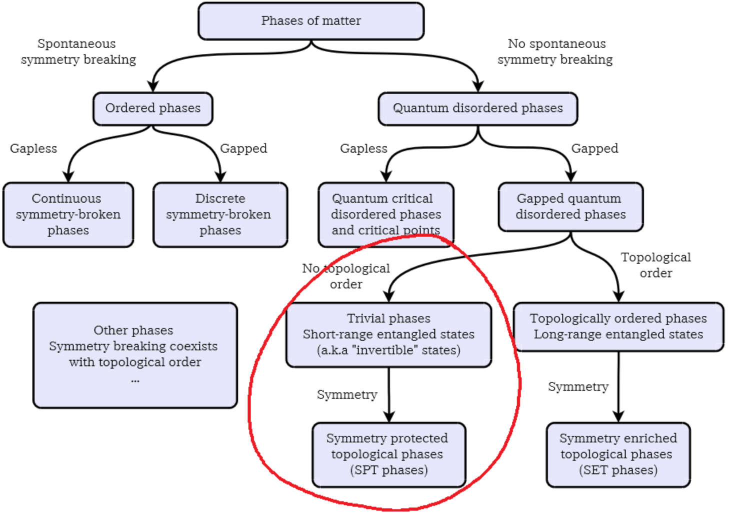
SPT Phases and Beyond

GGI Workshop: Topological Properties of Gauge Theories

August 31, 2021

Scope



[taken from last week's review talk by Shinsei]

Outline

- Entanglement in Gapped Systems
- Invertible Phases and SPTs
- Detecting Invertible Phases
- Free Fermions

Entanglement in Gapped Systems

Gapped Systems

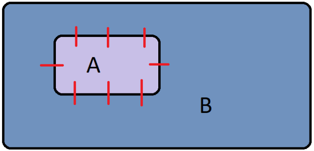
- Lattice: State space is
$$\mathcal{H} = \bigotimes_v \mathcal{H}_v \ .$$
- Locality: Hamiltonian is
$$H = \sum h_{\mathcal{R}} \ .$$
- Assume translation invariance: the data $\mathcal{H}_v$ and $h_{\mathcal{R}}$ are independent of $v, \mathcal{R}$.
- Data $(\mathcal{H}_v, \{h_{\mathcal{R}}\})$ defines a family of systems on various lattices.
- Gapped: the energy gap between the ground and first excited states remains nonzero in the thermodynamic limit (large system size $N \gg |\mathcal{R}|$).

Area law for entanglement entropy

- Gapped ground states are thought to have an area law for ent. entropy.
  - One dimension. [Hastings 07] [Arad, Kitaev, Landau, Vazirani 13]
  - Two dimensions. [Kitaev, Preskill 05]

$$\rho = \mathrm{Tr}_B(|\psi\rangle\langle\psi|) \ , \qquad S = -\,\mathrm{Tr}(\rho \log \rho)$$

$$S \to \alpha|\partial A| - \gamma + \ldots$$



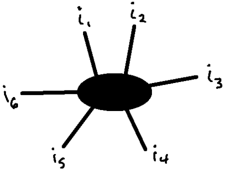
- Suggests that entanglement is "short-range" (except for the constant term γ).

- Ground states of gapped systems occupy a small corner of the exponentially large Hilbert space. Hope for a more efficient description than d^L values

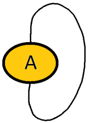
$$\langle i_1, \ldots, i_L|\psi\rangle = c_{i_1 \cdots i_L} \ .$$

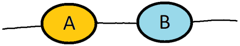
Tensor network diagrams

- Tensor $X_{i_1 \cdot i_n}$ is an n-dimensional array of numbers



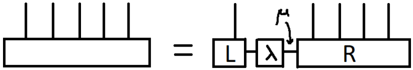
- Tensor contraction: indices on internals are summed over.
  - e.g. Trace of a matrix
  - e.g. Matrix multiplication





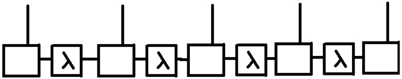
Entanglement in one dimension

- Schmidt decomposition captures the entanglement between sites:



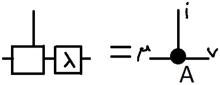
$$|\psi\rangle = \sum_{\mu \in V} \lambda_\mu |\psi_L^\mu\rangle \otimes |\psi_R^\mu\rangle$$

- Repeated Schimdt decomposition yields



Matrix product states

- The result is a matrix product state (MPS) built out of a tensors



- A^i is a matrix of rank D, the bond dimension.
  - D can be recovered from the 0^{th} Renyi entropy $S_0(\rho) = \log D$.
- Wavefunction is given by a product of matrices

$$\langle i_1 \cdots i_N | \psi_T \rangle = \text{Tr}\left[A_1^{i_1} \cdots A_N^{i_N} \right] \qquad \text{(closed chain)}$$

$$\langle i_1 \cdots i_N | \psi_T \rangle = \langle \mu_L | A_1^{i_1} \cdots A_N^{i_N} | \mu_R \rangle \qquad \text{(open chain)}$$

- Represented by NdD^2 numbers, rather than d^N

MPS and Gapped Hamiltonians

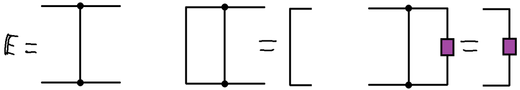
- The bond dimension D required to represent a *generic state* on N sites grows exponentially in N (since $d^N = NdD^2$ implies $D = \sqrt{N}e^{(N-1)\log d/2}$).
- Ground states of gapped local Hamiltonians have less entanglement.
  ... their approximate MPS have D finite. [White 1992] [Hastings, 2007]
  - Therefore entanglement entropy is bounded by a constant, reflecting an area law.
- Given a *translation-invariant* system, there is an MPS realization of its ground states with A_s independent of the site s. [Perez-Garcia, Verstraete, Wolf, Cirac 2007]
- Conversely, each MPS arises as a ground state of a parent Hamiltonian.
  - Assuming translation invariance, this Hamiltonian is gapped.
  - Assuming translation invariance, the ground state degeneracy is constant in N.
  - It is K-local with $K \sim 2\log D/\log d$. By blocking sites, achieve $K = 2$.

The MPS transfer matrix

- The MPS representation of a given state is not unique.
  - This "gauge freedom" is partially fixed by a canonical form. Remaining freedom is conjugation of A^i by a unitary matrix. [Perez-Garcia, Verstraete, Wolf, Cirac 2007]

- Transfer matrix

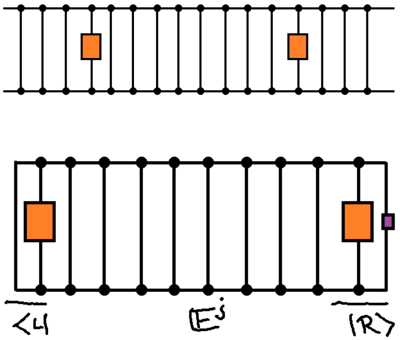
$$\mathbb{E} = \sum_i {A^i}^\dagger \otimes A^i$$



- Canonical form: $\mathbb{1}$ is a left fixed point of $\mathbb{E}$.

- Injective: $\mathbb{1}$ is the unique fixed point of $\mathbb{E}$.
  - Implies there is a unique right fixed point ρ, full rank.

Finite correlation length

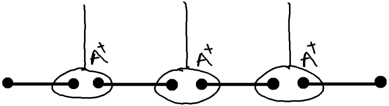


$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle L(\mathcal{O}_1) | \mathbb{E}^j | R(\mathcal{O}_2) \rangle \sim e^{j \log \lambda}$$

MPS as a 1D PEPS

Ingredients:

1. physical Hilbert space $\mathcal{H} \simeq \mathbb{C}^d$
2. virtual space $V \simeq \mathbb{C}^D$
3. MPS tensor $A : \mathcal{H} \to V \otimes V^*$



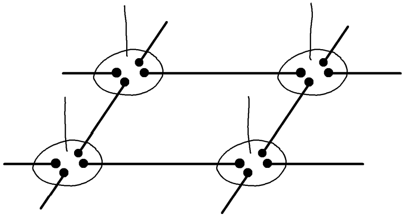
- Construction:
  - maximally entangled pair $|\omega\rangle \in V^* \otimes V$ on each edge
  - act on each vertex by $A^\dagger : V \otimes V^* \to \mathcal{H}$.

The result is a the MPS state

$$|\psi\rangle = (A_1^\dagger \otimes A_2^\dagger \otimes \cdots \otimes A_N^\dagger)(|\omega\rangle_{12} \otimes |\omega\rangle_{23} \otimes \cdots \otimes |\omega\rangle_{N1})$$
$$= \sum_{i_1 \dots i_N} \mathrm{Tr}\left[A^{i_1} \cdots A^{i_N}\right] |i_1 \cdots i_N\rangle \, .$$

PEPS in higher dimensions

- Maximally entangled pairs on edges. Projector on vertices.

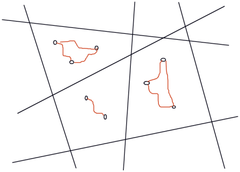


- Entanglement between two regions is characterized by the number of virtual legs across the cut $\implies$ Area law.

Invertible Phases and SPTs

Gapped Phases

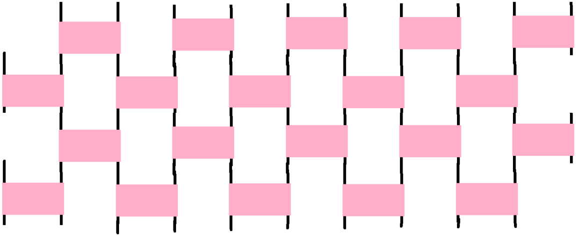
- Ancillas: $H = \sum_v h_v$ with unique ground state a product state $\otimes_v |0\rangle_v$.
- Phase equivalence: Two systems are in the same phase if they are related by appending ancillas and smoothly deforming without closing the gap.
    - Fragile phases: defined by deformations without ancillas.



- Intuition: observables suffer discontinuities at degeneracies in the spectrum. A gapped phase groups microscopic systems with identical low-energy physics.

Circuits

- Equivalently, two states are in the same phase if they are related by appending ancillas and evolving by a local quantum circuit of low depth.

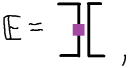


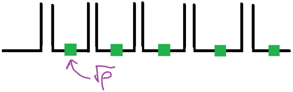
No phases in one dimension

- All 1D gapped ground states belong to the trivial phase. [Chen, Gu, Wen 10]
- Argument from MPS [Bridgeman, Chubb 17]

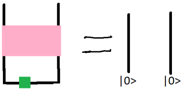
$$\mathbb{E}^k = |\rho\rangle\langle\mathbb{1}| + \mathcal{O}(\lambda^k)$$

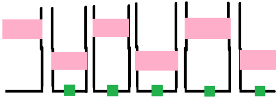
$\implies$ the MPS looks like





Disentangled by the circuit whose gates satisfy





- In 2D, there are many nontrivial phases. A large class of them (string-net models) have PEPS representations. [Williamson, Bultinck, Marien, Sahinoglu, Haegeman, Verstraete 16]

Stacking

- Two gapped systems may be stacked to produce a third gapped system:

$$A + B = C,$$

with

$$H_C = H_A \otimes \mathbb{1}_B + \mathbb{1}_A \otimes H_B \ .$$

- Commutative.
- Ground state degeneracy (GSD) is multiplicative:

$$\mathrm{gsd}(A + B) = \mathrm{gsd}(A)\mathrm{gsd}(B) \ .$$

- Stacking is compatible with phase equivalence:

$$A \sim A', B \sim B' \Rightarrow A \otimes B \sim A' \otimes B' \ ,$$

so phases can be stacked

$$[A] + [B] = [C] \ .$$

Invertible Phases

- Trivial phase: contains the ancilla. The unit of stacking:
$$[0] + [A] = [A] \; .$$
- Invertible phase: has in inverse
$$[A] + [A^{-1}] = [0] \; .$$
- Invertible phases form an abelian group.
- Invertibility $\Longrightarrow$ non-degenerate ground state
  - In 2+1d, no bulk anyonic excitations.
  - Almost trivial. However, has nontrivial (anomalous) boundary physics and response to probes.
  - Sometimes called "short-range entangled."
- Perhaps surprisingly, nontrivial invertible phases exist!
  - 2+1d bosonic E_8 state (chiral with $c = 8$)
  - In fermionic systems (where superselction enforced), 1+1d and 2+1d p+ip superconductors.
    - All systems of *free* fermions are invertible, and there are additional *intrinsically interacting* phases of fermions.

Invertible Phase Classification

- Classification is conjecturally related to cobordism groups.

  [Kapustin 14] [Kapustin, Thorngren, AT, Wang 15] [Gaiotto, Johnson-Freyd 19] [Freed, Hopkins 21]

- Motivations: topological actions, topological field theories, generalized cohomology arguments

$d = D + 1$	no symmetry	$T^2 = 1$	$T^2 = (-1)^F$	unitary $\mathbb{Z}_2$
1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_2^2$
2	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
3	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_8 \times \mathbb{Z}$
4	0	0	$\mathbb{Z}_{16}$	0
5	0	0	0	0
6	0	$\mathbb{Z}_{16}$	0	0
7	$\mathbb{Z}^2$	0	0	$\mathbb{Z}_{16} \times \mathbb{Z}^2$
8	0	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_{32}$	0
9	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2^4$
10	$\mathbb{Z}_2^2$	$\mathbb{Z}_2 \times \mathbb{Z}_8 \times \mathbb{Z}_{128}$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^4$

- Open questions surrounding whether certain high dimensional states, e.g. the ground state of the 3-Fermion Walker-Wang model, can be disentangled.

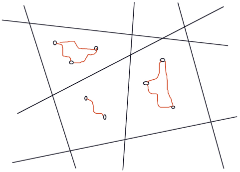
  [Haah, Fidkowski, Hastings 18]

Symmetry-enriched phases

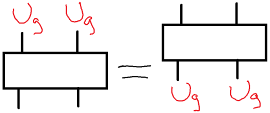
- Global symmetry G acts on-site as

$$g \mapsto U_g \otimes \cdots \otimes U_g \ .$$

- Restrict to symmetric systems and deformations.



- States in a phase are related by constant depth circuits of symmetric gates.



Symmetry-protected trivial/topological phases

- G-enriched phases can also be stacked. G-enrichments of invertible topological orders are themselves invertible and form an abelian group $\mathcal{I}_G^d$.
- Symmetry-enrichments of the trivial phase are called SPT phases. These are trivial in the absence of symmetry, so symmetry "protects" them.

$$\mathsf{SPT}_G^d \subseteq \mathcal{I}_G^d$$

- Many SPT phases fall into the group cohomology classification

$$H^{d+2}(G, \mathbb{Z}) \subseteq \mathsf{SPT}_G^d \ ,$$

while others are beyond cohomology SPT phases.

Symmetry fractionalization in one dimension

- In 1+1d, cohomology captures the symmetry fractionalization on the edge.

- Boundary action V_g may be projective

$$V_g V_h = \omega(g,h) V_{gh} \ , \qquad \omega : G \times G \to U(1)$$

[Chen, Gu, Wen 10] [Schuch, Perez-Garcia, Cirac 11] [Else, Nayak 14]

Group cohomology

- Associativity $(V_g V_h) V_k = V_g (V_h V_k)$ implies the 2-cocycle condition

$$\frac{\omega(g, hk)\omega(h, k)}{\omega(g, h)\omega(gh, k)} = \delta\omega(g, h, k) = 1 \ .$$

- Freedom $V_g \mapsto \beta(g) V_g$ shifts by 2-coboundary:

$$\omega(g, h) \mapsto \omega(g, h)\frac{\beta(g)\beta(h)}{\beta(gh)} = \omega(g, h)\delta\beta(g, h) \ .$$

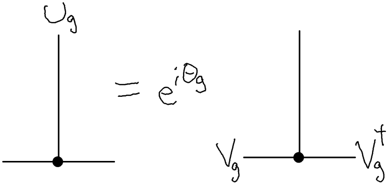
- Set of ω's modulo $\delta\beta$'s is group cohomology $[\omega] \in H^2(G, U(1))$.

- This was not the original argument historically.
  - Using MPS. [Pollmann, Turner, Berg, Oshikawa 10]
  - Classifying topological fixed points. [Chen, Gu, Wen 11]
  - Modern, formal argument. [Ogata 18] [Kapustin, Sopenko, Yang 20]

Fractionalization in MPS

- MPS Perspective. Symmetry means $U_g^N|\psi\rangle = |\psi\rangle$. Represent each as an MPS.
- Tensors A^i and $(U_g)^{ij}A_j$ define the same state $\Rightarrow$ related by a gauge trans.:

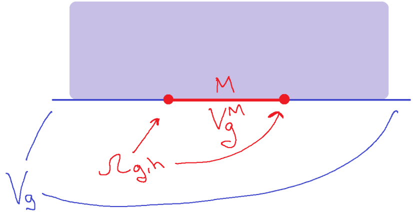
$$U_g^{ij}A_j = e^{i\theta_g}V_g A_i V_g^\dagger \ .$$



- The *weak invariant* $\theta_g \in \mathrm{SPT}_G^0$ is an artifact of translation-invariance.

The SPT invariant in 2D

- G acts on the boundary as V_g.



- Restrict V_g to a region M of the boundary. The group law for V_g^M only needs to hold up to $\Omega_{g,h}$ on $\partial M = \{a, b\}$...

$$V_g^M V_h^M = \Omega_{g,h} V_{gh}^M \ .$$

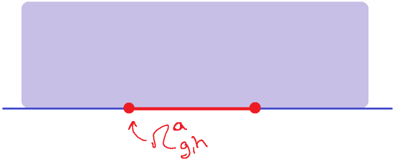
- Associativity of V_g^M means that $\Omega_{g,h}$ satisfies

$$\Omega_{g,h}\Omega_{gh,k} = V_g^M \Omega_{h,k} (V_g^M)^{-1} \Omega_{g,hk} \ .$$

Else-Nayak argument, continued

- Restrict $\Omega_{g,h}$ from $\partial M = \{a, b\}$ to $\{a\}$. The restriction $\Omega^a_{g,h}$ now obeys the relation only up to a phase:

$$\Omega^a_{g,h}\Omega^a_{gh,k} = \omega(g,h,k) V^M_g \Omega^a_{h,k} (V^M_g)^{-1} \Omega^a_{g,hk} \ .$$



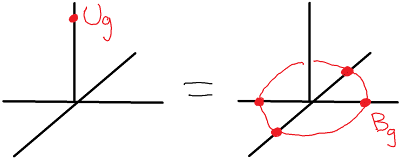
- Further analysis shows that ω is a 3-cocycle [Else, Nayak 14]

$$\frac{\omega(g,h,k)\omega(g,hk,l)\omega(h,k,l)}{\omega(gh,k,l)\omega(g,h,kl)} = \delta\omega(g,h,k,l) = 1$$

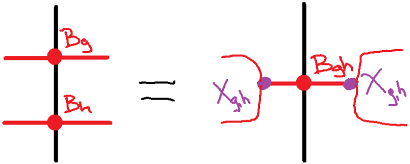
and is only defined up to a 3-coboundary $\delta\beta$. Conclude $[\omega] \in H^3(G, U(1))$.

Cohomology in PEPS

- Acting by U_g on the physical leg of the PEPS tensor creates a matrix product operator (MPO) loop on the virtual level.



- Virtual symmetry MPO tensors are related by a zipper tensor $X_{g,h}$, which plays the role of $\Omega^a_{g,h}$.

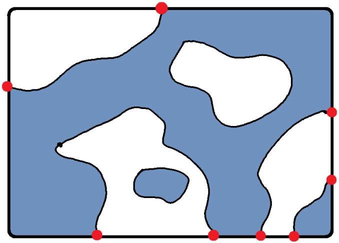


[Williamson, Bultinck, Marien, Sahinoglu, Haegeman, Verstraete 14]

Decorated domain walls (rough idea)

- Idea: build a d-dim $G \times H$ SPT phase out of a $(d-1)$-dim H SPT

- start in G-broken phase
- states have configurations of G domain walls
- proliferating the domain walls restores the G symmetry
- before proliferating, consistently decorate the domain walls with an H SPT phase



$$H^{d+2}(G \times H, \mathbb{Z}) = H^{d+2}(H) + H^1(G, H^{d+1}(H, \mathbb{Z})) + H^2(G, H^d(H, \mathbb{Z})) + \cdots$$

- Edge modes of the domain wall decorations contribute to the edge physics of the phase.

[Chen, Lu, Vishwanath 14] [Wen 15] [Wang, Gu 20] [Wang, Ning, Cheng 21] [many others]

Beyond cohomology

- In most general construction, lower dimensional defects (junctions of domain walls) are also decorated, subject to consistency conditions.

- Decorate defects with invertible topological orders.

- non-SPT decorations $\Longrightarrow$ beyond-cohomology SPT
  - by decorating domain walls with E_8, all bosonic SPTs in 3+1d can be constructed

- Fermionic phases:
  - Supercohomology: at most 0D defects are decorated (with complex fermions)
  - Beyond supercohomology: higher dimensional defects are decorated as well

- Response theory $Z[M, A]$ (examples):
  - Cohomology SPT phase: $\int F^2$
  - Beyond-cohomology SPT phase: $\int w_2 F$
  - Invertible topological order: $\int w_2^2$

Example: fermionic invertible phases in 1D

- Invertible topological orders:
  - 0D, complex fermion with odd (occupied) ground state.
  - 1D, the p+ip superconductor, modeled by the Kitaev Majorana chain.
- In 1D, $\mathbb{Z}_2^f \times G$-enriched invertible phases are classified by... [Fidkowski, Kitaev 09]
  - $\alpha \in H^2(G, U(1))$, bosonic SPT layer
  - $\beta \in H^1(G, \mathbb{Z}_2)$, decoration of domain walls by complex fermions
  - $\gamma \in H^0(G, \mathbb{Z}_2) = \mathbb{Z}_2$, Kitaev chain layer
  - $\gamma = 0$ are SPT phases, all belonging to supercohomology
- Different types of anomalies on the boundary:
  - α encodes the projectivity of the boundary action V_g
  - β (when $\gamma = 0$) encodes whether V_g is parity-odd on the boundary
  - $\gamma = 1 \Rightarrow$ dangling Majorana modes, boundary Hilbert space "ill-defined"

Detection of Invertible Phases

Order Parameters

- Under the Landau paradigm, phases are characterized by patterns of symmetry breaking.
    - Distinguished by local order parameters: ground state expectation values of local operators $\Delta(x)$ in a nontrivial representation of the symmetry.
    - $\langle \Delta(x) \rangle \neq 0$ indicates that the symmetry is broken (ordered phase).
- Topological states are locally indistinguishable from product states.
    - Must be detected by nonlocal order parameters.

String operators

- Consider and abelian symmetry group G.
- String operators [den Nijs, Rommelse 89] [Perez-Garcia, Wolf, Sans, Verstraete, Cirac 08] [Pollmann, Turner 12]

- End operators: let α label an irrep of G, χ_α its character

$$U_g^\dagger O_\alpha U_g = \chi_\alpha(g) O_\alpha \ .$$

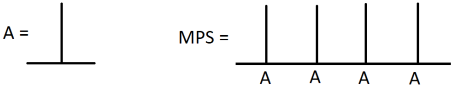
- Expectation values display a "pattern of zeros"

$$\langle s(g, O_\alpha) \rangle = 0 \quad \text{unless} \quad \chi_\alpha(h) \frac{\omega(g,h)}{\omega(h,g)} = 1 \text{ for all } h \ .$$

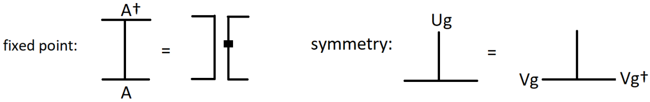
- Pattern determines ω/ω and therefore SPTO.

Detecting the pattern of zeros of an MPS state

- Represent the ground state as a matrix product state (MPS)



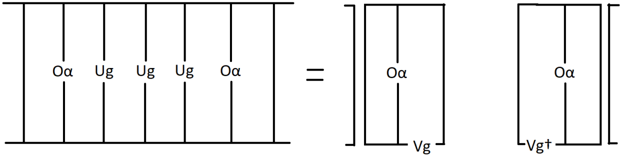
subject to conditions



- Projective representation $V_g V_h = \omega(g, h) V_{gh}$ encodes the phase invariant ω

Detecting the pattern of zeros of an MPS state

- Evaluate the expectation value by applying the relations on the MPS tensor:

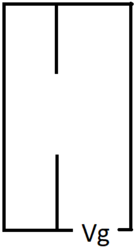


- Each end evaluates to $\text{Tr}[N_g O_\alpha]$.
- O_α transforms as α, while N_g transforms as ω/ω.
- Vanishes unless these characters are equal.

$$\langle s(g, O_\alpha)\rangle = 0 \quad \text{unless} \quad \chi_\alpha(h)\frac{\omega(g,h)}{\omega(h,g)} = 1 \text{ for all } h\ .$$

- If they are equal, *generically* $\text{Tr}[N_g O_\alpha] \neq 0$.

Ng =



Reconstructing the SPT invariant

- Claim: for abelian G, the ratios ω/ω (which we just measured) completely determine the cohomology class $[\omega]$.

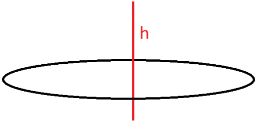
- Argument:
  - We show that the kernel of $\omega \mapsto \omega/\omega$ consists only of coboundaries.
  - Suppose $\omega/\omega = 1$. Then $V_g V_h = V_h V_g$, for all g, h.
  - By Schur's lemma, V_g is proportional to the identity: $V_g = \beta(g)\mathbb{1}$.
  - But then
  $$\beta(g)\beta(h)\mathbb{1} = V_g V_h = \omega(g,h)V_{gh} = \beta(gh)\omega(g,h)\mathbb{1} \ ,$$
  so $\omega = \delta\beta$.

- If G is non-abelian, other non-local order parameters are needed to fully reconstruct ω. [Pollmann, Turner 12]
  - These correspond to higher genus topologies. [Shiozaki, Ryu 17]

Detection of SPTO with twisted sector charges

- Twist the state by inserting an h flux:

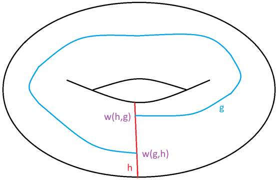
$$|\psi_h\rangle = \mathrm{Tr}\left[\, V_h\, A^{i_1} \cdots A^{i_N} \,\right]$$



- Ground state of twisted Hamiltonian H_h

- Act by U_g on each site:

$$U_g^{\otimes L} \cdot |\psi_h\rangle = \sum \mathrm{Tr}\left[\, V_g V_h V_g^{-1} A^{i_1} \cdots A^{i_N} \,\right]$$
$$= \frac{\omega(g,h)}{\omega(h,g)} |\psi_h\rangle$$



[Shiozaki, Ryu 17] [Kapustin, AT, You 17]

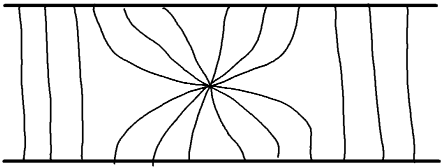
- ω/ω is the partition function of the torus $T^2_{g,h}$

Partial inversion

- Partial inversion operator: [Pollmann, Turner 12]

$$\mathcal{I} = \mathbb{1} \otimes |i_1 i_2 \cdot i_n\rangle\langle i_n \cdots i_2 i_1| \otimes \mathbb{1}$$

$$\text{sign}(\langle \mathcal{I} \rangle) = \pm 1$$

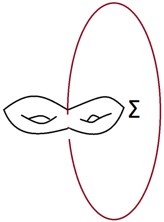


- Detects the $\mathbb{Z}_2$ classification of reflection invariant phases.

TQFT for topological phases

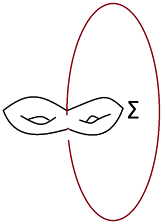
- Universal behavior ("the phase") of a lattice system = effective QFT
  - gapped bosonic phase ⇔ topological QFT
- Spacetime formulation: order parameters are encoded in path integrals
- Responses to gauge and gravitational probes
- Examples:
  - $\mathcal{Z}(T_{g,h}) = \omega/\omega$ captures the string order of 1D SPTs
  - GSD on a space Σ is the path integral $\mathcal{Z}(\Sigma \times S^1)$ on the "cylinder of Σ"

$$\mathcal{Z}(\Sigma \times S^1) = \mathrm{Tr}[\mathbb{1}_\Sigma] = \dim \mathcal{H}_\Sigma$$

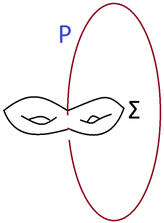


Incorporating fermions

- Boundary conditions (periodic and anti-periodic) for fermions = spin structure
  - gapped bosonic system ⇒ TQFT
  - gapped fermionic system ⇒ Spin-TQFT
- Once again, the Spin-TQFT encodes order parameters...



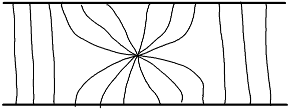
$$\mathcal{Z}(\Sigma \times S^1_{AP}) = \mathrm{Tr}[\mathbb{1}_\Sigma] = \dim \mathcal{H}_\Sigma$$

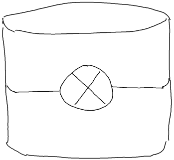


$$\mathcal{Z}(\Sigma \times S^1_{P}) = \mathrm{Tr}[P_\Sigma] = \dim \mathcal{H}^0_\Sigma - \dim \mathcal{H}^1_\Sigma$$

Incorporating time-reversal / reflection symmetry

- T-symmetry means the theory is insensitive to spacetime orientation, can be defined on spacetimes without orientation
  - gapped bosonic system with $\mathbb{Z}_2^T$ symmetry $\Rightarrow$ unoriented TQFT
  - gapped fermionic system with $\mathbb{Z}_2^T \times \mathbb{Z}_2^F$ symmetry $\Rightarrow$ Pin$^-$-TQFT
- Partial inversion order parameter interpreted as $\mathcal{Z}(\mathbb{R}P^2)$ [Shiozaki, Ryu 17]





- Fermionic invertible phases (in 2D and 3D) in the interacting periodic table have order parameters realized as $\mathcal{Z}(M)$ [Shapourian, Shiozaki, Ryu 17] [Shiozaki, Shapourian, Gomi, Ryu 18]
- Unitary invertible TQFTs are completely determined by their closed-spacetime partition functions [Freed, Moore 06] [Yonekura 18] , so in principle all order parameters of invertible phases should arise from them.

Free Fermions

Free and Interacting Fermion Systems

- Consider lattice systems of Fock space built from fermion operators $a_j^A, (a_j^A)^\dagger$.

free fermion system

$$H = a^\dagger \Xi a + a^\dagger \Delta a^\dagger + h.c.$$

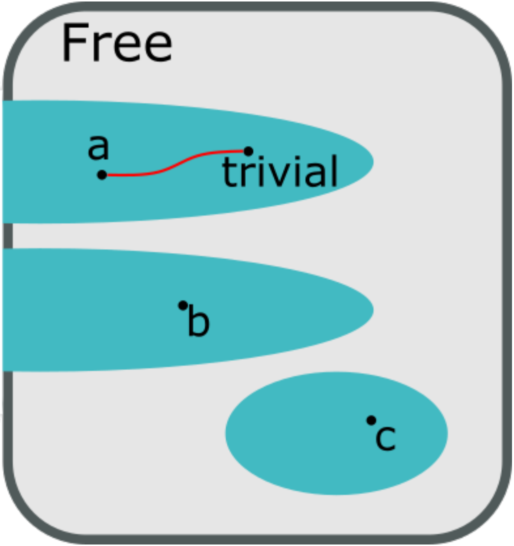
interacting fermion system

$$H = a^\dagger \Xi a + a^\dagger \Delta a^\dagger + h.c.$$
$$+ t a^\dagger a^\dagger a a + \cdots$$

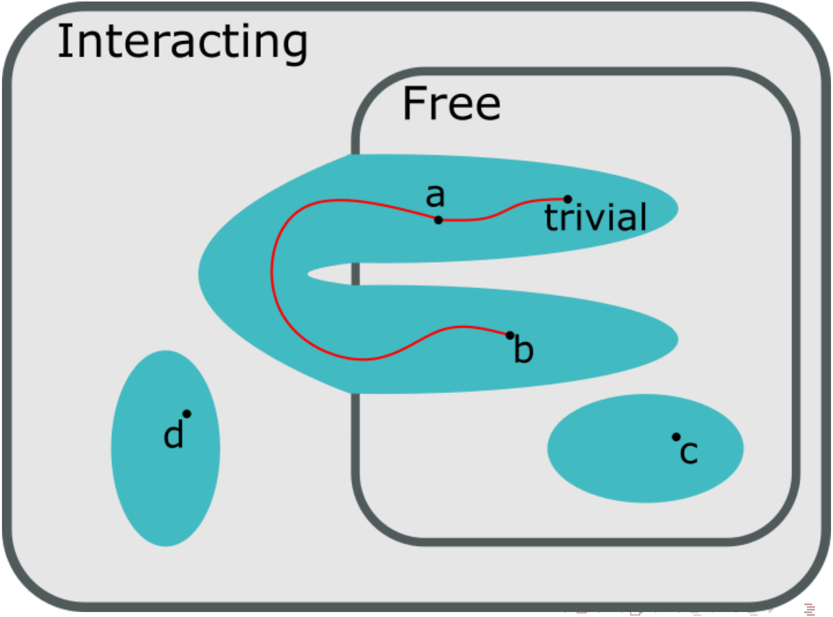
Questions:

- What is the classification of free fermion phases with a symmetry G? How does it relate to that of invertible interacting fermion phases with G?

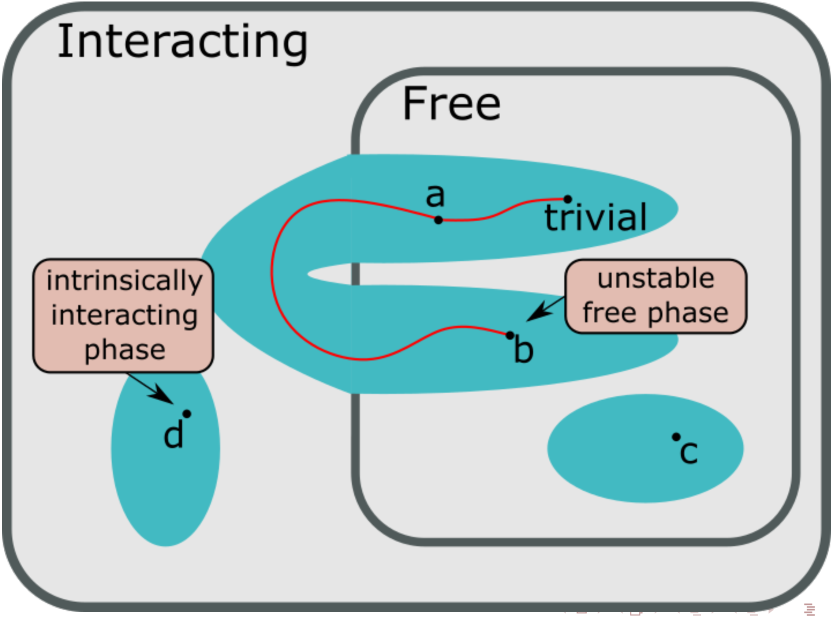
Free and Interacting Phases



Free and Interacting Phases



Free and Interacting Phases



The Effects of Interactions: $\mathbb{Z} \to \mathbb{Z}/8$ in 1D Class BDI

Consider the famous example of Fidkowski & Kitaev (2008)...

- Fermions $a_j^A, (a_j^A)^\dagger$, indexed by $j =$ 1D lattice site, $A = 1, \ldots, n$ species
- Real fermions

$$\Gamma^I_{2j-1} = a_j^A + (a_j^A)^\dagger, \qquad \Gamma^A_{2j} = -i(a_j^A - (a_j^A)^\dagger), \qquad \{\Gamma_I^A, \Gamma_J^B\} = 2\delta_{IJ}\delta^{AB}$$

- Time-reversal symmetry

$$T^2 = 1, \quad T a_J T^{-1} = -a_J$$

- Local translation-invariant free (quadratic) fermion Hamiltonian

$$\hat{H} = \frac{i}{2} \sum_j \left(u_{AB} \Gamma^A_{2j-1} \Gamma^B_{2j} + v_{AB} \Gamma^A_{2j} \Gamma^B_{2j+1} \right)$$

- Interested in values of parameters u, v such that $\hat{H}$ is gapped and T-symmetric.

The Effects of Interactions: $\mathbb{Z} \to \mathbb{Z}/8$ in 1D Class BDI

- Stable deformation classes ("phases")
  1. tensor by an ancilla system with product state ground state

$$\hat{H}_0 = \sum_j \left(a_j^\dagger a_j - \frac{1}{2} \right) = \frac{-i}{2} \sum_j \Gamma_{2j-1} \Gamma_{2j}, \qquad |\psi_{g.s.}\rangle = \otimes_j |0\rangle_j, \quad a_j |0\rangle_j = 0$$

  2. continuously deform the parameters while preserving the gap and symmetry
- Nontrivial Majorana chain

$$\hat{H}_1 = \frac{-i}{2} \sum_j \Gamma_{2j} \Gamma_{2j+1} = \frac{1}{2} \sum_j \left(-a_j^\dagger a_{j+1} - a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}^\dagger + a_{j+1} a_j \right)$$

- Free classification: $n \in \mathbb{N}$ (boundaries = difference classes $n - m \in \mathbb{Z}$)

$$\hat{H}_n = \sum_A^n \hat{H}_1^A = \frac{-i}{2} \sum_{j,A}^{A=n} \Gamma_{2j}^A \Gamma_{2j+1}^A$$

  - These exhaust all invertible phases.

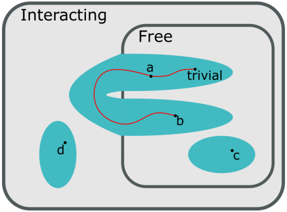
The Effects of Interactions: $\mathbb{Z} \to \mathbb{Z}/8$ in 1D Class BDI

- Now consider turning on local interactions

$$\hat{H} = \frac{i}{2} \sum_j \left(u_{AB} \Gamma^A_{2j-1} \Gamma^\beta_{2j} + v_{AB} \Gamma^A_{2j} \Gamma^B_{2j+1} \right) + t\Gamma\Gamma\Gamma\Gamma + \cdots$$

- In this larger parameter space, $\hat{H}_8$ is destabilized by interactions: $\hat{H}_8 \sim \hat{H}_0$.

$$\mathbb{Z} \to \mathbb{Z}/8$$



- If T-asymmetric terms are permitted, $\hat{H}_2 \sim \hat{H}_0$ at the quadratic level:

$$\mathbb{Z}/2 \to \mathbb{Z}/2$$

The Effects of Interactions: $\mathbb{Z} \to \mathbb{Z}/8$ in 1D Class BDI

- One dimensional phases of free fermions with $T^2 = 1$ time-reversal symmetry (i.e. Class BDI) are classified by a single $\mathbb{Z}$-valued invariant.

$$\nu \in \mathbb{Z} = \text{ number of dangling Majorana modes}$$

- Dangling modes may be *gapped out* 8-at-a-time by a quartic interaction. In other words, the $\nu = 8$ phase is destabilized by interactions. [Fidkowski, Kitaev 09]

$$\{ \text{ Free Phases } \} \longrightarrow \{ \text{ Interacting Phases } \}$$

$$\begin{array}{ccc} \mathbb{Z} & \longrightarrow & \mathbb{Z}/8 \\ \nu & \mapsto & \nu \text{ mod } 8 \end{array}$$

Classifications with 10-fold way symmetries

The K-theory [Schnyder, Ryu, Furusaki, Ludwig 08] [Kitaev 09] and cobordism classif.s of invertible phases yield similar results in all dimensions and 10-fold way symmetry classes.

$$KO_0 = \mathbb{Z}, \qquad \mathrm{Hom}(\mathrm{Tors}(\Omega_2^{\mathrm{pin}^-}), U(1)) = \mathbb{Z}/8$$

Free	0	1	2	3	4	5	6	7
BDI	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$		$\mathbb{Z}_2$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$	
DIII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$

$\longrightarrow$

Int.	0	1	2	3	4	5	6	7
BDI	$\mathbb{Z}_2$	$\mathbb{Z}_8$				$\mathbb{Z}_{16}$		$\mathbb{Z}_2^2$
D	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}^2$	
DIII		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{16}$				$\mathbb{Z}_2 \times \mathbb{Z}_{32}$

Note: consider only strong invariants; translation-invariance is not protected.

Intrinsically interacting phases. For example, class D systems in 6D.

$$\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$$

In low dimensions,

- Examples of intrinsically interacting crystalline SPT phases. [Lapa, Teo, Hughes 14]
- Examples of int. int. SPTs of on-site symmetries. [Chen, Kapustin, AT, You 19]

Summary

- Entanglement structure of gapped states enables tensor network methods.
- Under stacking, some gapped phases are invertible. These include SPTs.
- Invertible phases are characterized by the anomalies of their boundaries. For a large class of SPTs, the anomaly is captured by group cohomology.
- Invertible phases give rise to beyond cohomology SPT phases via decorated domain wall constructions. These phases have gravitational/mixed anomalies.
- Invertible phases are distinguished by non-local order parameters (such as string operators), which are closely related to topological partition functions.
- The free and interacting invertible phase classifications can be related. Unstable free phases and intrinsically interacting phases exist.