Topological phases of matter and quantum entanglement

Shinsei Ryu

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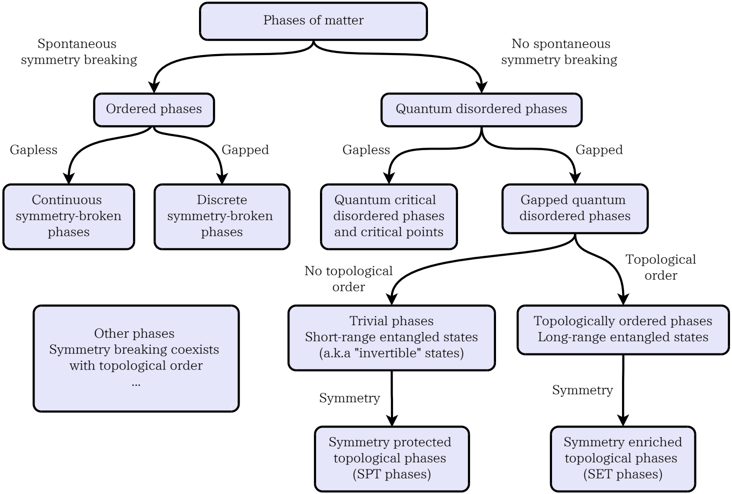
Phases of matter



Topologically-ordered phases

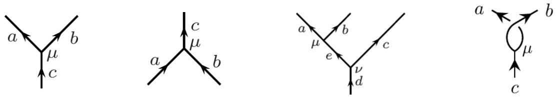
- Phases that support anyons ($\simeq$ support topology dependent ground state degeneracy)
- Not described by the symmetry-breaking paradigm. (I.e., Landau-Ginzburg type of theories) Instead, characterized by properties of anyons (fusion, braiding, etc.) (I.e., topological quantum field theories)

	SSB phases	Topologically-ordered phases
Ground states	Degeneracy w/ SSB	Topological degeneracy
Excitations	Nambu-Goldstone bosons	Anyons
Effective theory	Landau-Ginzburg	TQFT

- E.g., fractional quantum Hall states, $\mathbb{Z}_2$ quantum spin liquid, Kitaev spin liquid, etc.

Topologically-ordered phases in (2+1)D

- (bosonic) Topological order is believed to be fully characterized by a unitary modular tensor category (UMTC).
- Characterized by a finite set of anyons $\{1, a, b, \dots\}$, fusion, and braiding; $(N_{ab}^c, F_d^{abc}, R_c^{ab})$.



etc.

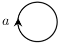
- E.g.

$$
\begin{aligned}
\text{Toric code :} \quad & 1 \times e = e, \quad 1 \times m = m, \quad 1 \times f = f, \\
& e \times m = f, \quad e \times f = m, \quad m \times f = e, \\
& e \times e = 1, \quad m \times m = 1, \quad f \times f = 1 \\
\text{Ising :} \quad & 1 \times a = a, \quad \sigma \times \sigma = 1 + \psi, \quad \sigma \times \psi = \sigma, \quad \psi \times \psi = 1
\end{aligned} \tag{1}
$$

(bosonic) Topological order in (2+1)D

- Quantum dimensions $\{1, d_a, d_b, \ldots\}$ $(d_a \geq 1)$ for each anyon:

$$d_a = \tag{2}$$



  Total quantum dimension $\mathcal{D} := \sqrt{\sum_a d_a^2}$.

- Modular T matrix, $T = \operatorname{diag}(1, \theta_a, \theta_b, \ldots)$ where

$$\theta_a = e^{2\pi i h_a} = \frac{1}{d_a} \tag{3}$$

  is the self-statistical angle of a; $\theta_a(h_a)$: the topological spin of a.

- Modular S matrix encodes the braiding between anyons, and defined by

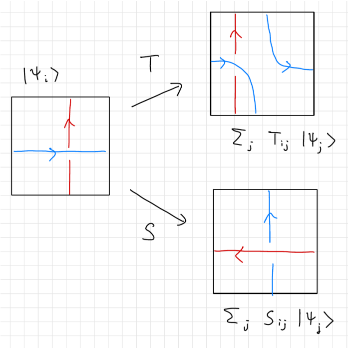
$$S_{ab} = \frac{1}{\mathcal{D}} \tag{4}$$

- Chiral central charge mod 8

$$e^{2\pi i c/8} = \frac{1}{\mathcal{D}} \sum_a d_a^2 \theta_a \tag{5}$$

Modular data (S and T) and ground states

- Ground state degeneracy depending on the topology of the space (topological ground state degeneracy), related to the presence of anyons. [Wen (89-90)] E.g., Ground state degeneracy on a spatial torus, $\{|\Psi_i\rangle\}$.
- Large diffeomorphism of the torus induces a transformation within degenerate multiplet.



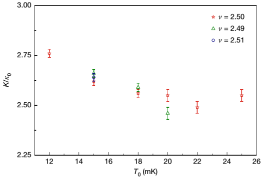
- The modular data may largely determine underlying topological order, but see [Mignard-Schauenburg (17);Wen-Wen (19)]

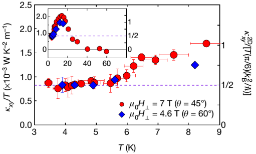
- There may be topologically ordered phases with the same braiding properties, but different values of c, the chiral central charge of the edge modes. They cannot be smoothly deformed to each other without closing the energy gap.

- Can be measured by the thermal conductance in the edge [Kane-Fisher (96)] :

$$\kappa = \frac{\pi k_B^2 T}{6} \times c$$

E.g. half-filled Landau level [Banerjee (18)] , Kitaev spin liquid [Kasahara (18)]



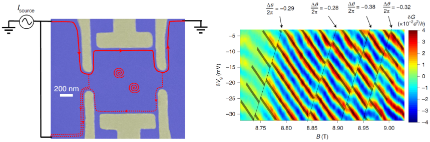


Other data

- With symmetry, there are other data. E.g., σ_{xy}
- Data for symmetry-enriched topologically-ordered phases (SET) $=$ G-crossed braided tensor category $\mathcal{C}_G^\times$ [Barkeshli et al (14), Etingof et al (10), ...]
  Fusion and braiding properties of symmetry defects together with anyons
- Geometrical data (Wen-Zee term, shift, Hall viscosity, ...).
- Other data?

Some questions

- What are the data characterizing topological phases of matter? Are there relations between different pieces of the data?
- How can we extract/measure such data? C.f. Direct observation of abelian braiding statistics [Nakamura et al (20)] Topological invariants? [C.f. Bonderson (21)]



- Can we extract these data from ground states? E.g., modular data [Wen (90); Keski-Vakkuri-Wen (93); Bais-Romers (11); Zhang et al (12); Cincio-Vidal (13); Moradi-Wen (15); You-Cheng (15); Zhu et al (17); Zhu et al (18) ...]

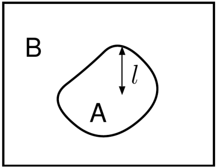
- Entanglement entropy (partial trace)
- Entanglement negativity (partial transpose)
  - Negativity for anyons
- (partial rotation)

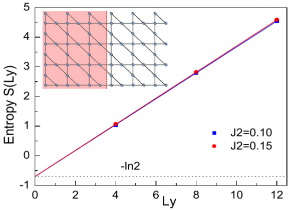
Topological entanglement entropy

[Levin-Wen, Kitaev-Preskill (05)]

- For topologically ordered phases in 2 spatial dimensions,

$$\begin{aligned} S_A &= -\mathrm{Tr}\,\rho_A \log \rho_A \quad (\rho_A = \mathrm{Tr}_B |GS\rangle\langle GS|) \\ &= const. \times \ell - \log \mathcal{D} \end{aligned}$$

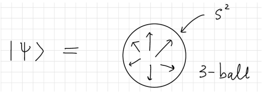


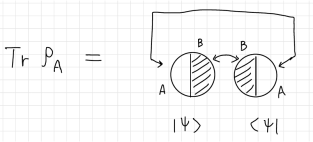


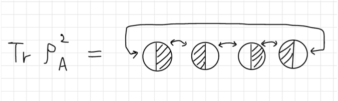
[Jiang-Wang-Balents (12)]

Bulk calculations

- TEE from surgery [Dong et al (08)]





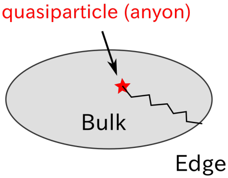


$$\frac{\mathrm{Tr}\,\rho_A^n}{(\mathrm{Tr}\,\rho_A)^n} = \frac{Z(S^3)}{[Z(S^3)]^n} = [Z(S^3)]^{1-n} = [\mathcal{S}_0{}^{0}]^{1-n} \qquad (6)$$

- UV divergent term is missing.

Bulk-boundary correspondence

- Bulk wfn $|\Psi_i\rangle \longleftrightarrow$ boundary partition function χ_i
- Bulk anyon $\longleftrightarrow$ twisted boundary conditions at edge:

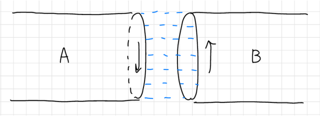


- Bulk S and T matrices acting on $|\Psi_i\rangle$ on *spatial torus* $\longleftrightarrow$ S and T matrices acting on boundary partition function χ_a on *spacetime torus*

$$\chi_a\left(e^{-\frac{4\pi\beta}{l}}\right) = \sum_{a'} \mathcal{S}_{aa'}\chi_{a'}\left(e^{-\frac{\pi l}{\beta}}\right) \tag{7}$$

[Witten (89), Moore-Seiberg (89), Elitzur-Moore-Schwimmer-Seiberg (89), Bos-Nair (89), Murayama (89), Dunne-Jackiw-Trugenberger (89), Cappelli-Zemba (96), ...]

Edge theory approach



- The reduced density matrix ρ_A obtained from a ground state $|GS\rangle$ by tracing out half-space can be obtained from conformal boundary state: [Qi-Katsura-Ludwig (12)]

$$(L_n - \overline{L}_{-n})|B\rangle = 0 \quad (\forall n \in \mathbb{Z}) \tag{8}$$

- Near the entangling boundary,

$$|GS\rangle \sim e^{-\epsilon H_{edge}}|B\rangle \tag{9}$$

so that the reduced density matrix is

$$\rho_A \propto \mathrm{Tr}_B \left[e^{-\epsilon H_{edge}}|B\rangle\langle B|e^{-\epsilon H_{edge}} \right] \tag{10}$$

- "Physical picture": healing the cut; gapped edge by potential "Formal picture": gauge invariance

Boundary states as gapped states

- Boundary states represent a highly excited state within the Hilbert space of a gapless CFT and can be viewed as gapped ground states.
  [Calabrese-Cardy (06), Miyaji-SR-Takayanagi-Wen (14), Cardy (17)]

- Boundary states are short-range correlated: Spatial correlation function $\langle B|e^{-\epsilon H}\mathcal{O}_1(x_1)\cdots\mathcal{O}_n(x_n)e^{-\epsilon H}|B\rangle/\langle B|e^{-2\epsilon H}|B\rangle$ factorizes in the limit of $\epsilon \to 0$.

- The GS of the free fermion $H = \int dx[-i\psi^\dagger\sigma_z\partial_x\psi + m\psi^\dagger\sigma_x\psi]$ is given by

$$|GS\rangle \propto e^{\sum_{k>0} \frac{m}{[m^2+k^2]^{\frac{1}{2}}+k}\left[\psi^\dagger_{Lk}\psi_{Rk}+\psi^\dagger_{R-k}\psi_{L-k}\right]}|0_L\rangle \otimes |0_R\rangle \tag{11}$$

  where $|0_{L,R}\rangle$ is the Fock vacuum of the left- and right-moving sector. In the limit $m/k \to \infty$, $|GS\rangle$ reduces to the boundary state.

- BCFT can be used to study (1+1)d SPTs to extract $H^2(G, U(1))$
  [Cho-Shiozaki-SR-Ludwig (16)]

TEE from edge

- Ishibashi boundary state:

$$|h_a\rangle\rangle \equiv \sum_{N=0}^{\infty} \sum_{j=1}^{d_{h_a}(N)} |h_a, N; j\rangle \otimes \overline{|h_a, N; j}\rangle \tag{12}$$

  Different (Ishibashi) boundary states correspond to different ground states

- Topological sector dependent normalization (regularization):

$$|\mathfrak{h}_a\rangle\rangle = \frac{e^{-\epsilon H}}{\sqrt{\mathfrak{n}_a}} |h_a\rangle\rangle \quad \text{so that} \quad \langle\langle \mathfrak{h}_a | \mathfrak{h}_b \rangle\rangle = \delta_{ab}. \tag{13}$$

- Reduced density matrix:

$$\rho_{A,a} = \mathrm{Tr}_B(|\mathfrak{h}_a\rangle\rangle\langle\langle \mathfrak{h}_a|) = \sum_{N,j} \frac{1}{\mathfrak{n}_a} e^{-\frac{8\pi\epsilon}{l}(h_a + N - \frac{c}{24})} |h_a, N; j\rangle\langle h_a, N; j|. \tag{14}$$

TEE from edge

- Trace of the reduced density matrix:

$$\mathrm{Tr}_A\left(\rho_{A,a}\right)^n = \frac{1}{\mathfrak{n}_a^n}\chi_a\left(e^{-\frac{8\pi n\epsilon}{l}}\right) = \frac{\chi_a\left(e^{-\frac{8\pi n\epsilon}{l}}\right)}{\chi_a\left(e^{-\frac{8\pi\epsilon}{l}}\right)^n} \tag{15}$$

- Modular transformation

$$\chi_a\left(e^{-\frac{8\pi n\epsilon}{l}}\right) = \sum_{a'} \mathcal{S}_{aa'}\chi_{a'}\left(e^{-\frac{\pi l}{2n\epsilon}}\right) \to \mathcal{S}_{a0} \times e^{\frac{\pi c l}{48 n\epsilon}} \quad (l/\epsilon \to \infty), \tag{16}$$

i.e., only the identity field I, labeled by "0" here, survives the limit. [C.f. Boundary entropy (Affleck-Ludwig g); Kitaev-Preskill(05), Fendley-Fisher-Nayak(06)]

- Hence, in the thermodynamic limit $l/\epsilon \to \infty$:

$$\mathrm{Tr}_A\left(\rho_{A,a}\right)^n = \frac{\sum_{a'} \mathcal{S}_{aa'}\chi_{a'}\left(e^{-\frac{\pi l}{2n\epsilon}}\right)}{\left[\sum_{a'} \mathcal{S}_{aa'}\chi_{a'}\left(e^{-\frac{\pi l}{2\epsilon}}\right)\right]^n} \to e^{\frac{\pi c l}{48\epsilon}\left(\frac{1}{n}-n\right)}\left(\mathcal{S}_{a0}\right)^{1-n}, \tag{17}$$

- Final result:

$$
\begin{aligned}
S_A^{(n)} &= \frac{1+n}{n} \cdot \frac{\pi c}{48} \cdot \frac{l}{\epsilon} - \ln \mathcal{D} + \frac{1}{1-n} \ln d_a^{1-n} \\
S_A^{\mathsf{vN}} &= \frac{\pi c}{24} \cdot \frac{l}{\epsilon} - \ln \mathcal{D} + \ln d_a
\end{aligned} \tag{18}
$$

where $\mathcal{S}_{a0} = d_a/\mathcal{D}$ is the quantum dimension.

Entanglement in mixed states?

- How to quantify quantum entanglement between A and B when $\rho_{A\cup B}$ is *mixed* ? E.g., finite temperature; A, B is a part of bigger system.
- The entanglement entropy is an entanglement measure only for pure states. It is not monotone under LOCC.
- *Entanglement negativity* and *logarithmic negativity*, using *partial transpose* [Peres (96), Horodecki-Horodecki-Horodecki (96), Vidal-Werner (02), Plenio (05) ...]

$$\mathcal{N}(\rho) := \sum_{\lambda_i<0} |\lambda_i| = \frac{1}{2}\left(||\rho^{T_B}||_1 - 1\right),$$

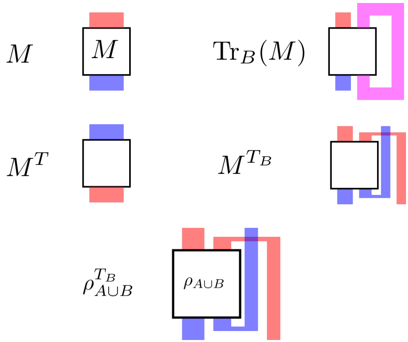
$$\mathcal{E}(\rho) := \log(2\mathcal{N}(\rho) + 1) = \log ||\rho^{T_B}||_1 .$$

- Good entanglement measure since LOCC monotone.
- For mixed states, negativity can extract quantum correlations only.

- Definition: for an operator M, its partial transpose M^{T_B} is

$$\langle e_i^{(A)} e_j^{(B)} | M^{T_B} | e_k^{(A)} e_l^{(B)} \rangle := \langle e_i^{(A)} e_l^{(B)} | M | e_k^{(A)} e_j^{(B)} \rangle$$

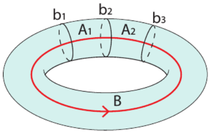
where $|e_i^{(A,B)}\rangle$ is the basis of $\mathcal{H}_{A,B}$.



Negativity for topological liquid

[Lee-Vidal (13), Castelnovo (13), Wen-Matsuura-SR (16), Wen-Chang-SR (16) Lim-Asasi-Teo-Mulligan (21)]

- Generic state on a torus: $|\psi\rangle = \sum_a \psi_a |\mathfrak{h}_a\rangle\!\rangle$



- Mutual information and negativity:

$$I_{A_1 A_2} = \frac{\pi c}{12}\frac{l_2}{\epsilon} - 2\ln\mathcal{D} + 2\sum_a |\psi_a|^2 \ln d_a - \sum_a |\psi_a|^2 \ln|\psi_a|^2 \qquad (19)$$

$$\begin{aligned}\mathcal{E}_{A_1 A_2} &= \lim_{n_e\to 1} \ln \mathrm{Tr}\,(\rho^{T_2}_{A_1\cup A_2})^{n_e} \\ &= \frac{\pi c}{16}\frac{l_2}{\epsilon} - \ln\mathcal{D} + \ln\Big(\sum_a |\psi_a|^2 \ln d_a\Big) \qquad (20)\end{aligned}$$

$\mathcal{E}$ is dependent on ψ_a only for non-Abelian topological order.

Finite T negativity for topological liquid

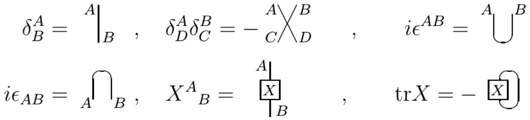
- At finite T, topological order may be destroyed by proliferation of excitations. (2+1)d toric code supports no TO at $T > 0$ while (4+1)d toric code does support TO at $T > 0$ [Dennis-Kitaev-Landahl-Preskill (02)]
- Unlike entropy, negativity exhibits area law: [Hart-Castelnovo (18), Lu-Hsieh-Grover (19)]

$$\mathcal{E} = \text{Area law} - \mathcal{E}_{top} \tag{21}$$

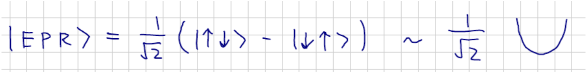
$\mathcal{E}_{top}$ is sensitive to finite T topological transition.

Quantum entanglement by diagramatics

- Penrose's graphical calculus for $SU(2)$:



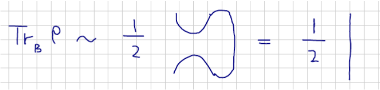
- EPR (Bell) pair $|\mathrm{EPR}\rangle$:



Quantum entanglement by diagramatics

- Density matrix ρ:

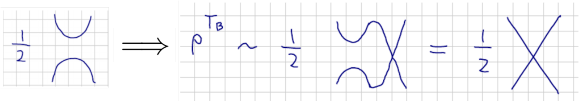
- Partial trace and reduced density matrix $\mathrm{Tr}_B\rho$:



- Entanglement for anyonic systems [Hikami (08), Kato-Furrer-Murao (14), Pfeifer (14), Bonderson-Knapp-Patel (17)]

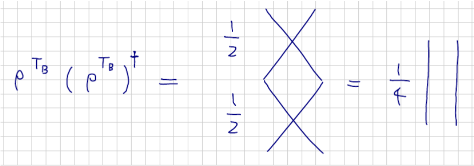
Partial transpose and negativity

- Partial transpose ρ^{T_B}:



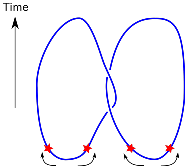
- Entanglement negativity $\mathcal{E}$:

$$\mathcal{E} = \ln \mathrm{Tr} \sqrt{\rho^{T_B} \rho^{T_B \dagger}} = \ln 2$$

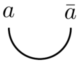


Partial transpose for anyons?

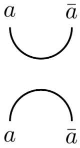
- World lines for anyons



- "EPR pair" for anyons



- Density matrix and partial transpose [Shapourian-Mong-SR (20)]

$\rho = \dfrac{1}{d_a}$ 

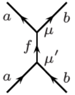
$\rho^{T_2} = \dfrac{1}{d_a}$ 

Anyonic partial transpose

- Density matrix of anyons

$$\tilde{\rho}_{ab} = \sum_{f,\mu,\mu'} \frac{[p^f]_{\mu\mu'}}{d_f} \left| a, b, \mu; f \right\rangle \left\langle a, b, \mu'; f \right|$$

$$= \sum_{f,\mu,\mu'} \frac{[p^f]_{\mu\mu'}}{\sqrt{d_a d_b d_f}}$$



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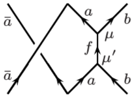
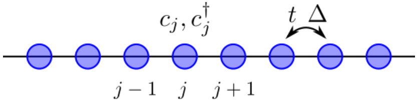
- *Proposal*: define partial transpose for anyons as: [Shapourian-Mong-SR (20)]

$$\tilde{\rho}_{ab}^{T_A} = \sum_{f,\mu,\mu'} \frac{[p^f]_{\mu\mu'}}{\sqrt{d_a d_b d_f}}$$

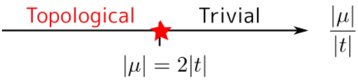


- The Kitaev chain

$$H = \sum_j \Big[-tc_j^\dagger c_{j+1} + \Delta c_{j+1}^\dagger c_j^\dagger + h.c. \Big] - \mu \sum_j c_j^\dagger c_j$$



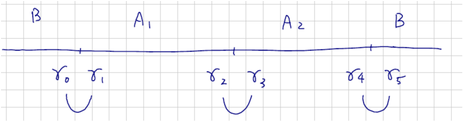
- Phase diagram: there are only two phases:



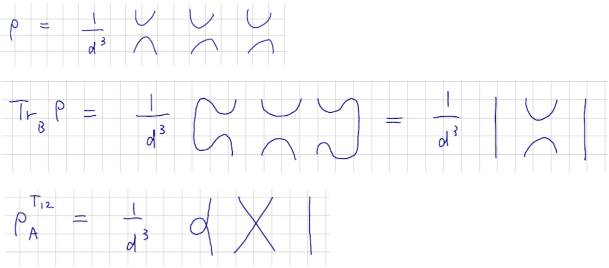
- Topologically non-trivial phase is realized when $2|t| \geq |\mu|$.

The Kitaev chain

- Setup:

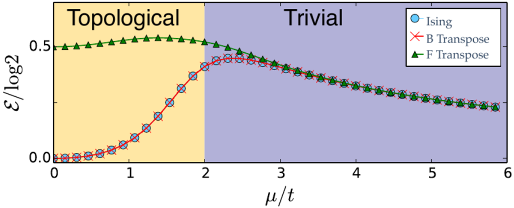


- Negativity can be calculated as $\mathcal{E} = \log \sqrt{2}$:



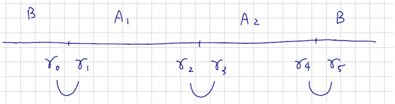
Numerics

- Numerics: fermionic v.s. bosonic partial transpose



- (Blue circles and Red crosses) are computed by using Jordan-Wigner transformation and bosonic partial transpose
- (Green and Orange triangles) are computed by using fermionic partial transpose
- At critical point: agrees with the CFT prediction by [Calabrese-Cardy-Tonni].

Aside: partial transpose and SPT invariant

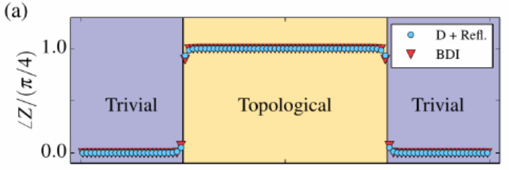


- Using the same setup of the Kitaev chain, but considering

$$Z = \mathrm{Tr}[\rho_{A_1 \cup A_2} \rho_{A_1 \cup A_2}^{T_1}] = e^{2\pi i \nu/8}$$

  gives the $\mathbb{Z}_8$ topological invariant of time-reversal symmetric topological superconductors. [Fidkowski-Kitaev(10)]

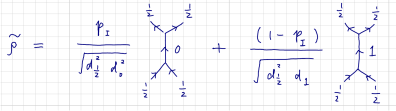
- Numerics:

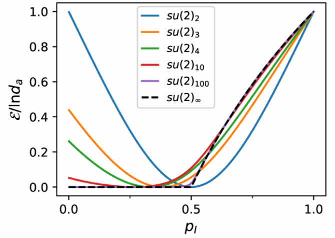


- The effective action interpretation: $Z(\mathbb{R}P^2, \eta) = e^{2\pi i \mathrm{Brown}(\eta)/8}$ where η is one of two Pin_- structures on $\mathbb{R}P^2$.

Anyonic entanglement negativity

- Negativity $\mathcal{E}$ for two spin $1/2$ anyons in $su(2)_k$ [Shapourian-Mong-SR]





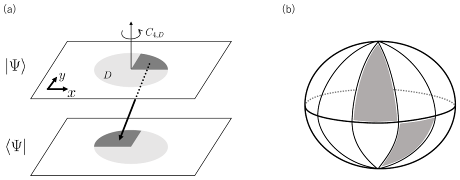
- C.f. Broken curve: $\rho = \frac{4p-1}{3}|EPR\rangle\langle EPR| + \frac{1-p}{3}I$ for regular spin $1/2$
- The subspace of vanishing anyonic negativity forms a zero measure set (in particular when no multiplicity fusion).
- The anyonic negativity is LOCC monotone.

Partial rotation and lens space

- The expectation value of partial n-fold rotation acting on a subregion D is given by the partition function on the lens space [Shiozaki-Shapourian-SR(17)] :

$$\langle C_{n,D}\rangle = \langle GS|C_{n,D}|GS\rangle \sim Z(L(n,1)) \tag{22}$$

C.f. higher-central charges [Kaidi et al (21)]

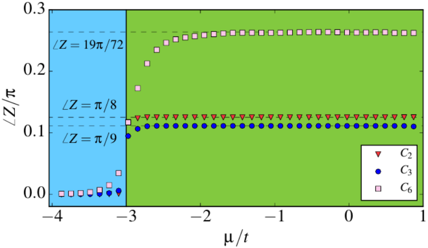


- "Cut-and-glue" calculation: C.f. momentum polarization [Tu-Zhang-Qi (12)]

$$\begin{aligned}\langle C_{n,D}\rangle &\sim \frac{\mathrm{Tr}\left[e^{-iP\frac{L}{n}}e^{-\epsilon H}\right]}{\mathrm{Tr}\left[e^{-\epsilon H}\right]} = \frac{e^{\frac{2\pi i}{n}\left(\langle L_0\rangle - \frac{c}{24}\right)}\sum_a \chi_a\left(\frac{i\epsilon}{L}-\frac{1}{n}\right)}{\sum_a \chi_a\left(\frac{i\epsilon}{L}\right)} \\ &= \frac{e^{\frac{2\pi i}{n}\left(\langle L_0\rangle - \frac{c}{24}\right)}\sum_{a,b}(\mathcal{S}\mathcal{T}^n\mathcal{S})_{ab}\chi_b\left(\frac{iL}{n^2\epsilon}+\frac{1}{n}\right)}{\sum_{a,b}\mathcal{S}_{ab}\chi_b\left(\frac{iL}{\epsilon}\right)}\end{aligned} \tag{23}$$

- Example: chiral p-wave SC

$$\langle C_{n,D}\rangle \sim \begin{cases} e^{-\frac{(n^2+2)\pi i}{24n}+\cdots} & n:\text{even} \\ \sqrt{2}^{-1} e^{-\frac{(n^2-1)\pi i}{24n}+\cdots} & n:\text{odd} \end{cases} \tag{24}$$



- Can be used to study crystalline SPTs [Shiozaki-Shapourian-SR(17)]; Can detect SPTs classified by $H^3(G,U(1))$; [Tiwari-Chen-Shiozaki-SR(17)]

Summary/Outlook

- Using partial operations, we can put topological phases on interesting spacetime manifolds to extract topological data.
- We can compute these by using either bulk or edge theory.
- Other entanglement measures; Charged entanglement/Symmetry-resolved entanglement, Linking entanglement, Reflected entropy, odd entropy, etc.
- Experiments [Islam et al (15)] [Kaufman et al (16)] [Lukin et al (18)] [Brydges (19)]