Lattice Field Theory

a primer of methods and Results

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Summary :

- LECTURE 1 : Theoretical Background
- LECTURE 2 : Montecarlo Methods + Results

References :

Montvay, Münster
Smit
Gattringer, Lang
Rothe
Lattice Field Theory

\+ Research Papers

Weinberg
Wightman Streater
QFT

QFT and Path Integrals

⊙ **Ingredients :** 1) A Hilbert space $\mathcal{H}$ 2) A unitary rep. of Poincaré group (Λ, a)

3) $\hat{\phi}(x)$ op. valued distributions

2a) $U(\mathbb{1}, a) = \exp(ia \cdot P)$ with spectrum of $\hat{P}^\mu$ in the forward light cone

3a) $U(\Lambda, a)\hat{\phi}(x)\, U^{-1}(\Lambda, a) = \hat{\phi}(\Lambda x + a)$

3b) $[\phi(x), \phi(y)] = 0$, $(x-y)^2 < 0$

2b) $\exists ! \, |0\rangle \,/\, U(\Lambda, a)|0\rangle = |0\rangle \quad \forall \Lambda, a$

⊙ **Theorem** : Given $W(x_1, \dots, x_n) = \langle 0 | \hat{\phi}(x_1) \dots \hat{\phi}(x_n) | 0 \rangle$, we can reconstruct the whole field theory $(\mathcal{H}, P_\mu, \dots)$ [The W are called Wightman functions.]

[See PCT, spin, statistics and all that - Streater-Wightman]

QFT and Path Integrals

$$W(x_1, \dots, x_n) = (-i)^n \prod_{i=1}^{n} \frac{\delta}{\delta J(x_i)} Z[J] \Big|_{J=0}$$

$$Z[J] = \int \mathcal{D}\phi \, e^{\frac{i}{\hbar} S[\phi] + i\int dx \, J(x)\, \phi(x)}$$

Example: Källen-Lehman representation of the 2pts function [QFT vol 1 - Weinberg]

Problems:
1) "$\mathcal{D}\phi \, e^{iS[\phi]/\hbar}$" needs to be precisely defined
2) A perturbative computation might not capture the interesting physics
3) UV divergences

Lattice Regularization solves all these problems!

Euclidean QFT

- Under the assumptions above, W can be continued analytically to the whole complex plane. In particular to euclidean space-time : $\begin{cases} x_E^4 = i x^0 \\ \vec{x}_E = \vec{x} \end{cases}$
where euclidean Green functions are defined:

$$G_E(\vec{x}_1, x_1^4, \dots, \vec{x}_n, x_n^4) = W((-ix_1^4, \vec{x}_1), \dots)$$

- How to go back to Minkowsky space? $\longrightarrow$ Positivity conditions of Osterwalder and Schroder [Axioms for Eucl. Green functions, Osterwalder Schroder]

- Path integral rep.

$$G_E(x_1^E, \dots, x_n^E) = \frac{1}{Z_E} \int \mathcal{D}\phi \, e^{-S_E/\hbar} \phi(x_1^E) \cdots \phi(x_n^E) \,, \qquad Z_E = \int \mathcal{D}\phi \, e^{-\frac{1}{\hbar} S_E[\phi]}$$

S_E euclidean action

Euclidean P.I.

$$G_E(x_1,\dots,x_n) = \frac{1}{Z_E}\int \mathcal{D}\phi\, e^{-S_E/\hbar} \quad , \quad Z_E = \int \mathcal{D}\phi\, e^{-\frac{1}{\hbar}S_E[\phi]}$$

- Analogy with Stat. Mech. : $Z_E = \int \mathcal{D}\phi\, e^{-\frac{1}{\hbar}S_E[\phi]}$ is formally equal to a classical canonical partition function of a system with (reduced) Hamiltonian $S_E/\hbar$

- At finite temperature $Z = \mathrm{Tr}\{e^{-\beta\hat{H}}\} = \int d\phi \langle\phi| e^{-\beta\hat{H}} |\phi\rangle = \int \mathcal{D}\phi \exp\left\{-\int_0^{\beta\hbar} dx_4 \, L_E\right\}$

  $\phi(\bar{x},0) = \pm\,\phi(\bar{x},\beta\hbar)$

$\longrightarrow$ Montecarlo simulation of QFT at temperature $T = \frac{1}{\beta\hbar} = \frac{1}{L_t}$

Lattice Regularization

Define a hypercubic lattice: $\Lambda = \{ n_\mu a \,/\, n_\mu \in \mathbb{Z}, \mu = 1, \dots, 4 \}$, a lattice spacing

Then:

1) $\int d^4x^E \to a^4 \sum_{n \in \Lambda}$ 2) $(f, g) = a^4 \sum_{n \in \Lambda} f(n) g(n)$

3) $\partial_\mu^E \to \Delta_\mu^+ f = \frac{1}{a}\left(f(n + a\hat{\mu}) - f(n) \right)$, $\Delta_\mu^- f = \frac{1}{a}\left(f(n + a\hat{\mu}) - f(n) \right)$

forward derivative backwards derivative

4) $f(x) = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ikx} \tilde{f}(k) \to f(n) = \int_{-\pi/a}^{\pi/a} \frac{dk}{2\pi} e^{ikna} \tilde{f}(k)$ (1st Brillouin Zone), $\tilde{f}\left(\frac{\pi}{a}\right) = \tilde{f}\left(-\frac{\pi}{a}\right)$

Note: $\tilde{f}(k) = a \sum_{n=-\infty}^{\infty} f(n) e^{-ikna}$

⚠

5) $\mathcal{D}\phi \to \prod_n d\phi(n)$ Path Integration well defined!

Useful: $(\Delta^+ f, \Delta^+ g) = (-\Delta^- \Delta^+ f, g)$ [Exercise for the reader]

Lattice Regularization - Free scalar field

$$S_E = \int d^4x^E \left\{ \frac{1}{2} \left(\partial_\mu^E \phi \, \partial_\mu^E \phi \right) + \frac{1}{2} m_0^2 \phi^2 \right\}$$

- Discretize :

$$S_E^L = \frac{1}{2} \sum_\mu \left(\Delta_\mu^+ \phi , \Delta_\mu^+ \phi \right) + \frac{1}{2} m_0^2 \left(\phi, \phi \right)$$

$$\left(\Delta_\mu^+ \phi , \Delta_\mu^+ \phi \right) = - \left(\Delta_\mu^- \Delta_\mu^+ \phi , \phi \right)$$

$$\Rightarrow \quad \Delta_\mu^- \Delta_\mu^+ \phi = \Delta_\mu^- \frac{1}{a} \left(\phi(n+a\hat{\mu}) - \phi(n) \right) = \frac{1}{a} \frac{1}{a} \left\{ \phi(n+a\hat{\mu}) - \phi(n) - \phi(n) + \phi(n-a\hat{\mu}) \right\}$$

$$= \frac{1}{a^2} \left\{ \phi(n+a\hat{\mu}) + \phi(n-a\hat{\mu}) - 2\phi(n) \right\}$$

Therefore :

$$S_E^L = a^4 \sum_n \left\{ -\frac{1}{2a^2} \sum_\mu \left(\phi(n+a\hat{\mu}) + \phi(n-a\hat{\mu}) \right) \phi(n) + \frac{1}{2} \left(\frac{8}{a^2} + m_0^2 \right) \phi(n)^2 \right\}$$

$$= a^2 \frac{1}{2} \sum_n \left(- \sum_\mu \left(\phi(n+a\hat{\mu}) + \phi(n-a\hat{\mu}) \right) \phi(n) + \frac{1}{2} \left(\frac{8}{a^2} + m_0^2 \right) \phi(n)^2 \right.$$

Lattice Regularization - Free Scalar field

- Since $[S_E] = [\hbar] = 1$ $\qquad \phi = \frac{\hat{\phi}}{a}$, $m_0 = \frac{\hat{m}_0}{a}$

Then

$$S_E^L = \frac{1}{2} \sum_{nm} \hat{\phi}(n)\, K(n,m)\, \hat{\phi}(m)$$

where:

$$K(n,m) = -\sum_\mu \left(\delta(n+a\hat{\mu}, m) + \delta(n-a\hat{\mu}, m) \right) + (8+\hat{m}_0^2)\, \delta(m,n)$$

$$\Rightarrow \quad Z_E = \int \prod_n d\hat{\phi}(n) \exp\left\{ -\frac{1}{2} \sum_{nm} \hat{\phi}(n) K(n,m) \hat{\phi}(m) \right\} \propto \frac{1}{(\det K)^{1/2}}$$

and

$$G(n,m) = \langle \phi(n)\, \phi(m) \rangle = \frac{1}{a^2} \langle \hat{\phi}(n)\, \hat{\phi}(m) \rangle = \frac{1}{a^2} K^{-1}(n,m)$$

From $K^{-1}(n,m)$: 1 particle spectrum!

Lattice Regularization - The Free propagator

$$\delta(n,m) = \int_{-\pi/a}^{\pi/a} \frac{d^4\hat{k}}{(2\pi)^4} e^{ik(n-m)a}$$

$\odot$ $\sum_m K(n,m)\, K^{-1}(m,\ell) = \delta(n,\ell)$, $K^{-1}(m,\ell) = \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} e^{ik(m-\ell)a}\, \hat{\tilde{G}}(k)$

Then

$$\int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \sum_m K(n,m)\, e^{ik(m-\ell)a} \hat{\tilde{G}}(k) = \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4}\, e^{ik(n-\ell)a}$$

Now

$$K(n,m)\, e^{ik(m-\ell)a} = \left[-\sum_\mu \delta(n+a\hat{\mu},m) + \delta(n-a\hat{\mu},m) + (8+\hat{m}^2)\,\delta(n,m) \right] e^{ik(m-\ell)a}$$

$$= \left[-\sum_\mu e^{ik(n+a\hat{\mu}-\ell)} + e^{ik(n-a\hat{\mu}-\ell)} + (8+\hat{m}_0^2)\, e^{ik(n-\ell)} \right]$$

$$= e^{ik(n-\ell)} \left[-\sum_\mu \left(e^{ik_\mu a} + e^{-ik_\mu a} - 2 \right) + \hat{m}_0^2 \right] = e^{ik(n-\ell)a} \left[2\sum_\mu (1-\cos k_\mu a) + \hat{m}_0^2 \right]$$

Thus

$$\hat{\tilde{G}}(k) = \left[4\sum_\mu \sin^2 \frac{k_\mu a}{2} + \hat{m}_0^2 \right]^{-1} = \frac{1}{a^2}\, \tilde{G}(k)$$

Euclidean QFT on the Lattice

$$\hat{\tilde{G}}(k) = \left[4 \sum_\mu \sin^2 \frac{k_\mu a}{2} + \hat{m}_0^2\right]^{-1} = \frac{1}{a^2} \tilde{G}(k)$$

⊙ Naïve continuum limit : $\lim_{a\to 0} \tilde{G}(k) = \lim_{a\to 0} \dfrac{a^2}{4 \sin^2 \frac{k_4 a}{2} + 4 \sum_i \sin^2 \frac{k_i a}{2} + \hat{m}_0^2}$

$$\left(\frac{2}{a}\right) \sin^2 \frac{k_\mu a}{2} = k_\mu^2 \left(\frac{2}{k_\mu a}\right)^2 \sin^2 \frac{k_\mu a}{2} \longrightarrow k_\mu^2$$

$$\Rightarrow \quad \lim_{a\to 0} \tilde{G}(k) = \frac{1}{k_4^2 + \vec{k}^2 + m_0^2} \quad \text{as expected}$$

⊙ Interacting case : Use K-L rep. to obtain $m_R = f(\hat{m}_0, a)$

$$\tilde{G}(x^4, \vec{k}) = \int_{-\pi/a}^{\pi/a} \frac{dk_4}{2\pi} e^{-ik_4 x^4} \hat{\tilde{G}}(k_4, \vec{k}) \underset{x^4\to\infty}{\sim} e^{-E_R(\hat{m}_0, a, \vec{k})}$$

[Exercise in contour integration, Smit]

The continuum limit

- Naïve continuum limit : $a \to 0$, bare quantities fixed

Then $\quad O = \hat{O}\, a^{d_0} \longrightarrow \begin{cases} +\infty & \text{if } d_0 < 0 \\ 0 & \text{if } d_0 > 0 \end{cases}$ TRIVIAL

- Proper continuum limit : $a \to 0$, physical quantities fixed.

<u>Example</u> : Free theory : $m_R = \dfrac{\hat{m}_R}{a} = m_0$, Interacting theory : $m_R = \dfrac{\hat{m}_R}{a} = a^{-1} f(\hat{m}_0)$

If m_R finite as $a \to 0$, then $\hat{m}_R \to 0$ as $a \to 0$.

Since $\hat{m}_R = 1/\hat{\xi}$, $\hat{\xi}$ correlation length , $\hat{m}_R \to 0 \iff \hat{\xi} \to \infty$ 2nd order phase transition

<u>Note</u> : 1) A priori we do not know if and where (in bare parameter space) the transition takes place.

2) Neither do we know to what theory we flow!

- How do we compute m_R ? 1) PERTURBATION THEORY

2) NUMERICALLY (LECT. 2)

The continuum limit

- w.r.t the continuum: Regularization with cutoff $\frac{1}{a}$ $\Leftrightarrow$ discretization
  Renormalization $\Leftrightarrow$ continuum limit

  Note: The lattice regularization breaks at least Poincaré symmetry (But not gauge symmetry)

- Relation between a and bare parameters: $\quad \mathcal{O} = a^{d_0}\,\hat{\mathcal{O}}(\hat{m}_0) \Rightarrow a = F(\mathcal{O}, \hat{m}_0)$

  Example: $\quad m_R = a^{-1}\,\hat{m}_R(m_0) \Rightarrow a = F(m_R, \hat{m}_0)$

  $\longrightarrow$ Scale setting: knowledge of $\mathcal{O}$ allows knowledge of a!

- Then $\quad \mathcal{O}' = a^{d_0}\,\hat{\mathcal{O}}'(\hat{m}_0, a) \Rightarrow \dfrac{\mathcal{O}'}{a^{d_0}} = \hat{\mathcal{O}}'(\hat{m}_0, a) \overset{a \ll 1}{=} \hat{\mathcal{O}}'(\hat{m}_0, 0) + \underbrace{\left.\dfrac{\partial \hat{\mathcal{O}}'}{\partial a}\right|_{a=0} a + \dfrac{1}{2}\left.\dfrac{\partial^2 \hat{\mathcal{O}}'}{\partial a^2}\right|_{a=0} a^2 + \dots}_{\Delta_a \mathcal{O}}$

  $\hat{\mathcal{O}}'(\hat{m}_0, 0)$ continuum limit, $\Delta_a \mathcal{O}$ discretization error.

- Another systematic error: Finite size effects.

Fermion Fields - Regularization

$$\{\gamma^E_\mu, \gamma^E_\nu\} = 2\delta_{\mu\nu}$$

$$\gamma^E_\mu = \gamma^{E\,\dagger}_\mu$$

- Proceed as in the case of the scalar field, except that now $\psi, \bar\psi$ are GRASSMANN

$$\psi(x) \to \psi(n)$$

$$\bar\psi(x) \to \bar\psi(n)$$

$$S_E[\psi,\bar\psi] = \int d^4x_E\, \bar\psi\,(\gamma_\mu \partial_\mu + m_0)\,\psi \quad ,$$

$$S^L_E[\psi,\bar\psi] = a^4 \sum_\Lambda \bar\psi(n)\left(\gamma_\mu \frac{\Delta^+_\mu + \Delta^-_\mu}{2} + m_0\right)\psi(n)$$

$$\frac{1}{2}(\Delta^+_\mu + \Delta^-_\mu)\,\psi(n) = \frac{1}{2a}\left(\psi(n+a\hat\mu) - \psi(n) + \psi(n) - \psi(n-a\hat\mu)\right) = \frac{1}{2a}\left(\psi(n+a\hat\mu) - \psi(n-a\hat\mu)\right)$$

$$S^L_E[\psi,\bar\psi] = a^4 \sum_\Lambda \bar\psi(n)\left[\sum_\mu \gamma_\mu\left(\psi(n+a\hat\mu) - \psi(n-a\hat\mu)\right)\frac{1}{2a} + m_0\,\psi(n)\right]$$

$$= a^3 \sum_\Lambda \bar\psi(n) \sum_\mu \gamma_\mu\left(\psi(n+a\hat\mu) - \psi(n-a\hat\mu)\right)\frac{1}{2} + m_0 a\, \bar\psi(n)\psi(n)$$

$$\psi(n) = a^{-3/2}\hat\psi(n) \quad , \quad \bar\psi(n) = a^{-3/2}\hat{\bar\psi}(n) \quad , \quad m_0 = \frac{\hat m_0}{a}$$

$$= \sum_{n,m} \hat{\bar\psi}(n)\, D(n,m)\, \hat\psi(m) \quad , \quad D(n,m) = \frac{1}{2}\sum_\mu \gamma_\mu\{\delta(n+a\hat\mu, m) + \delta(n-a\hat\mu, m)\} + \hat m_0\,\delta(n,m)$$

- As a result $Z = \int \mathcal{D}\bar\psi\,\mathcal{D}\psi\, e^{-\sum_{nm}\hat{\bar\psi}\,D\,\hat\psi} \propto \det D$ and $\langle \hat\psi(n)\,\hat{\bar\psi}(m)\rangle = \hat D^{-1}(n,m)$

$$= a^3\, D^{-1}(n,m)$$

Fermion Fields - Propagator

- $$\sum_m D(n,m)\, G(m,\ell) = \delta(n,\ell)$$

$$\int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4} \sum_m D(n,m)\, e^{ik(m-\ell)a}\, \tilde{G}(k) = \int_{-\pi/a}^{\pi/a} \frac{d^4k}{(2\pi)^4}\, e^{ik(n-\ell)a}$$

[Exercise : Perform the calculation]. Solution : $\tilde{G}(k) = \dfrac{\hat{m}_0 - i\gamma_\mu \sin k_\mu a}{\hat{m}_0^2 + \sum_\mu \sin^2 k_\mu a}$

- Now set $\hat{m}_0 = 0$, then $\tilde{G}(k) = \dfrac{i\gamma_\mu \sin(k_\mu a)}{\sum_\mu \sin^2(k_\mu a)} = \dfrac{i\gamma_\mu \sin(k_\mu a)}{\sin^2(k_4 a) + \sum_i \sin^2(k_i a)}$

  The pole of the propagator is at $k_\mu = 0, \pm \pi/a$!

- In the naive continuum limit, we have a theory of $2^4 = 16$ fermions

  This is an example of **Fermion Doubling** : We flow to a different theory than expected !

Fermion Fields - Fermion Doubling

- The doubling also happens for $m_0 \neq 0$, One can see it from the large x^4 behaviour of $G(x^4; \vec{k})$, as before [Exercise! See Smit]

- Other discretization are possible: ok if naive cont. lim correct

- Nielsen - Ninomiya Theorem: Given $S_E^L[\psi, \bar{\psi}] = \sum_{nm} \hat{\bar{\psi}}(n)\, D(n,m)\, \hat{\psi}(m)$,

we cannot have simultaneously

1) D local $\to$ D^{-1} periodic analytic function of p_μ

$\downarrow$

Violation means singular derivatives and singular Ward Id.

3) $D(p)$ invertible for $p \neq 0$

2) $\tilde{D}(p) = i\gamma_\mu p_\mu + O(ap^2)$

} Single Dirac Fermion as $a \to 0$

4) $\{\gamma_5, D\} = 0$ Chiral symmetry

- Examples: Naive Fermions $\to$ violate 3

Wilson Fermions $\to$ violate 4

Staggered Fermions $\to$ violate 1

⋮

Fermion Fields - Wilson Fermions

⊙ Add to the naïve action the term $\Delta S_E^L = \frac{r}{2} \sum_n \hat{\bar{\psi}}(n)\, \Delta_\mu^- \Delta_\mu^+ \psi(n)$

Then:

$$D^W(n,m) = (\hat{m}_0 + 4r)\,\delta(n,m) - \frac{1}{2}\sum_\mu (r-\gamma_\mu)\,\delta(m,n+a\hat{\mu}) + (r+\gamma_\mu)\,\delta(m,n-a\hat{\mu})$$

$\longrightarrow$ Chiral symmetry is broken even for $\hat{m}_0 = 0$

⊙ After a short calculation one has: $M(\vec{p}) = M + \frac{2r}{a}\sum_\mu \sin^2\left(\frac{p_\mu a}{2}\right)$

So at fixed p_μ: 1) $\lim_{a\to 0} M(p) = M$

2) If $p_\mu = \pm\frac{\pi}{a}$, $\lim_{a\to 0} M(p) = +\infty$

DECOUPLING OF THE DOUBLERS

⊙ However, Chiral sym. explicitly broken even at $\hat{m}_0 = 0$.

Gauge invariance

- The action $$S[\psi,\bar{\psi},A_\mu] = \int d^4x \left\{ \bar{\psi}(\not{D}+m_0)\psi - \frac{1}{4g_0^2}\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}) \right\}$$
is invariant under the gauge transformation, where $D_\mu = \partial_\mu - iA_\mu$

and $\psi'(x) = \Omega(x)\psi(x)$, $\bar{\psi}'(x) = \bar{\psi}(x)\Omega^\dagger(x)$, $A'_\mu(x) = \Omega(x)\left(A_\mu(x) - i\partial_\mu\right)\Omega^\dagger(x)$

$$\Omega(x) = \exp(i\alpha^a(x)T^a) \in \mathcal{G}, \quad T^a \in g, \quad [T^a, T^b] = i f^{ac}{}_{b}\, T^c$$

and

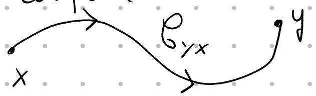
$$F^{\mu\nu} = [D^\mu, D^\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + i g_0 [A_\mu, A_\nu]$$

- Gauge invariance $\longleftrightarrow$ parallel transport

$\psi(x)$ transforms in an IRREP of $\mathcal{G}$ at x.

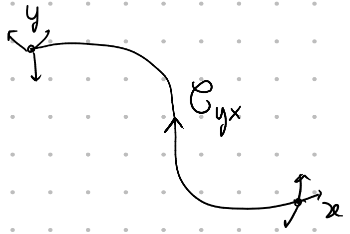
To compare $\psi(x)$ and $\psi(y)$ — $\psi(y)$ and $U(\mathcal{C}_{yx})\psi(x)$ can be compared

$\uparrow$ PARALLEL TRANSPORTER



Parallel Transport and the Wilson action

- Parallel transporters: Path ordered $U(\mathcal{C}_{yx}) = P\exp\left\{ i\int_{\mathcal{C}_{yx}} A_\mu \, dx^\mu \right\}$



Properties:

$$U(\emptyset) = \mathbb{1}$$

$$U(\mathcal{C}_2 \circ \mathcal{C}_1) = U(\mathcal{C}_2)\,U(\mathcal{C}_1)$$

$$U(-\mathcal{C}) = U^{-1}(\mathcal{C}) \quad \left(= U^\dagger(\mathcal{C})\right)$$

Moreover, under a gauge transformation: $U(\mathcal{C}_{yx}) = \Omega(y)\,U(\mathcal{C}_{yx})\,\Omega(x)^{-1}$

- We can build several gauge invariant

1) $\bar{\psi}(x)\,U(x,y)\,\psi(y)$

2) Wilson loops: $W(\Gamma) = \mathrm{Tr}\{U(\Gamma)\} = \mathrm{Tr}\left\{P\exp\left\{ i\oint_\Gamma A_\mu \, dx^\mu \right\}\right\}$

and gauge covariant objects

1) $D_\mu\psi(x) = \lim_{h\to 0}\frac{1}{h}\left(U(x, x+h)\,\psi(x+h) - \psi(x)\right)$

[Ex. demonstrate gauge invariance]

Gauge invariant Fermions

- Parallel transporters are easily discretized: $U_\mu(n) = U(n, n+a\hat{\mu}) = \exp\left\{ i g_0 \int_n^{n+a\hat{\mu}} A_\alpha dx^\alpha \right\}$

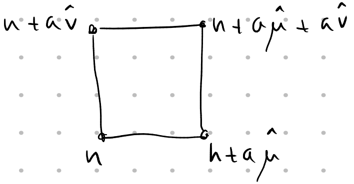
One can then easily discretize the Dirac + gauge action:

$\rightarrow$ $U_\mu(n)\hat{\psi}(n+a\hat{\mu}) - U_\mu^\dagger(n)\hat{\psi}(n-a\hat{\mu})$ instead of $\hat{\psi}(n+a\hat{\mu}) - \hat{\psi}(n-a\hat{\mu})$

Therefore:

$$D^U(n,m) = -\frac{1}{2}\sum_\mu \left[U_\mu(n)(r-\gamma_\mu)\,\delta(m, n+a\hat{\mu}) + U_\mu^\dagger(n)(r+\gamma_\mu)\,\delta(m, n-a\hat{\mu}) \right] + (\hat{m}_0 + 4r)$$

The Gauge Action



- The gauge system by itself is interesting!

- What is the gauge action? Consider the simplest (non-trivial) gauge invariant object with only $\{U_\mu(n)\}$: $U(\partial p) = U_{\mu\nu}(n) = U_\mu(n)\, U_\nu(n+a\hat{\mu})\, U_\mu^\dagger(n+a\hat{\mu}+a\hat{\nu})\, U_\nu^\dagger(n+a\hat{\nu})$

  The Wilson action is given by: $S_W = \beta \sum_{n,\mu>\nu} \left(\mathbb{1} - \frac{1}{2N} \mathrm{Re\,Tr}\, U_{\mu\nu}(n) \right)$

- Naive continuum limit: $U_\mu(n) = e^{iaA_\mu} = e^{iB_\mu(n)}$

This allow us to compute the "map" between a and g_0

$$B_\nu(n+\mu) \sim B_\nu(n) + a\Delta_\mu^+ B_\nu(n),$$

$$B_{-\mu}(n+\hat{\mu}+\hat{\nu}) \sim -B_\mu(n+\nu) \sim -B_\mu(n) - a\Delta_\nu^+ B_\mu(n)$$

$$B_{-\nu}(n+\hat{\nu}) \sim B_\nu(n)$$

$$U_{\mu\nu}(n) \sim e^{iB_\mu(n)}\, e^{iB_\nu(n) + ia\Delta_\mu^+ B_\nu(n)}\, e^{-iB_\nu(n) - ia\Delta_\nu^+ B_\mu(n)}\, e^{iB_\nu(n)}$$

Using the BCS formula: $e^x e^y = e^{x+y+\frac{1}{2}[x,y]}$ we obtain [Exercise!]

$$U_{\mu\nu}(n) \sim \exp\{ ia(\Delta_\mu^+ B_\nu - \Delta_\nu^+ B_\mu) - [B_\mu, B_\nu] \} \sim \exp\{ ia^2 F_{\mu\nu} \}, \quad F_{\mu\nu} = F_{\mu\nu}^i \tau^i$$

The Gauge action - Naïve Continuum Limit

⊙ Expanding S_W in powers of a:

$$S_W \simeq \beta \sum_{n,\mu,\nu} \Big(\underbrace{\mathbb{1} - \mathrm{Re}\,\mathrm{Tr}\,(\mathbb{1}}_{0} + \underbrace{i a^2 F_{\mu\nu}}_{0} - \frac{1}{2} a^4 F_{\mu\nu}^2 + \dots) \Big) \simeq \beta\, a^4 \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{N} \sum_{n,\mu,\nu} \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right)$$

$$\xrightarrow[a\to 0]{} \frac{1}{4 g_0^2} \int d^3x\, \mathrm{Tr}\left(F_{\mu\nu} F^{\mu\nu}\right) \quad \text{iff} \quad \frac{\beta}{8N} = \frac{1}{4 g_0^2} \Rightarrow \beta = \frac{2N}{g_0^2}$$

⊙ The parallel transporters are elements of the gauge group: $Z_{YM} = \int \mathcal{D}U\, e^{-S_W[U]}$

Where $\mathcal{D}U = \prod_{(n,\mu)} dU_\mu(n) \longleftarrow$ Haar (invariant) measure on $\mathcal{G}$: $d(\Omega\, U_\mu(n)) = d(U_\mu(n))$ $\forall \Omega \in \mathcal{G}$

$\longrightarrow$ The path integral is a multidimensional integral on $\mathcal{G} \otimes \dots \otimes \mathcal{G}$

$\longrightarrow$ No need for F.P. ghosts

Gauge Fields - The Wilson action

- In a theory of fermions and gauge fields:

$$\langle O(\psi,\bar\psi,U)\rangle = \frac{1}{Z}\int \mathcal{D}U\,\mathcal{D}\psi\,\mathcal{D}\bar\psi\; O(\psi,\bar\psi,U)\, e^{-S_W - S_D}$$

This is the object that is usually computed with MC simulations [Lecture 2]

- The lattice regularization is a nonperturbative regularization of euclidean QFT

Scalar, Fermion, Gauge fields

↳ ⚠ Nielsen - Ninomiya Theorem

- The Lattice introduce two systematic errors

1) Finite lattice spacing a
2) Finite Volume

} In any numerical calculation, these must be "sent" to 0.

LATTICE FIELD THEORY: A survey of methods and Results

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LECTURE 2.

- Montecarlo methods [Fermions Schematically]
- Glueballs in $Sp(2N)$ gauge theories

Introduction

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi\, \mathcal{D}U \;\; O(\psi, \bar{\psi}, U)\, e^{-S_W - S_{YM}}$$

- Several methods to compute $\langle O \rangle$: i) Strong Coupling Approximation $\left(\beta = \frac{2N}{g_0^2} \ll 1 \right) \rightarrow$ Not analytically connected to continuum limit

  ii) Weak Coupling Approximation (Lattice Perturbation Theory)

  iii) Markov Chain Montecarlo Simulation $\boxed{MCM}$

- MCM : Family

$$\langle O \rangle = \lim_{N \to \infty} \sum_{i=1}^{N} O(C_i) \qquad C_i \text{ cnfs of the system} : \{\psi, \bar{\psi}, U\}$$

  if $\{C_i\}$ are distributed according to $\pi(c)\,dC = e^{-E(c)}\,dC$ where $E(c) = S_W + S_{YM}$.

- How to sample $e^{-E(c)}\,dC$? $\{C\}$ VERY high dimensional space

  Dynamic MCM : Define a stochastic process p s.t. $\{ \cdots \xrightarrow{p} C_i \xrightarrow{p} \cdots \xrightarrow{p} C_e \xrightarrow{p} \cdots \}$

  with $\{C_i\}$ distributed with $\pi(c)$.

Markov Chains

- Consider a discrete (for simplicity) state space $\mathcal{C}$

  Markov chain on $\mathcal{C}$: A sequence $\{C_i\}_{i=1,\dots,N}$ such that State C_{t+1} only depends on state C_t.

- Not any Markov chain: Let p_{ij} = probability$(C_i \to C_j)$

1) Ergodicity : $\forall C_i, C_j$, $\exists n \geq 0$ such that $p_{ij}^n > 0$ $\rightarrow$ From each state you can get to any other state.

1) distribution $\pi(c) = e^{-E(c)}$ must be equilibrium distribution : $\pi(C_i) = \sum_j p_{ij}\pi(C_j)$

Note : 1) The stochastic process needs not be physical, the "simulation time" is not physical time.

2) Irrespective of $\pi(c)$ ⊛, averages will have fluctuations that behave as $\sim \frac{1}{\sqrt{N}}$ for $N \to \infty$

3) What about Autocorrelations ? Efficiency ? [The algorithm may produce the correct distribution but may be terribly inefficient]

⊛ Central limit Theorem

Autocorrelations

- In general: $\langle O\rangle \pm \sigma_{\langle O\rangle} = \langle O\rangle \pm \frac{\sigma_O}{\sqrt{N}}$ where $\langle O\rangle = \sum_j \pi_j O(c_j)$, $\sigma_O^2 = \sum_j (O_j - \langle O\rangle)^2$

$\longrightarrow$ estimators: $\bar{O} = \frac{1}{N}\sum_i O_i$, $\mathrm{var}(O) = \frac{1}{N-1}\sum_i (O_i - \bar{O})^2$, $\{O_i\}$ obtained in simulations

- Autocorrelations: $\mathrm{var}(\bar{O}) = \langle (\bar{O} - \langle O\rangle)^2\rangle = \left\langle \left[\frac{1}{N}\sum_i (O_i - \langle O\rangle)\right]^2 \right\rangle$

$$= \left\langle \frac{1}{N^2} \sum_i (O_i - \langle O\rangle) \sum_j (O_j - \langle O\rangle) \right\rangle = \frac{1}{N^2}\sum_{ij} \langle (O_i - \langle O\rangle)(O_j - \langle O\rangle)\rangle$$

$\rightarrow$ If data is not correlated: $\langle O_i O_j\rangle = \langle O_i\rangle\langle O_j\rangle$ and $\sigma_{\bar{O}} = \frac{\sigma_O}{\sqrt{N}}$

$\rightarrow$ If data is correlated, $\sigma_{\bar{O}} \simeq \sigma_O \sqrt{\frac{2\tau_{int}}{N}} = \frac{\sigma_O}{\sqrt{N_{eff}}}$

[Ex. do the calculation]

[τ_{int} Integrated autocorrelation time]

$$\tau_{int} = \frac{1}{2} + \sum_{t=1}^{N} \frac{\langle (O_i - \langle O\rangle)(O_{i+t} - \langle O\rangle)\rangle}{\langle O^2\rangle - \langle O\rangle^2}$$

Note: Each observable has its own τ_{int}

Montecarlo Simulation - The Metropolis-Hastings algorithm

[No sinks, No sources]

Detailed Balance

- Sufficient condition for equilibrium : $\pi_\ell p_{\ell m} = \pi_m p_{m\ell} \quad \forall \ell, m$ (1)

  "enters ℓ" "exits ℓ"

- Metropolis-Hastings : General Framework : propose + accept

[Prob. of going from i to j]

$$(2) \quad \begin{cases} p_{ij} = \underset{\uparrow \text{"propose a change"}}{p_{ij}^{(0)}} \; \underset{\nwarrow \text{"accept the change"}}{a_{ij}} \quad (i \neq j) \\ p_{ii} = \underset{\uparrow \text{"propose no change"}}{p_{ii}^{(0)}} + \sum_{j \neq i} \underset{\uparrow \text{"propose change"}}{p_{ij}^{0}} \underset{\nwarrow \text{"change not accepted"}}{(1 - a_{ij})} \end{cases}$$

- Plugging (2) in (1) : $a_{ij} = \dfrac{\pi_j p_{ji}^{(0)}}{\pi_i p_{ij}^{(0)}} a_{ji}$. In general $a_{ij} = f\left(\dfrac{\pi_j p_{ji}^{(0)}}{\pi_i p_{ij}^{(0)}} \right) = f(z)$

  so this is equivalent to $f(z) = z \, f\left(\frac{1}{z}\right)$. Possible (not unique) solution : $f(z) = \min(1, z)$

  ↳ Ex : $f(z) = \dfrac{z}{1+z}$

MH algorithm - The Wilson Gauge Action

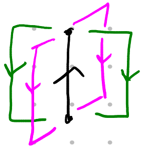
- For a pure gauge theory: $Z_{YM} = \int \mathcal{D}U \, e^{-S_W}$ with $S_W = \beta \sum_{pl.} \left(\mathbb{1} - \frac{1}{2N} \mathrm{Re\,Tr}\, U_{pl} \right)$

- Local variables are U_{link} and $\mathcal{C} = \bigotimes_{i=1}^{N_{links}} \mathcal{G}$.

- The action is local: If only U_ℓ changes, then $\Delta S_\ell = \Delta S(U_\ell)$

$$\Delta S_\ell = -\frac{1}{2N} \mathrm{Re\,Tr} \left\{ \Delta U_\ell \, \Sigma \right\}$$

where Σ = staple



staple

- Proposed link?

$$\begin{cases} U^{new} = (\mathbb{1} + \varepsilon^a T^a)\, U^{old} & \varepsilon^a \text{ random, small} \\ a(U^{new}, U) = \min\left(1, e^{-\Delta S_\ell}\right) & \end{cases}$$

Heuristically: "propose a random $U^{new} = X U^{old}$, with X close to $\mathbb{1} \in \mathcal{G}$, then accept or reject with probability $\min\left(1, e^{-\Delta S_\ell}\right)$"

$\longrightarrow$ Note that $\left\langle \frac{\#\,\text{accepted}}{\#\,\text{proposed}} \right\rangle \searrow$ when $\|\varepsilon\| \nearrow$: compromise!

Montecarlo Simulations - Fermion Fields

We have seen that $S_F(\psi, \bar\psi) = a^4 \sum_{x,y\in\Lambda} \bar\psi(x)\, D(x,y)\, \psi(y)$

where ψ and $\bar\psi$ are independent grassman variables, thus

$$Z = \int \mathcal{D}\psi\, \mathcal{D}\bar\psi\; e^{-S_F[\psi,\bar\psi]} = \det D$$

In a theory with both fermions and gauge fields:

$$\langle O[\psi,\bar\psi,U]\rangle = \frac{1}{Z}\int \mathcal{D}U\, \mathcal{D}\psi\, \mathcal{D}\bar\psi\; O[\psi,\bar\psi,U]\, e^{-S_F[\psi,\bar\psi,U]-S_W[U]}$$

Ex. n-pts function $$= \frac{1}{Z}\int \mathcal{D}U\; D^{-1}[U]\cdots D^{-1}[U]\, \det D[U]\; e^{-S_W[U]}$$

Remember $Z[\eta,\bar\eta] = \det D\; e^{+\int d^4x_E\, \bar\eta\, D^{-1}\, \eta}$

Montecarlo Simulations - The sign Problem

- Highly expensive to compute $\det D$, D^{-1}, ... because every pair of site is coupled and D is an enormous matrix. [Moreover, $\det D$ highly fluctuating]

- Det D is real but not necessarily positive : $D^\dagger = \gamma_5 D \gamma_5$

$$\det D^\dagger = \det(\gamma_5 D \gamma_5) = \det D \qquad \text{since } \gamma_5 = \mathbb{1}$$

[$\longrightarrow$ Sign Problem]

[Cannot use it as a statistical weight]

Note : For 2 deg. fermions, $N_f = 2$

$$Z = \int \mathcal{D}\psi_1 \mathcal{D}\bar{\psi}_1 \mathcal{D}\psi_2 \mathcal{D}\bar{\psi}_2 \, \mathcal{D}U \, e^{-\sum_{i=1,2} \bar{\psi}_i D \psi_i - S_W[U]} = \int \mathcal{D}U (\det D)^2 e^{-S_W[U]}$$

- Commonly used updating scheme :

1) $$\det(DD^\dagger) \propto \int \mathcal{D}[\phi_R] \mathcal{D}[\phi_I] \, e^{-\phi^\dagger (DD^\dagger)^{-1} \phi} \qquad \phi = \phi_R + i\phi_I \quad \text{PSEUDO FERMIONS}$$

2) $$\langle O \rangle_Q = \frac{1}{Z_Q} \int \mathcal{D}Q \, e^{-S[Q]} O(Q) = \frac{1}{Z_{QP}} \int \mathcal{D}Q \mathcal{D}P \, e^{-\frac{1}{2}P^2 - S[Q]} O(Q) = \langle O \rangle_{PQ}$$

3) Integrate $(Q,P) = (U,P)$ with $H(P,Q) = \frac{1}{2}P^2 + S[Q]$ for a time Δt, then use as new proposed configuration in a Metropolis step.

Montecarlo Simulations

Some general observations

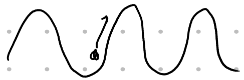
1) In general, simulations with dynamical fermions are expensive.

2) Quenched limit $\longrightarrow$ it is not possible to extract fermions from the Dirac sea. Only valence fermions

3) Local Algorithms $\to$ Critical slowing down : $\tau_{int} \sim \min(\xi, L)^z$

z dynamical critical exponent

Example: Topological Freezing



4) Sign Problem : If $\mu \neq 0$, $\gamma_5 D(\mu) \gamma_5 = D^\dagger(-\mu)$

$\longrightarrow$ use a different μ_f for each flavour? $\left[\begin{pmatrix} D(\mu_f) & 0 \\ 0 & D(-\mu_f) \end{pmatrix}\right]$

$\longrightarrow$ use Imaginary μ ... OPEN PROBLEM

A few remarks on Confinement,
strings and the glueballs spectrum.

Confinement

- So far, no explanation from first principles
  - → But many pheno models and some success: (2+1) U(1) LGT
    dual supercond. (monopole condensation ...)
    ⋮

- Lattice: perfect tool!
  - → Static $q\bar{q}$ potential
  - → Glueball spectrum
  - → Universal quantities
    - ↘ $\frac{m(2^{++})}{m(0^{++})}$

  } Clues to understand confining mechanism

  Lot of material for SU(N) theories [Teper, Lucini, Athenodorou, ...]

Confinement, strings & Glueballs

Regge Traj.
Sp. of Quarkonia $\Big\} \Rightarrow$ $V_{q\bar{q}}(R) \underset{R\to\infty}{\sim} \sigma R$, σ string tension. Might give insight into CONFINING MECHANISM

- Effective String theory : String formation breaks transverse translational symmetry
  → Constraints on an action in powers of $\left(\frac{1}{\sigma R L}\right)$. Finite R correction to σ [Aharony]

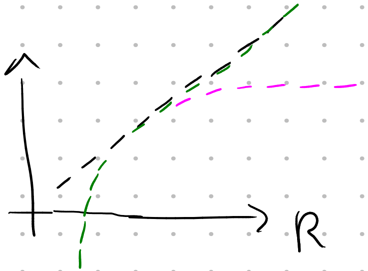
- Rich spectrum of pure glue states : GLUEBALLS + mixing with $q\bar{q}$ states [P.D.G]
  ↳ Very difficult to detect !!

- On the lattice :

  1) Produce confined configs 2) Compute at finite a, L [and set the scale]

  3) Extrapolate to $L\to\infty$, $a\to 0$

Ex. $V(R)$



- Quenched Approximation : Sea quarks introduce :
  - noise
  - String breaking
  - glueballs - meson mixing

Measurement of Masses

⊙ $|\psi\rangle = \hat{O}|\Omega\rangle$, $\hat{O}$ with desired quantum numbers : Ex. $\bar{\psi} U \psi$, $W(\Gamma)$, ...

↳ □, ⊟, ⊞, ...

$$C(t) = \langle \Omega | O^\dagger(t) O(t) | \Omega \rangle = \sum_n \langle \Omega | \hat{O}^\dagger(0) | n \rangle \langle n | \hat{O}(0) | \Omega \rangle e^{-E_n t}$$

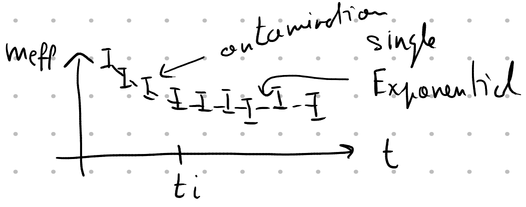
$$C(t) \sim |C_0|^2 e^{-E_0 t} \left(1 + \left| \frac{C_1}{C_0} \right|^2 e^{-(E_1 - E_0)t} + \dots \right) , \quad C_n = \langle n | \hat{O}(0) | \Omega \rangle$$

Note: 2) Noise is constant with t, signal $\sim e^{-E_0 t}$; $\frac{\text{signal}}{\text{noise}} \sim e^{-E_0 t}$

3) C_n : Overlaps between $|n\rangle$ and $\hat{O}(0)|\Omega\rangle$

4) At finite t, contaminations from $E_i > E_0$

$$m_{eff} = -\frac{1}{a} \log \frac{C(t+a)}{C(t)}$$



$\Rightarrow$ [to find E_0, fit $|C_0|^2 e^{-E_0 t}$ for $t > t_i$]

The Variational Method

$$m_{var}^{RP} = -\frac{1}{a}\log\left\{\max_{|\psi\rangle}\frac{\langle\psi|\hat{T}|\psi\rangle}{\langle\psi|\psi\rangle}\right\} \quad \text{where} \quad \hat{T} = e^{-a\hat{H}}$$

- To find $|\psi\rangle$, define variational basis $\{O_i^{RP}\}_i$, $\quad C_{ij}(t) = \sum_a v_i^{a*} v_j^a e^{-m_a^{(RP)} t}$

- C_{ij} is real and symmetric, can be diagonalized. $\quad C(t)\vec{v} = \lambda(t)\vec{v}$
  - ↳ We take C eigenstates : Re and Im
- At large t, diagonalization unstable, so instead solve the G.E.V.P

$$C(t)\vec{\xi} = \lambda(t,t_0)\,C(t_0)\vec{\xi}$$

where $\vec{\xi}$ are the optimal eigenfunctions. $\quad D(t,t_0) = C(t) - \lambda C(t_0) = \operatorname{diag}\left\{e^{-E_n t} - \lambda e^{-E_n t_0}\right\}$

So $\quad \lambda_n(t,t_0) = \exp\{-E_n(t-t_0)\}$

- Set $t_0 = 1$, $t = 0$ and diagonalize.

$$\tilde{C}_{ii}(t) = A\cosh E_0\left(t - \frac{L_t a}{2}\right)$$

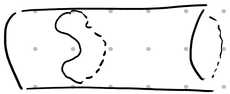
L_t time extent
a lattice spacing

$\vec{\xi}$ "Form Factors"

(cosh because of periodic boundary conditions)

Pure Gauge States

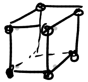
$$W(\Gamma) = \mathrm{Tr}(U_\Gamma)$$

⊙ 2 types: contractible:  , noncontractible  ↖ p.b.c

[Exercise: $U_4 \to z U_4$]

contractible ↓ Glueballs; noncontractible ↓ Torelons

→ G. and T do not mix because they belong to diff rep. of $C(\mathcal{G})$

⊙ Broken Poincaré invariance: Subduced representations. [→ The symmetries of ] [Landau-Lifschitz QM]

→ Discrete translations and finite volume: momenta $\in \left(-\frac{\pi}{a}, \frac{\pi}{a}\right]$ and quantized

→ Octahedral group O_h, IRREPs

$A_1 \to J = 0, 4, \dots$ $\quad E \to J = 2, \dots$ $\quad T_2 \to J = 2, 3, 4, \dots$

$A_2 \to J = 3, \dots$ $\quad T_1 \to J = 1, 3, 4, \dots$

→ Add P and C 20 states $A_1^{\pm\pm}, A_2^{\pm\pm}, E^{\pm\pm}, T_1^{\pm\pm}, T_2^{\pm\pm}$ [channels]

⊙ Torelons: Focus on g.s. Then $m(\text{Torelon}) \underset{L\to\infty}{\sim} \sigma L$ → extract σ [Useful for scale setting]

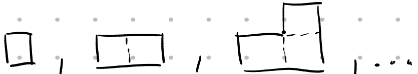
↳ for finite L corrections see Eff. string Theory [Aharony]

Glueball Operators

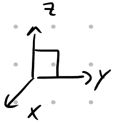
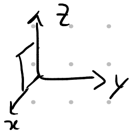
- Glueball operators : $O_\Gamma(\vec{x},t) = W(\Gamma)$
  - $\rightarrow$ Definite momentum : $O_\Gamma(\vec{p},t) = \sum_{\vec{x}} e^{-ip\bar{x}} O_\Gamma(\vec{x},t) \implies O_\Gamma(t) = \sum_{\vec{x}} O_\Gamma(\vec{x},t)$ $\quad \vec{p}=\vec{0}$
  - $\rightarrow$ Under O_h, O_Γ has the same properties as Γ :

$$U(r)\,\hat{O}_\Gamma\,U^{-1}(r) = \hat{O}_{r\Gamma} \qquad r \in O_h$$

where $\Gamma : [\hat{f}_1, \dots, \hat{f}_L]$, $\sum_{i=1}^{L} \hat{f}_i = 0$

Ex.  , $\dots$

L=4 $\quad$ L=6 $\quad$ L=8

Ex. $R_{\hat{z}}^{\pi/2}$ () = 

- Let $\mathcal{M}$ be a Rep of O_h : $\mathcal{M}\,\Gamma = R[\hat{f}_1, \dots, \hat{f}_L] = [R\hat{f}_1, \dots, R\hat{f}_L]$

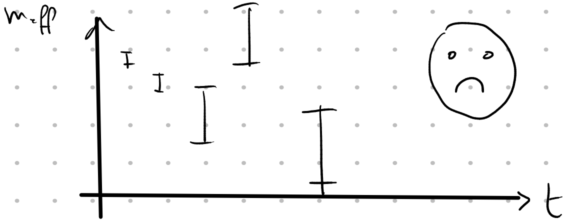
- To find basis of IRREP of O_h use projection method [Hamermesh] or other [Berg-Billoire]

<u>Result :</u> $0^{++} : (\square + \square + \square) +$ parity and charge conj.

$\dots$

$2^{++} : (-2\,\square + \square + \square) + ''$

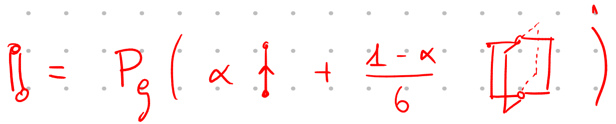
UV fluctuations & Smearing



"Signal drowned into noise before single exponential behaviour kicks in"

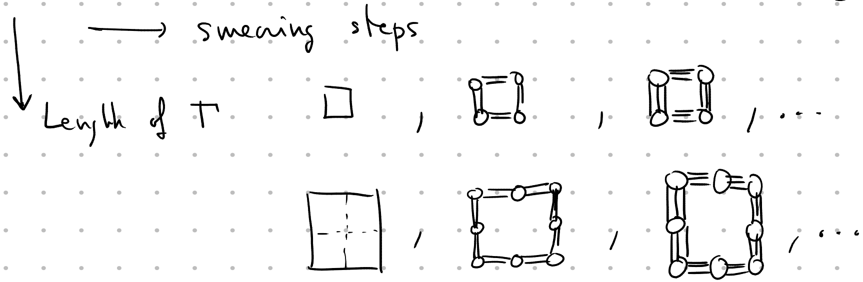
- Measurement dominated by UV fluctuations → improve overlap with large scale fluctuations
- Solution: SMEARING

$$\begin{cases} X_i(n) = \alpha\, U_i(n) + \dfrac{1-\alpha}{6} \displaystyle\sum_{i \neq j} U_j(n)\, U_i(n+j)\, U_j^\dagger(n+i) \\ \tilde{U}_i(n) = \mathrm{Proj}_{\mathfrak{g}} \{ X_i(n) \} \end{cases}$$

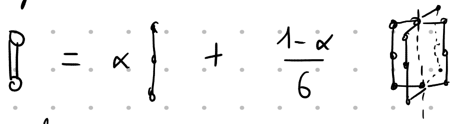


Smearing is a smoothening operation → It removes UV fluctuations

Example: Variational basis for glueballs



- Blocking is similar



Example:

Variational basis: $N \times N(\text{smearing}) \times N(\text{blocking})$

operators $\sim 10^2$ [Can be automatized]

Sp(2N) Glueballs

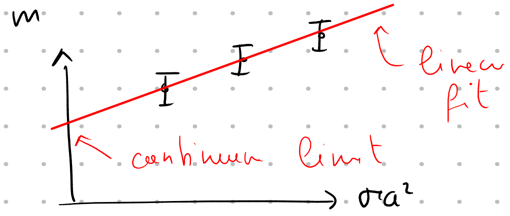
- Sp(2N) gauge theories interesting for | Composite Higgs [Potentially UV complete C.H.] → [Ferretti-Karateev]
  | Large-N → [Lovelace]

- Heatbath Algorithm [See Gattringer-Lang]
- The spectrum was obtained for $N = 1, 2, 3, 4$
- The σ was used to set the scale. [Teukolsky + E.F.T]

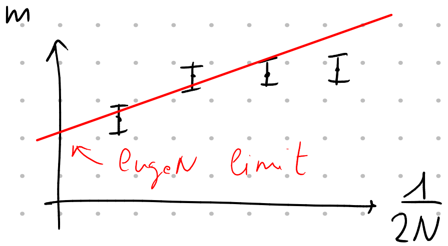
$$\frac{m(R^P)}{\sqrt{\sigma}}(a) = \frac{m(R^P)}{\sqrt{\sigma}}(0) + C_1(R^P)\,\sigma a^2$$

continuum limit

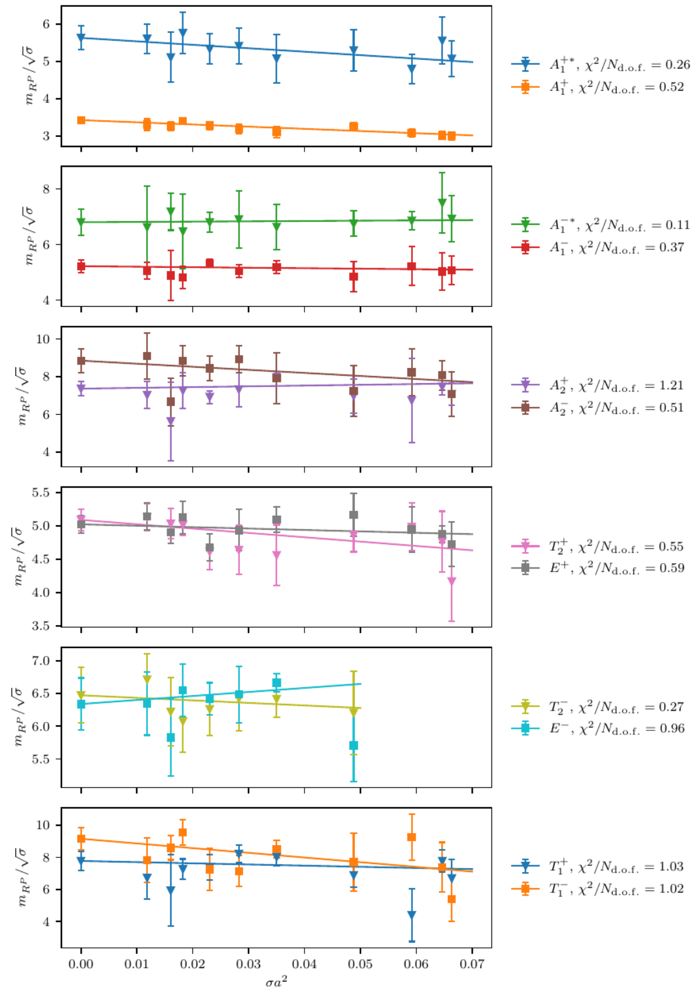
discretization error



$$\frac{m(R^P)}{\sqrt{\sigma}}(N) = \frac{m(R^P)}{\sqrt{\sigma}}(\infty) + C_2(R^P)\,\frac{1}{N}$$

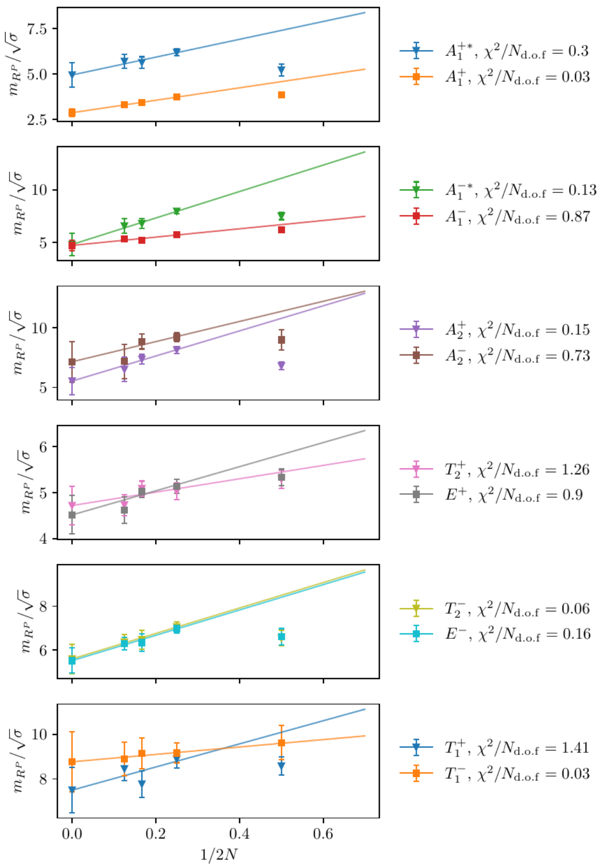


$\leftarrow$ Sp(6), continuum limits

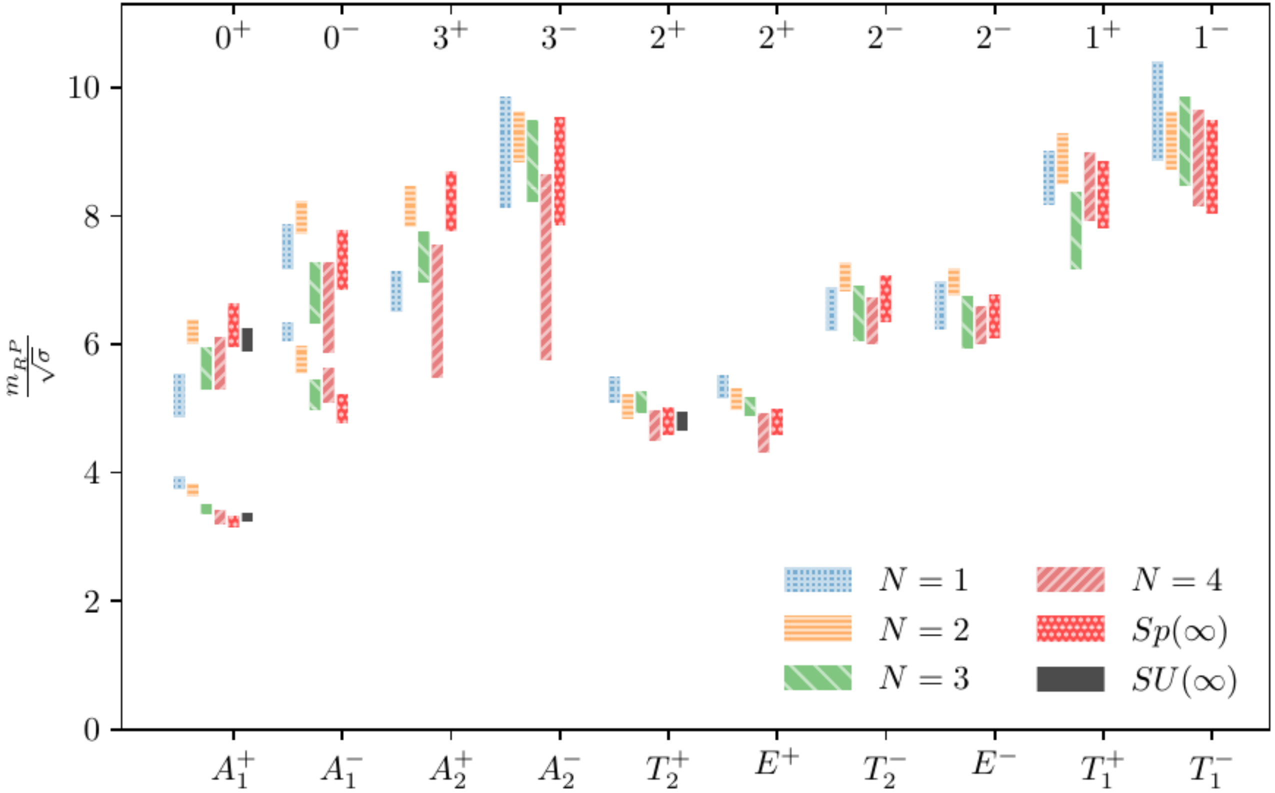


degenerate as $a \to 0$

Large N $\rightarrow$



Find Result :



0^+
0^-
3^+
3^-
2^+
2^+
2^-
2^-
1^+
1^-
$\frac{m_{R^P}}{\sqrt{\sigma}}$
0
2
4
6
8
10
A_1^+
A_1^-
A_2^+
A_2^-
T_2^+
E^+
T_2^-
E^-
T_1^+
T_1^-
$N = 1$
$N = 2$
$N = 3$
$N = 4$
$Sp(\infty)$
$SU(\infty)$