Lieb-Schultz-Mattis type theorems for Majorana models with discrete symmetries

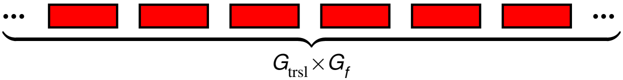
Christopher Mudry[1]
Oemer Aksoy [1] Apoorv Tiwari[2]

[1]Paul Scherrer Institute; Switzerland

[2]Paul Scherrer Institute and University of Zurich, Switzerland

"Galileo Galilei Institute for Theoretical Physics"
September 1 2021

Main results: Theorem (1)



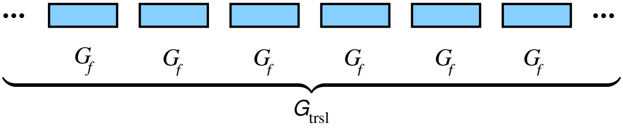
Theorem (1)

For any *d-dimensional Majorana* *lattice Hamiltonian $\widehat{H}$ such that*

1. *$\widehat{H}$ is* ***local****,*
2. *$\widehat{H}$ has $G_{\mathrm{trsl}} \times G_f$ symmetry with G_f a* ***fermionic*** *symmetry,*
3. *$\widehat{H}$ has a* ***nondegenerate****, $G_{\mathrm{trsl}} \times G_f$* ***symmetric****, and* ***gapped*** *ground state,*

then there must be an even number of Majorana degrees of freedom per repeat unit cell.

Main results: Theorem (2)



Theorem (2)

For any one-dimensional Majorana lattice Hamiltonian $\widehat{H}$ such that

1. *$\widehat{H}$ is **local**,*
2. *$\widehat{H}$ has $G_{\rm trsl} \times G_f$ symmetry, with G_f an internal (on-site) **fermionic** symmetry,*
3. *$\widehat{H}$ has **nondegenerate**, $G_{\rm trsl} \times G_f$ **symmetric**, and **gapped** even- or odd-parity injective fermionic matrix product state (FMPS) as ground state,*

*then G_f is realized by a **trivial projective representation**.*

Contraposition: If $d = 1$-$\widehat{H}$ (i) is local, (ii) admits the global symmetry group $G_{\rm trsl} \times G_f$, with the fermionic symmetry group G_f an internal symmetry that is realized by a nontrivial projective representation, then $\widehat{H}$ cannot have a nondegenerate, gapped, and $G_{\rm trsl} \times G_f$-symmetric ground state that can be described by an even- or odd-parity injective fermionic matrix product state (FMPS).

Main results: Comments

1. The thermodynamic limit is implicit in both theorems.
2. The direct product structure of the symmetry group $G_{\mathrm{trsl}} \times G_f$ is crucial in Theorems 1 and 2.
3. Theorem 1 holds in any dimension without any restriction on G_f.
4. Theorem 1 for one-dimensional lattices can be proved with the help of Theorem 2.
5. When G_f is continuous, its projective representation on the local Fock space can be trivial. **If so**, the contraposition of Theorem 2 **is not** predictive. However, one can use complementary arguments, such as the adiabatic threading of a gauge flux (Laughlin 1981, Oshikawa 2000), to decide if the gapped ground state is degenerate. The full power of Theorem 2 is unleashed when G_f is a finite group.
6. A weaker form of Theorem 2 holds in any dimension if it is assumed that G_f is Abelian and can be realized locally using unitary operators.

A review of the Lieb-Schultz-Mattis theorem

One motivation by Lieb, Schultz, and Mattis in 1961 was to find an analytical argument that could decide if the nearest-neighbor antiferromagnetic quantum spin-1/2 Heisenberg chain supports antiferromagnetic long-range order at zero temperature.

Although they could not answer this question rigorously (Mermin and Wagner in 1966 proved rigorously that the ground state does not support antiferromagnetic long-range order), they could show rigorously that the antiferromagnetic quantum spin-1/2 *XY* Hamiltonian has a gapless spectrum with all correlation functions of spins decaying algebraically in space.

In their appendices, Lieb, Schultz, and Mattis also established two theorems, the second of which is now called the Lieb-Schultz-Mattis theorem:

Theorem (3)

The ground state of the nearest-neighbor antiferromagnetic quantum spin-1/2 Heisenberg chain made of N sites is annihilated by the total spin operator

$$\widehat{\boldsymbol{S}} := \sum_{j=1}^{N} \widehat{\boldsymbol{S}}_j \tag{1}$$

for any even integer N.

Theorem (Lieb-Schultz-Mattis)

The nearest-neighbor antiferromagnetic quantum spin-1/2 Heisenberg chain made of N sites and obeying periodic boundary conditions supports an excited eigenstate with an energy of order $1/N$ *above the nondegenerate ground state for* ***any even integer*** N.

Remark (1)

The (original) proof of the LSM theorem **does not apply** to the nearest-neighbor antiferromagnetic quantum spin-1 Heisenberg chain. It is still possible to show for $S = 1$ that there exists a state with an energy of order $1/N$ above the ground state for any N, however, it is not possible to show that this state is orthogonal to the ground state.

Remark (2)

LSM only prove the existence of **at least one** excited state with an energy that collapses like $1/N$ to that of the ground state in the thermodynamic limit $N \to \infty$. This state might not be isolated so that there is no guarantee that the thermodynamic limit of the ground state and this excited state are distinct. Hence, **neither** the existence of a gapless continuum of states **nor** the degeneracy of gapped ground states in the thermodynamic limit have been shown.

Remark (3)

The (original) proof by LSM makes use of the global $\mathrm{SU}(2)$ symmetry, of time-reversal symmetry, and of Theorem 3 (i.e., the fact that the ground state is nondegenerate).

Remark (4)

The same constructive proof applied when $d > 1$ would imply the bound N^{d-2} between the ground state and the excited states. This bound is thus useless when $d > 1$.

The qualitative difference (Haldane 1983) between half-integer and integer antiferromagnetic quantum spin-S Heisenberg chains motivated the following refinements of the Lieb-Schultz-Mattis Theorem 4:

1. Affleck and Lieb in 1986 showed that any half-integer antiferromagnetic quantum spin-S Heisenberg chain with translation and global internal $\mathrm{U}(1)$ spin invariance has a nondegenerate ground state for any finite chain made of an **even** number N of sites with a gap of order $1/N$. In the thermodynamic limit, the ground state manifold is **either** degenerate and gapped **or** nondegenerate and gapless.
2. Oshikawa, Yamanaka, and Affleck in 1997 replaced the condition of Affleck and Lieb that S is a **half integer** with the condition that $\nu := \frac{M^z}{N} + S$ is **not an integer**.
3. An analogous theorem was proven by Yamanaka, Oshikawa, and Affleck in 1997 for any local lattice model of interacting electrons for which the electronic charge is conserved, translation symmetry holds, and the ratio ν between the (conserved) total number of electrons N_{f} and the number of sites N on the ring is not an integer.
4. All these papers had always chosen Hamiltonians for which **either** reversal of time **or** inversion in space were symmetries. This assumption was shown by Koma in 2000 to be superfluous.

The most recent and general extension of the Lieb-Schultz-Mattis Theorem is due to Tasaki in 2018. Its proof relies on three steps:

1. First, variational state**s** are constructed.
2. Second, the energy expectation value**s** for these variational states are shown to collapse to the ground-state energy in the thermodynamic limit.
3. Third, the variational state**s** are shown to be orthogonal with each other and with the ground state for any fixed number of degrees of freedom. Finally, the conditions are given (no spontaneous symmetry breaking of translation symmetry) under which th**ese** orthogonalitie**s** survive the thermodynamic limit.

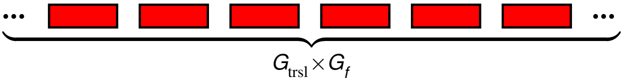
The variational state**s** in the first step are constructed by **deformations in position space of a ground state** that are **local** and **smooth**. Here, the existence of an internal (not spatial) global **continuous** symmetry of the Hamiltonian is crucial.
The degree of smoothness of these local deformation**s** is controlled by the **length** of the one-dimensional lattice hosting the quantum degrees of freedom and the **continuity** of the internal (not spatial) global symmetry of the Hamiltonian. The longer the length of the one-dimensional lattice, the smoother the local deformation**s** in position space of the ground state are and the closer the energy expectation value**s** of the variational state**s** relative to the ground state energy are. The **locality of the Hamiltonian** is needed to control the separation in energy between the variational and ground states. The conditions for **gapped degenerate ground states in the thermodynamic limit** are given: **either** infinite degeneracy **or** spontaneous symmetry breaking of translation symmetry.
The proof of orthogonality in the third step hinges on the filling fraction ν **not being integer valued** and the existence of **translation symmetry** in addition to a continuous symmetry. **No more information from the Hamiltonian is needed to complete this step of the proof.**

One **logical contraposition** of Tasaki's extension of the Lieb-Schultz-Mattis Theorem applies to a **local** lattice Hamiltonian that is invariant under **translation** of the lattice repeat unit cell by one lattice spacing and invariant under a **continuous internal** (not spatial) symmetry group. It also presumes the existence of a positive real-valued number ν, the filling fraction of the repeat unit cell. It states that, if the ground-state manifold is **finitely** degenerate and separated by a gap from all excited states in the thermodynamic limit, then **either** translation symmetry is spontaneously broken **or** ν is an integer.

Question 1: Can the condition that the Hamiltonian is invariant under an internal (not spatial) continuous symmetry be weaken by demanding that the internal symmetry group is no more than a discrete group?

Question 2: Can this discrete group accomodate the conservation of fermion parity?

Proof of Theorem 1 in any dimension $d \geq 1$



Theorem 1 holds for any spatial dimension d and any fermionic symmetry group G_f:

if $\widehat{H}$ is a ***d*-dimensional** Majorana lattice Hamiltonian such that

1. $\widehat{H}$ is local,
2. $\widehat{H}$ has $G_{\mathrm{trsl}} \times G_f$ symmetry,
3. $\widehat{H}$ has a nondegenerate, $G_{\mathrm{trsl}} \times G_f$ symmetric, and gapped ground state,

then there must be an **even** number of Majorana degrees of freedom per repeat unit cell.

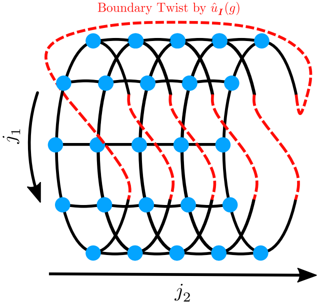
Assume an odd number of Majorana degrees of freedom per repeat unit cell. We are going to contradict the assumption of nondegeneracy of the gapped ground state. We consider a d-dimensional lattice Λ such that at each repeat unit cell labeled by $\boldsymbol{j} \in \Lambda$, there exists a Majorana spinor $\hat{\chi}_{\boldsymbol{j}}$ with $2n+1$ components $\hat{\chi}_{\boldsymbol{j},l=1,\cdots,2n+1}$.

The total number of sites $|\Lambda|$ in the lattice is **even** by assumption. On lattice Λ, we impose the **tilted** translation symmetry group

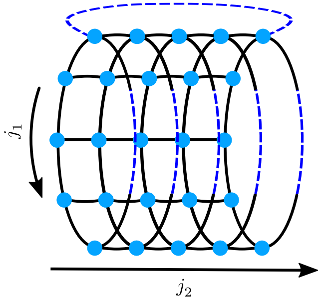
$$G^{\mathrm{tilt}}_{\mathrm{trsl}} := \left\{ t^n \,\middle|\, n = 1, \cdots, |\Lambda| \right\} \equiv \mathbb{Z}_{N_{\hat{1}} \cdots N_{\hat{d}}} \equiv \mathbb{Z}_{|\Lambda|} \tag{2}$$

as opposed to the periodic boundary conditions

$$G_{\mathrm{trsl}} := \mathbb{Z}_{N_{\hat{1}}} \times \mathbb{Z}_{N_{\hat{2}}} \times \cdots \times \mathbb{Z}_{N_{\hat{d}}}. \tag{3}$$



VS



In terms of the Majorana spinors $\hat{\chi}_{\boldsymbol{j}}$, the total fermion parity operator $\widehat{P}$ has the representation

$$\widehat{P} := \mathrm{i}^{|\Lambda|/2} \prod_{\boldsymbol{j}\in\Lambda} \prod_{l=1}^{2n+1} \hat{\chi}_{\boldsymbol{j},l}. \tag{4}$$

Conjugation of the fermion parity operator $\widehat{P}$ by the tilted translation operator $\widehat{T}_{\hat{1}}$ delivers

$$\widehat{T}_{\hat{1}}\,\widehat{P}\,\widehat{T}_{\hat{1}}^{-1} = (-1)^{|\Lambda|-1}\widehat{P} = -\widehat{P}, \tag{5}$$

where we arrived at the last equality by noting that $|\Lambda|$ is an **even** integer. The factor $(-1)^{|\Lambda|-1}$ arises since each spinor $\hat{\chi}_{\boldsymbol{j}}$ consists of an **odd** number of Majorana operators:

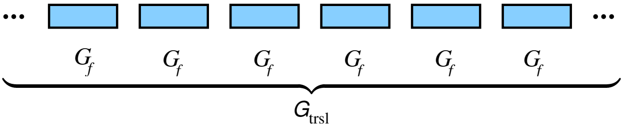
$\widehat{T}_{\hat{1}}\widehat{P}\widehat{T}_{\hat{1}}^{-1} = -\widehat{P} \Longleftrightarrow \widehat{T}_{\hat{1}}\,\left(\widehat{A}\widehat{B}\cdots\widehat{Y}\widehat{Z}\right)\widehat{T}_{\hat{1}}^{-1} = \widehat{B}\cdots\,\widehat{Y}\widehat{Z}\widehat{A} = (-1)^{26-1}\widehat{A}\widehat{B}\cdots\widehat{Y}\widehat{Z}$ for pairwise anticommuting $\widehat{A},\cdots,\widehat{Z}$.

The anticommuting algebra (5) implies a minimal two-fold degeneracy of all energy eigenvalues, in **contradiction** with our assumption that the gapped ground state is nondegenerate.

The algebra (5) was shown by Hsieh 2016 for a one dimensional Majorana chain and interpreted as the presence of Witten's quantum-mechanical supersymmetry (Witten 1982).

The reasoning why the **difference** between **periodic** boundary conditions and **tilted** boundary conditions is conjectured to be **irrelevant in the thermodynamic limit** goes back to Yao and Oshikawa 2021. The same conjecture is used in the proof of Theorem 2.

Proof of Theorem 2 in any dimension $d \geq 1$ when G_f is Abelian and unitarily represented



We prove Theorem 2:
if $\widehat{H}$ is a d-dimensional Majorana lattice Hamiltonian such that

1. $\widehat{H}$ is local,
2. $\widehat{H}$ has $G_{\mathrm{trsl}} \times G_f$ symmetry, with G_f an **Abelian**$^{(\text{if } d>1)}$ internal fermionic symmetry group to be **unitarily**$^{(\text{if } d>1)}$ represented,
3. $\widehat{H}$ has a nondegenerate, $G_{\mathrm{trsl}} \times G_f$ symmetric, and gapped ground-state,

then G_f must be realized by a trivial projective representation.

Our method is inspired by the one used by Yao and Oshikawa 2021 for quantum spin Hamiltonians.

- Consider a d-dimensional lattice Λ with periodic boundary conditions in each linearly independent direction $\hat{\mu} = \hat{1}, \cdots, \hat{d}$ such that Λ realizes a d-torus.
- Let each repeat unit cell be labeled as $\boldsymbol{j}$ and host a local fermionic Fock space $\mathcal{F}_{\boldsymbol{j}}$ that is generated by a Majorana spinor $\hat{\chi}_{\boldsymbol{j}}$ with $2n$ components $\hat{\chi}_{\boldsymbol{j},l=1,\cdots,2n}$. The fermionic Fock space attached to the lattice Λ is $\mathcal{F}_{\Lambda}$.
- Impose the global symmetry corresponding to the central extension G_f of G by $\mathbb{Z}_2^F$ whereby G_f is **assumed to be Abelian** and realized by **unitary** operators.
- Impose translation symmetry. If the d-dimensional lattice Λ has $N_{\hat{\mu}}$ repeat unit cell in the $\hat{\mu}$-direction and thus the cardinality

$$|\Lambda| \equiv \prod_{\hat{\mu}=\hat{1}}^{\hat{d}} N_{\hat{\mu}}, \tag{6}$$

the translation group is

$$G_{\mathrm{trsl}} \equiv \mathbb{Z}_{N_{\hat{1}}} \times \mathbb{Z}_{N_{\hat{2}}} \times \cdots \times \mathbb{Z}_{N_{\hat{d}}}. \tag{7}$$

The representation of the translation group (7) is generated by the unitary operator $\widehat{T}_{\hat{\mu}}$ whose action on the Majorana spinors is

$$\widehat{T}_{\hat{\mu}}\,\hat{\chi}_{\boldsymbol{j}}\,\widehat{T}_{\hat{\mu}}^{-1}=\hat{\chi}_{\boldsymbol{j}+\boldsymbol{e}_{\hat{\mu}}},\qquad \widehat{T}_{\hat{\mu}}^{-1}=\widehat{T}_{\hat{\mu}}^{\dagger}, \tag{8a}$$

along the $\hat{\mu}$-direction ($\boldsymbol{e}_{\hat{\mu}}$ is a basis-vector along the $\hat{\mu}$-direction).

Imposing periodic boundary conditions implies

$$\left(\widehat{T}_{\hat{\mu}}\right)^{N_{\hat{\mu}}}=\widehat{1},\qquad \hat{\mu}=\hat{1},\cdots,\hat{d}. \tag{8b}$$

The representation $\widehat{U}(g)$ of $g \in G_f$ can be projective, i.e., it is defined by

$$\widehat{U}(g) := \bigotimes_{\boldsymbol{j}\in\Lambda} \hat{u}_{\boldsymbol{j}}(g) \tag{9}$$

with

$$\hat{u}_{\boldsymbol{j}}(e) = \hat{1}, \tag{10a}$$

$$\hat{u}_{\boldsymbol{j}}(g)\,\hat{u}_{\boldsymbol{j}}(h) = e^{\mathrm{i}\phi(g,h)}\,\hat{u}_{\boldsymbol{j}}(g\,h), \tag{10b}$$

$$\left[\hat{u}_{\boldsymbol{j}}(g)\,\hat{u}_{\boldsymbol{j}}(h)\right]\hat{u}_{\boldsymbol{j}}(f) = \hat{u}_{\boldsymbol{j}}(g)\left[\hat{u}_{\boldsymbol{j}}(h)\,\hat{u}_{\boldsymbol{j}}(f)\right], \tag{10c}$$

whereby the projective phase must be compatible with associativity

$$\phi(g,h) + \phi(g\,h,f) = \phi(g,h\,f) + \underbrace{\mathfrak{c}(g)}_{=1}\,\phi(h,f), \tag{11}$$

[$\mathfrak{c}(g) = 1$ as we do not allow antiunitary symmetries].

Remark

Even if G_f is Abelian, its realization at the quantum level might not be, say if $\exists g,h \in G_f$ such that $\phi(g,h) \neq \phi(h,g)$!!!

How do we know that we cannot gauge away the phase factor $\phi(g,h)$ in Eq. (10b)? Answering this question is equivalent to enumerating all gauge inequivalent classes of functions

$$\begin{aligned} \phi: G_f \times G_f &\to [0,2\pi), \\ (g,h) &\mapsto \phi(g,h), \end{aligned} \tag{12a}$$

obeying (11) under the equivalence relation $\phi \sim \phi'$, whereby $\phi \sim \phi'$ iff

$$\exists \xi: G_f \to [0,2\pi), \qquad \phi(g,h) - \phi'(g,h) = \xi(g) + \mathfrak{c}(g)\,\xi(h) - \xi(g\,h). \tag{12b}$$

In cohomology, the set of functions ϕ obeying the condition (11) are called 2-cocycles from the group $G_f \times G_f$ to the group $U(1)_{\mathfrak{c}}$. When they are gauge-equivalent to the function $\phi = 0$, i.e., when they are pure gauge, they are called 2-coboundaries. The set $\{[\phi]\}$ of gauge equivalent classes under the similarity relation $\phi \sim \phi'$ is the quotient space of 2-cocycles by 2-coboundaries. This quotient space forms an Abelian group denoted

$$H^2\left(G_f, U(1)_{\mathfrak{c}}\right).$$

It is known as the **second cohomology group**.
By convention, the neutral element of the second cohomology group corresponds to a group representation of G_f since the projective phase can be set consistently to zero in Eq. (10). This is what we call the **trivial projective representation** in Theorem 2.

By assumption, the combined symmetry group is the Cartesian product group

$$G_{\text{total}} \equiv G_{\text{trsl}} \times G_f. \tag{13}$$

Any translation- and G_f-invariant local Hamiltonian acting on $\mathcal{F}_\Lambda$ can be written in the form

$$\widehat{H}_{\text{pbc}} := \sum_{\hat{\mu}=\hat{1}}^{\hat{d}} \sum_{n_{\hat{\mu}}=1}^{N_{\hat{\mu}}} \left(\widehat{T}_{\hat{\mu}}\right)^{n_{\hat{\mu}}} \hat{h}_{\boldsymbol{j}} \left(\widehat{T}^{\dagger}_{\hat{\mu}}\right)^{n_{\hat{\mu}}}, \tag{14a}$$

where $\hat{h}_{\boldsymbol{j}}$ is a **local** Hermitian operator **centered** at an **arbitrary** repeat unit cell $\boldsymbol{j}$. More precisely, it is a **finite-order** polynomial in the Majorana operators **centered** at $\boldsymbol{j}$ that is also invariant under all the non-spatial symmetries, i.e.,

$$\hat{h}_{\boldsymbol{j}} = \widehat{U}(g)\,\hat{h}_{\boldsymbol{j}}\,\widehat{U}^{-1}(g) = \left(\hat{h}_{\boldsymbol{j}}\right)^{\dagger} \tag{14b}$$

for any $g \in G_f$.

Instead of extracting spectral properties of Hamiltonian $\widehat{H}_{\mathrm{pbc}}$ directly, we shall do so with the family of Hamiltonians **indexed** by $g \in G_f$ and given by

$$\widehat{H}^{\mathrm{tilt}}_{\mathrm{twis}}(g) := \sum_{a=1}^{|\Lambda|} \left(\widehat{T}_{\hat{1}}(g)\right)^{a} \, \hat{h}^{\mathrm{tilt}}_{1} \, \left(\widehat{T}^{-1}_{\hat{1}}(g)\right)^{a}, \qquad (15)$$

where $\hat{h}^{\mathrm{tilt}}_{1}$ is a G_f-symmetric and local Hermitian operator and $\widehat{T}_{\hat{1}}(g)$ is the "g-twisted translation operator" to be defined shortly. One verifies that $\widehat{H}^{\mathrm{tilt}}_{\mathrm{twis}}(g)$ commutes with $\widehat{T}_{\hat{1}}(g)$ and with $\widehat{U}(h)$ for all $g, h \in G_f$. We shall derive LSM-like constraints for $\widehat{H}^{\mathrm{tilt}}_{\mathrm{twis}}(g)$.

If $\widehat{H}^{\mathrm{tilt}}_{\mathrm{twis}}(g)$ has a ground state that is nondegenerate, $G_{\mathrm{trsl}} \times G_f$-symmetric, and gapped, the same **could** be true of $\widehat{H}_{\mathrm{pbc}}$, for $\widehat{H}^{\mathrm{tilt}}_{\mathrm{twis}}(g)$ only differs from $\widehat{H}_{\mathrm{pbc}}$ by sub-extensively many local terms in the thermodynamic limit. If so, Theorem 2 applies to both $\widehat{H}^{\mathrm{tilt}}_{\mathrm{twis}}(g)$ and $\widehat{H}_{\mathrm{pbc}}$.

As a warm up, we first consider the one-dimensional case, i.e., $\Lambda \cong \mathbb{Z}_N$. Symmetry-twisted boundary conditions are implemented by defining the symmetry twisted translation operator

$$\widehat{T}_{\hat{1}}(g) := \hat{u}_1(g)\, \widehat{T}_{\hat{1}}. \tag{16a}$$

It follows that

$$\widehat{T}_{\hat{1}}(g)\, \hat{\chi}_j\, \widehat{T}^{-1}_{\hat{1}}(g) = \begin{cases} (-1)^{\rho(g)}\, \hat{\chi}_{j+1}, & \text{if } j \neq N, \\ \hat{u}_1(g)\, \hat{\chi}_1\, \hat{u}^{-1}_1(g), & \text{if } j = N, \end{cases} \tag{16b}$$

for $j = 1, \cdots, N$, where $\rho(g) \in \{0, 1\} \equiv \mathbb{Z}_2$ encodes the fact that $\hat{u}_1(g)$ either commutes with the fermion-parity operator when $\rho(g) = 0$ or anticommutes with the fermion-parity operator when $\rho(g) = 1$.

We then consider any Hamiltonian of the form (15) where the operator $\hat{h}^{\text{tilt}}_1$ in Eq. (15) is nothing but the operator $\hat{h}_{\boldsymbol{j}}$ in Eq. (14a) with Λ restricted to a one-dimensional lattice.

The effect of turning on such a background field is that it delivers the operator algebra

$$\left[\widehat{T}_{\hat{1}}(g)\right]^N = \widehat{U}(g), \qquad g \in G_f \tag{17a}$$

and (!!!**key step of the proof**!!!)

$$\widehat{U}(h)^{-1}\,\widehat{T}_{\hat{1}}(g)\,\widehat{U}(h) = e^{\mathrm{i}\chi(g,h)}\,\widehat{T}_{\hat{1}}(g), \quad h \in G_f, \tag{17b}$$

where the phase $\chi(g,h) \in [0, 2\pi[$ is **gauge invariant** and given by

$$\chi(g,h) := \phi(h,g) - \phi(g,h) + (N-1)\pi\,\rho(h)[\rho(g)+1]. \tag{17c}$$

The same algebra with $\rho(g') \equiv 0$ for all $g' \in G_f$ was obtained by Yao and Oshikawa in 2020 and 2021.

Non obvious fact: The phase $\chi(g,h)$ is vanishing if and only if the second cohomology class $[\phi]$ is trivial.

If $\chi(g, h)$ is nonvanishing, one-dimensional representations of (17) are not allowed. The ground state of any Hamiltonian of the form (15), i.e., $\widehat{H}^{\mathrm{tilt}}_{\mathrm{twis}}(g)$, is then either degenerate or spontaneously breaks the symmetry in the thermodynamic limit. We have derived the LSM Theorem 2 for the Abelian group G_f that is represented unitarily when symmetry-twisted boundary conditions apply.

The same LSM-no-go Theorem should also apply to $\widehat{H}_{\mathrm{pbc}}$ according to **Yao-Oshikawa conjecture:** *When a local quantum many-body Hamiltonian with lattice translation invariance and a global (continuous or discrete) symmetry has a gapped spectrum with a nondegenerate ground state under periodic boundary conditions, the same must be true under any symmetry-twisted boundary conditions.*

In rederiving Theorem 2, we have taken (i) the group G_f to be Abelian and (ii) representation $\hat{u}_j(g)$ to be unitary for all $g \in G_f$. There exist several challenges in relaxing both of these assumptions.

1. When G_f is non-Abelian, one cannot consistently define a symmetry-twisted Hamiltonian $\widehat{H}^{\rm tilt}_{\rm twis}(g)$, that is invariant under both global symmetry transformations $\widehat{U}(h)$ and symmetry-twisted translation operators $\widehat{T}_{\hat{1}}(g)$ without imposing stricter constraints on local operators $\hat{h}^{\rm tilt}_{1}$ than Eq. (15).
2. The challenges with imposing antiunitary symmetry-twisted boundary conditions with the group element $g \in G_f$ are the following.
    1. First, complex conjugation is applied on all the states in the Fock space $\mathcal{F}_{\Lambda}$. This means that Hamiltonian $\widehat{H}^{\rm tilt}_{\rm pbc}$ can differ from Hamiltonian $\widehat{H}^{\rm tilt}_{\rm twis}(g)$ through an extensive number of terms when $\mathfrak{c}(g) = -1$, in which case it is not obvious to us how to safely tie some spectral properties of Hamiltonian $\widehat{H}^{\rm tilt}_{\rm twis}(g)$ and Hamiltonian $\widehat{H}^{\rm tilt}_{\rm pbc}(g)$.
    2. Second, not all representations of the group G_f are either even or odd under complex conjugation, in which case conjugation of $\widehat{T}_{\hat{1}}(g)$ by $\widehat{U}(h)^{-1}$ need not result anymore in a mere phase factor multiplying $\widehat{T}_{\hat{1}}(g)$ when $\mathfrak{c}(g) = -1$.

In view of these difficulties with interpreting antiunitary symmetry-twisted boundary conditions, we observe that the FMPS construction of LSM-type constraints is more general than the one using symmetry-twisted boundary conditions.

Our strategy when $d > 1$ is to construct the counterpart of Eqs. (16) and (17). To this end, we are going to trade the translation symmetry group (7), which is a polycyclic group when $d > 1$, for the cyclic group

$$G^{\mathrm{tilt}}_{\mathrm{trsl}} \equiv \mathbb{Z}_{N_{\hat{1}}\cdots N_{\hat{d}}} \tag{18}$$

and define the combined symmetry group

$$G^{\mathrm{tilt}}_{\mathrm{total}} \equiv G^{\mathrm{tilt}}_{\mathrm{trsl}} \times G_f. \tag{19}$$

The intuition underlying the construction of the symmetry-tilted translation symmetry group $G^{\mathrm{tilt}}_{\mathrm{trsl}}$ is provided by Fig. 1.

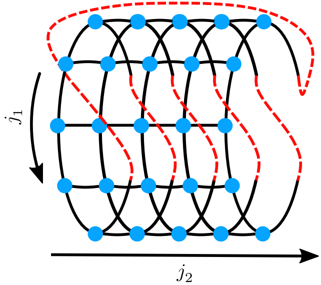


Figure: Example of a path that visits all the sites of a two-dimensional lattice **that decorates the surface of a torus.**

Summary

Whenever the Majorana degrees of freedom within a single repeat unit cell realize **a nontrivial projective representation of G_f**, then the lattice Hamiltonian **cannot have** a nondegenerate, gapped, and symmetric ground state that can be described by an even- or odd-parity injective FMPS (for $d > 1$ we must assume that G_f is Abelian and all its elements are represented by unitary operators).

The idea of the proof when G_f is Abelian and represented unitarily was to assume a nondegenerate symmetric gapped ground state for a family of Hamiltonians labeled by $g \in G_f$ such that all Hamiltonians commute with $\widehat{T}_{\mathrm{trsl}}(g)$ and $\widehat{U}(h)$ for all $g, h \in G_f$ and show a contradiction when there exists a pair $g, h \in G_f$ for which $\widehat{T}_{\mathrm{trsl}}(g)$ does not commute with $\widehat{U}(h)$.