-2p, 1-2p) \;,\; \vec{c} = 0$$



$$E_0 = \sqrt{p}\,\mathbb{I} \;\;,\;\; E_1 = \sqrt{1-p}\,Z$$

giving the only non-zero coefficients: $\alpha_0 = \sqrt{p}$, $\alpha_{11} = \sqrt{1-p}$ which yield

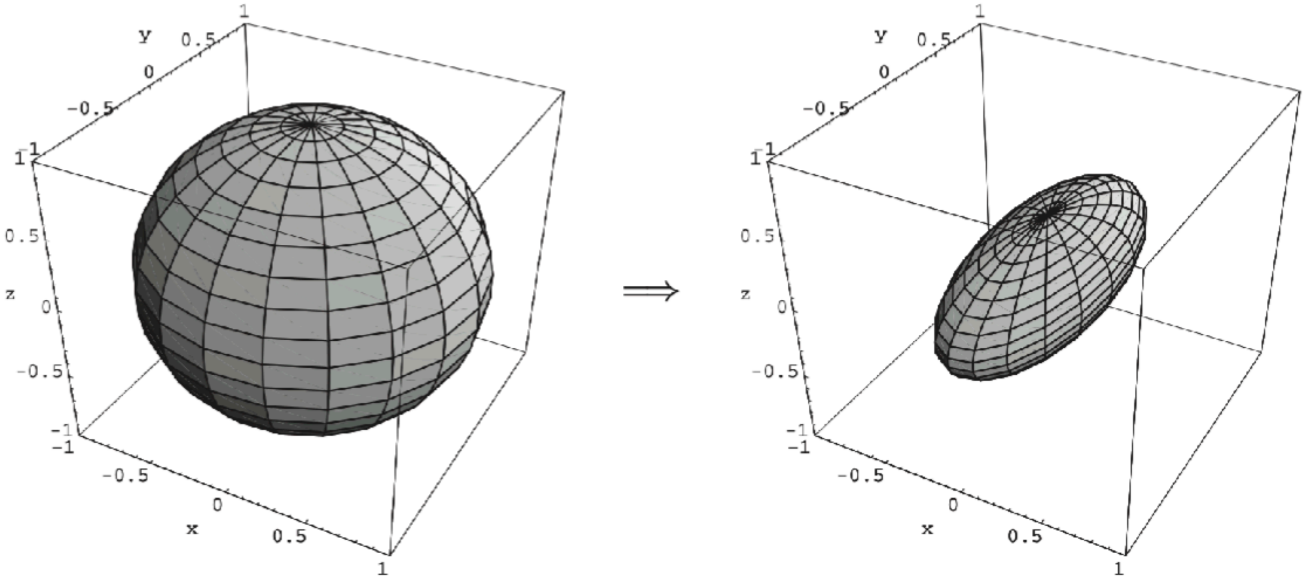
$$M = diag(1-2p, 1-2p, 1) \;,\; \overrightarrow{c} = 0$$



$$E_0 = \sqrt{p}\,\mathbb{1} \;\;,\;\; E_1 = \sqrt{1-p}\,Y \qquad\qquad (ZX = Y)$$

giving the only non-zero coefficients: $\alpha_0 = \sqrt{p}$, $\alpha_{11} = \sqrt{1-p}$ which yield

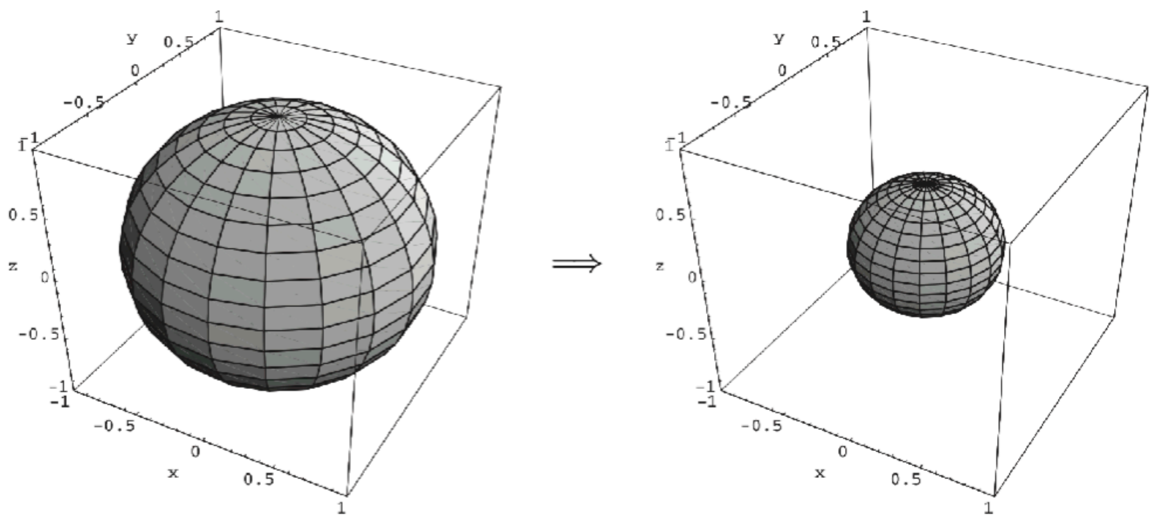
$$M = diag(1-2p, 1, 1-2p) \;,\; \vec{c} = 0$$



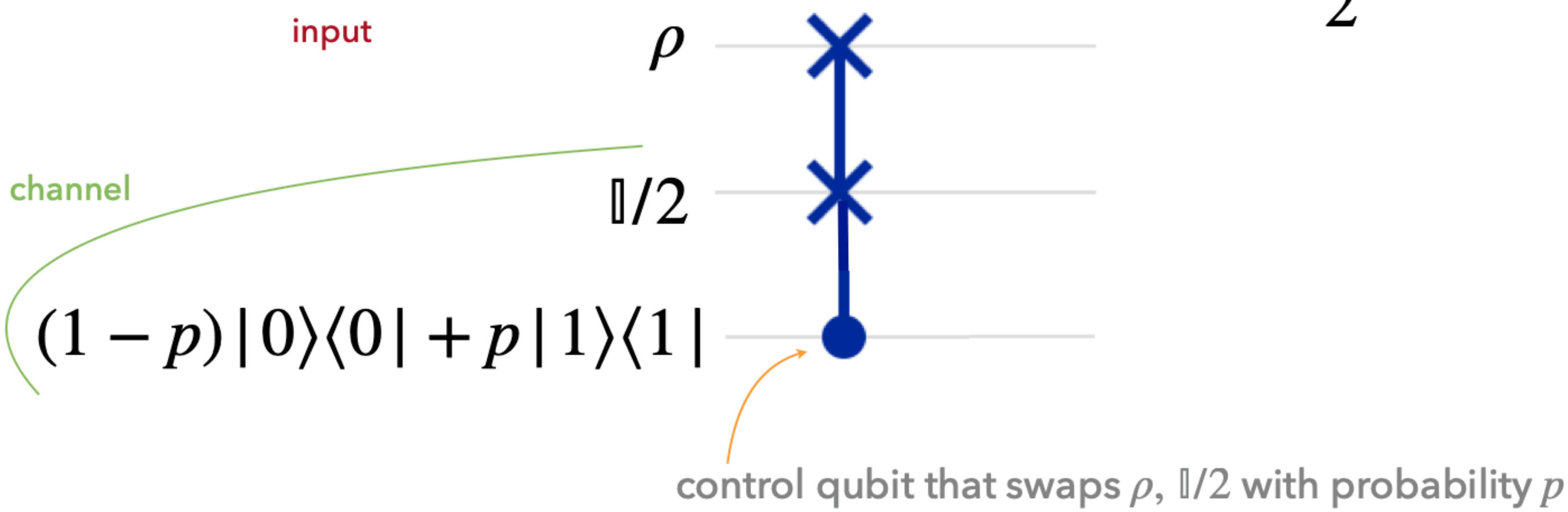
$$\mathscr{E}(\rho) = p\,\frac{\mathbb{I}}{2} + (1-p)\rho$$

$$\left(\frac{\mathbb{I}}{2} = \frac{\rho + X\rho Y + Y\rho Y + Z\rho Z}{4}\right)$$

$$E_0 = \sqrt{1 - \frac{3}{4}p\,\mathbb{I}}\ ,\quad E_1 = \frac{\sqrt{p}}{2}X,\ \ E_2 = \frac{\sqrt{p}}{2}Y,\ \ E_3 = \frac{\sqrt{p}}{2}Z$$



$$\mathcal{E}(\rho) = p\frac{\mathbb{I}}{2} + (1-p)\rho$$



$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}$$

$$\rho = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} \mapsto E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger = \begin{pmatrix} |a|^2 + \gamma |b|^2 & \sqrt{1-\gamma}ab^* \\ \sqrt{1-\gamma}a^*b & (1-\gamma)|b|^2 \end{pmatrix}$$

$$|\,Q\rangle = a\,|\,0\rangle + b\,|\,1\rangle \quad , \quad |\,0\rangle = vacuum \quad , \quad |\,1\rangle = 1\ photon$$

E_0 leaves $|\,0\rangle$ unchanged but reduces the coefficient of $|\,1\rangle$

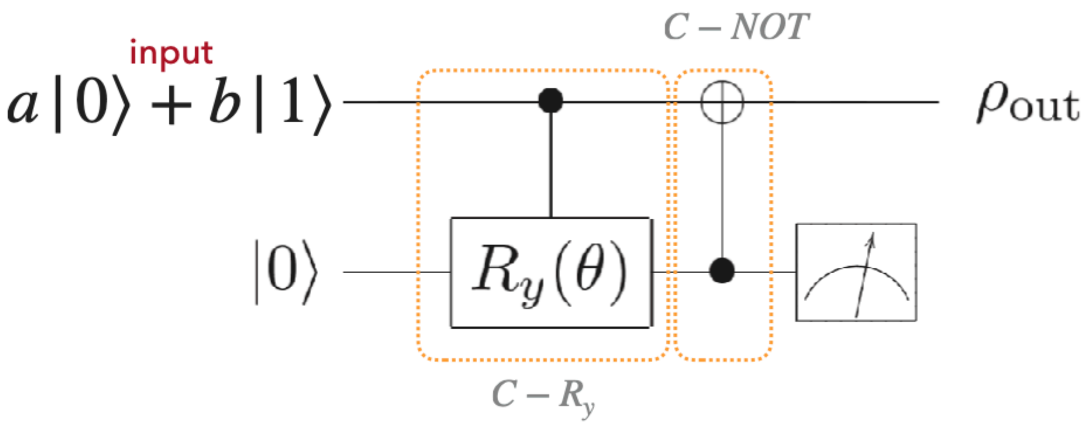
E_1 transforms $|\,1\rangle$ into $|\,0\rangle$

} it is more unlikely to find 1 photon

Remark. This can be realised, e.g. with a beamsplitter

Circuit representation

$$\gamma = \sin^2(\theta/2)$$

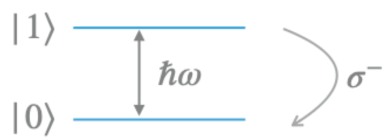


Time-evolution of an open system is given by a differential equations (that replaces Schroedinger equation)

$$\frac{\partial}{\partial t}\rho = -\frac{1}{\hbar}[H, \rho] + \sum_j \left(2L_j\rho L_j^\dagger - \{L_j^\dagger L_j, \rho\}\right)$$

which is called Lindblad equation.

Example. 2-level atom with $H = -\frac{1}{2}\hbar\omega\sigma_z$ $L = L_1 = \sqrt{\gamma'}\sigma^-$



equivalent to amplitude damping with $\gamma = 1 - e^{-2t\gamma'}$

- For physicists, very complete as introduction to both quantum information and computation

MA Nielsen, IL Chuang, Quantum Computation and Quantum Information (Cambridge)

- For physicits, deep in theoretical problems

J. Preskill
Lecture Notes for Physics 229: Quantum Information and Computation
(visit Preskill's webpage)

- From a great physicist, it requires almost no knowledge of quantum theory

N.D. Mermin
Quantum Computer Scince: an Introduction

- Very pedagogical, with exercises and solutions

D. McMahon
Quantum computing explained
(Wiley)