Violation of the Wiedemann-Franz law and multi-particle scattering in the Topological Kondo model

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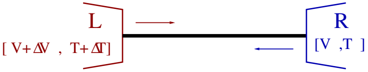
Overview

- Main goal: direct connection between the emergence of local Majorana modes and violation of the Wiedemann-Franz law
- Heat transport and the Wiedemann-Franz law
- Charge and heat transport at a junction of noninteracting quantum wires
- Junctions of interacting quantum wires
- Majorana fermions and the topological Kondo effect
- Topological Kondo effect and violation of the Wiedemann-Franz law
- Conclusions and further perspectives

Charge and heat flow across a wire connected to biased reservoirs



- $\Delta V \Delta T \neq 0 \Rightarrow$ electric current I_e and heat current I_h flow
- Onsager relations

$$I_e = G\Delta V + GS\Delta T$$
$$I_h = G\Pi\Delta V + (K + GS\Pi)\Delta T$$

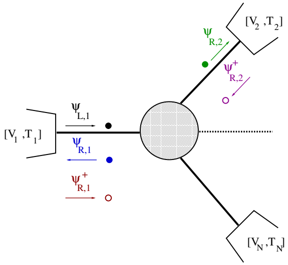
- Electric conductance $G = \left(\frac{I_e}{\Delta V}\right)_{\Delta T=0}$; Thermopower $S = -\left(\frac{\Delta V}{\Delta T}\right)_{I_e=0}$; Peltier coefficient $\Pi = \left(\frac{I_h}{I_e}\right)_{\Delta T=0}$; Thermal conductance $K = \left(\frac{I_h}{\Delta T}\right)_{I_e=0}$

Onsager coefficients

- We will be focusing onto particle-hole exchange symmetric systems and will work within linear response regime $\Rightarrow S = \Pi = 0$
- Accordingly, we will either compute the heat, or the *energy* current $I_\epsilon = I_h - VI_e$: given our assumptions they coincide within linear response regime
- Wiedemann-Franz law (Fermi liquid, low-T regime, $k_B T/\mu \ll 1$)

$$\frac{K}{TG} = \frac{\pi^2 k_B^2}{3e^2} \approx 2.4 \times 10^{-8} \, \frac{W\Omega}{K^2}$$

Noninteracting leads, single-particle scattering at the junction



- Incoming particle (hole)
- Normal reflection as a particle (hole)
- Andreev reflection as a hole (particle)
- Normal transmission as a particle (hole)
- Crossed Andreev reflection as a hole (particle)

System Hamiltonian – S-matrix

- For the sake of simplicity: junction of N spinless quantum wires:

$$H_{\mathrm{Lead}} = -iv \int_0^{\ell} dx \sum_{j=1}^{N} \{\psi^\dagger_{R,j}\partial_x \psi_{R,j} - \psi^\dagger_{L,j}\partial_x \psi_{L,j}\}$$

- Boundary Hamiltonian $H_B \Rightarrow$ **S**-matrix (in Nambu spinor basis):

$$\psi_{R,j}(0) = r_{j,j}\psi_{L,j}(0) + a_{j,j}\psi^\dagger_{L,j}(0) + \sum_{j'\neq j=1}^{N} \{t_{j,j'}\psi_{L,j'}(0) + c_{j,j'}\psi^\dagger_{L,j'}(0)\}$$

- **S**-matrix in Nambu basis ($j, j' = 1, \ldots, N$):

$$S_{j,j'} = r_{j,j'}\delta_{j.j'} + (1-\delta_{j,j'})t_{j,j'}$$
$$S_{j,j'+N} = a_{j,j'}\delta_{j.j'} + (1-\delta_{j,j'})c_{j,j'}$$
$$S_{j+N,j'} = S^*_{j,j'+N} \;\; , \; S_{j+N,j'+N} = S^*_{j,j'}$$

Charge and heat currents at the noninteracting junction (I)

- Charge and heat current operators (noninteracting fermions):

$$\begin{aligned} j_{\mathrm{el},j} &= ev\{\psi^\dagger_{R,j}\psi_{R,j} - \psi^\dagger_{L,j}\psi_{L,j}\} \\ j_{\mathrm{th},j} &= -iv^2\{\psi^\dagger_{R,j}\partial_x\psi_{R,j} - \psi^\dagger_{L,j}\partial_x\psi_{L,j}\} \end{aligned}$$

- Landauer-Büttiker approach + expansion around equilibrium + unitarity of the **S**-matrix ⇒ charge- and heat-conductance tensors:

$$\begin{aligned} I_{\mathrm{el},j} &= \sum_{j'=1}^{N} G_{j,j'}\Delta V_{j'} \; , \; G_{j,j'} = \frac{e^2}{2\pi}\{|t_{j,j'}|^2 - |a_{j,j'}|^2 - \delta_{j,j'}\} \\ I_{\mathrm{th},j} &= \sum_{j'=1}^{N} K_{j,j'}\Delta T_{j'} \; , \; K_{j,j'} = \frac{\pi k_B^2 T}{6}\{|t_{j,j'}|^2 + |a_{j,j'}|^2 - \delta_{j,j'}\} \end{aligned}$$

- Unitarity of **S**-matrix ⇒ $\sum_{j=1}^{N} K_{j,j'} = \sum_{j'=1}^{N} K_{j,j'} = 0$

Charge and heat currents at the noninteracting junction (II)

- Charge conservation at the junction $\Rightarrow a_{j,j'} = c_{j,j'} = 0$. As a result, we obtain the "lorenz ratio matrix"

$$\mathcal{L}_{j,j'} = \frac{K_{j,j'}}{T G_{j,j'}} = \frac{\pi^2 k_B^2}{3e^2} \equiv L_0$$

that is, the *Wiedemann-Franz law*, stating that the charge-carrying excitations are the same as the heat-current excitations

- However, typically, Andreev processes $\Rightarrow$ violation of the charge conservation at the junction. In this case, the "disentanglement" between the charge- and the heat- carriers implies a violation of the Wiedemann-Franz law

$$\mathcal{L}_{j,j'} = \frac{K_{j,j'}}{T G_{j,j'}} = \left[\frac{|t_{j,j'}|^2 + |a_{j,j'}|^2 - \delta_{j,j'}}{|t_{j,j'}|^2 - |a_{j,j'}|^2 - \delta_{j,j'}} \right] L_0$$

Charge and heat currents at the noninteracting junction (III)

- In the noninteracting case, either the charge is conserved and the WFL holds, or the charge is not conserved, with a corresponding breakdown of the WFL
- Recovering the WFL *in the presence* of charge conservation $\Rightarrow$ interacting systems
- The interaction can either concern the *bulk* of the leads, or the junction (multi-particle scattering processes), or both
- Mean to single out the effects of the bulk from the ones of the junction dynamics

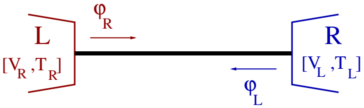
Luttinger liquid model for interacting quantum wires

- Adding a nonzero bulk interaction ⇒ (spinless) Luttinger liquid model Hamiltonian:

$$
\begin{aligned}
H_{\mathrm{Leads}} &= u \int_0^{\ell} dx \sum_{j=1}^{N} \{(\partial_x \varphi_{R,j})^2 + (\partial_x \varphi_{L,j})^2\} \\
j_{\mathrm{el},j} &= e u \sqrt{\frac{g}{\pi}} \{\partial_x \varphi_{R,j} - \partial_x \varphi_{L,j}\} \\
j_{\mathrm{th},j} &= u^2 \{(\partial_x \varphi_{R,j})^2 - (\partial_x \varphi_{L,j})^2\} \\
\psi_{R/L,j} &= \Gamma_j e^{i\sqrt{\pi}\left[\frac{1}{\sqrt{g}} \pm \sqrt{g}\right]\varphi_{R,j} + \left[\frac{1}{\sqrt{g}} \mp \sqrt{g}\right]\varphi_{L,j}}
\end{aligned}
$$

- With u, g Luttinger parameters, $\varphi_{R,j}, \varphi_{L,j}$ chiral bosonic fields ($[\varphi_{R,j}(x), \varphi_{R,j'}(x')] = -[\varphi_{L,j}(x), \varphi_{L,j'}(x')] = \frac{i\delta_{j,j'}}{4}\epsilon(x - x')$), Γ_j Klein factors ($\{\Gamma_j, \Gamma_{j'}\} = 2\delta_{j,j'}$)

Bulk interaction and violation of the Wiedemann-Franz Law in a single wire (I)



- $_R$ ($_L$) modes are shot in from a left(right)-hand reservoir at voltage bias $V_R(V_L)$ and at temperature $T_{R(L)} = 1/(\beta_{R(L)} k_B)$
- Partition function for the biased system

$$\mathcal{Z}[\{V_R, V_L\}, \{T_R, T_L\}] = \mathrm{Tr}\left\{e^{-u\beta_R \int dx\, (\partial_x \varphi_R)^2 + \beta_R \int e\sqrt{\frac{g}{\pi}} V_R \partial_x \varphi_R}\right\} \times \mathrm{Tr}\left\{e^{-u\beta_L \int dx\, (\partial_x \varphi_L)^2 - \beta_L \int e\sqrt{\frac{g}{\pi}} V_L \partial_x \varphi_L}\right\}$$

Bulk interaction and violation of the Wiedemann-Franz Law in a single wire (II)

- Employing spatial homogeneity in a length-ℓ system

$$I_{\rm el} = \frac{e}{\ell}\sqrt{\frac{g}{\pi}} \int_0^\ell dx \, \langle \partial_x \varphi_R(x) + \partial_x \varphi_L(x) \rangle$$
$$I_{\rm th} = \frac{u^2}{\ell} \int_0^\ell dx \, \langle \partial_x (\varphi_R(x))^2 - (\partial_x \varphi_L(x))^2 \rangle$$

- Taking the derivatives of the partition function

$$e\sqrt{\frac{g}{\pi}} \int_0^\ell dx \, \langle \partial_x \varphi_{R/L}(x) \rangle = \pm \frac{1}{\beta_{R/L}} \frac{\partial \ln \mathcal{Z}[\{V_R, V_L\}, \{T_R, T_L\}]}{\partial V_{R/L}}$$
$$u^2 \int_0^\ell dx \, \langle \partial_x (\varphi_{R/L}(x))^2 \rangle = u^{-1} \frac{\partial \ln \mathcal{Z}[\{V_R, V_L\}, \{T_R, T_L\}]}{\partial \beta_{R/L}}$$

Bulk interaction and violation of the Wiedemann-Franz Law in a single wire (III)

- Retaining only linear contributions in the applied biases

$$I_{\rm el} = \frac{e^2 g}{2\pi} (V_R - V_L) \Rightarrow G = \frac{e^2 g}{2\pi}$$
$$I_{\rm th} = \frac{\pi}{12\beta_R^2} - \frac{\pi}{12\beta_L^2} \Rightarrow K = \frac{\pi k_B^2 T}{6}$$

- Violation of WFL [C. L. Kane and M. Fisher, PRL76, 3192 (1996)]

$$\mathcal{L} = \frac{K}{TG} = \frac{1}{g} L_0$$

- This effect is independent of the junction dynamics and disappears with Fermi liquid reservoirs [I. Safi and H. Shulz, Rev. B 52, R17040 (1995); D. Maslov and M. Stone, Phys. Rev. B 52, R5539 (1995)]

Fixed points of junctions of N interacting quantum wires

- In many relevant cases a fixed point is fully determined in terms of the $N \times N$ *splitting matrix* ρ such that

$$\varphi_{R,j}(0) = \sum_{j'=1}^{N} \rho_{j,j'} \varphi_{L,j'}(0)$$

- Chiral theory for the *unfolded*, left-handed fields
  $\varphi_j(x) = \theta(x)\varphi_{L,j}(x) + \theta(-x) \sum_{j'=1}^{N} \rho_{j,j'} \varphi_{R,j}(-x)$, $-\ell < x < \ell$
- Current operators

$$j_{\mathrm{el},j}(x) = e\sqrt{\frac{g}{\pi}} \{\partial_x \varphi_j(x) - \sum_{j'=1}^{N} \partial_x \varphi_{j'}(-x)\}$$

$$j_{\mathrm{th},j}(x) = u^2 \{-(\partial_x \varphi_j(x))^2 + [\sum_{j'=1}^{N} \partial_x \varphi_{j'}(-x)]^2\}$$

ρ-matrix and WFL

- Biasing each φ_j field at voltage V_j and at temperature T_j and retaining only contributions linear in the applied biases

$$G_{j,j'} = \frac{e^2 g}{2\pi} \{\rho_{j,j'} - \delta_{j,j'}\}$$
$$K_{j,j'} = \frac{\pi k_B^2 T}{6} \{[\rho_{j,j'}]^2 - \delta_{j,j'}\}$$

- Renormalization of the Lorenz ratio

$$\mathcal{L}_{j,j'} = \frac{K_{j,j'}}{T G_{j,j'}} = \frac{1}{g} \{\delta_{j,j'} + \rho_{j,j'}\} L_0$$

- The factor g^{-1} is the same as for a single, homogenous wire: it is fully determined by the bulk interaction. The factor depending on ρ encodes all the multi-particle scattering processes at the junction

Structure of the ρ matrix

- g-dependent relation between ρ and $\hat{\rho}$ (splitting matrix with Fermi liquid reservoirs)

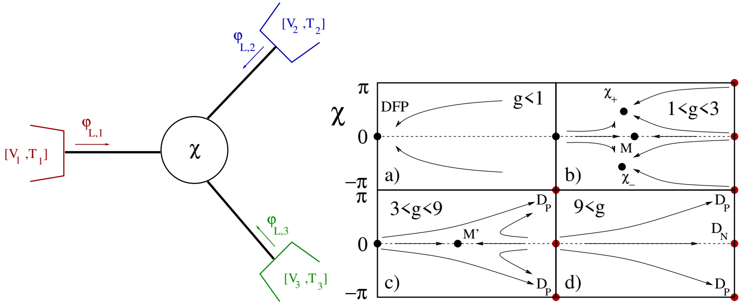
$$[M_R]_{j,j'} = \frac{1+g}{2\sqrt{g}}\hat{\rho}_{j,j'} + \frac{1-g}{2\sqrt{g}}\delta_{j,j'}$$

$$[M_L]_{j,j'} = \frac{1-g}{2\sqrt{g}}\hat{\rho}_{j,j'} + \frac{1+g}{2\sqrt{g}}\delta_{j,j'}$$

$$\rho_{j,j'} = \sum_{j''=1}^{N} [M_R]_{j,j''} [M_L]^{-1}_{j'',j'}$$

- Resolving for $\hat{\rho}$ at a given $\rho \Rightarrow$ ripping out the effects depending on the bulk interaction in the leads $\Rightarrow$ identifying genuine multi-particle scattering processes at the junction

The $N = 3$ junction of interacting quantum wires



- $$H_B = -2J \sum_{j=1}^{3} \Gamma_j \Gamma_{j+1} \ \cos\left[\sqrt{\pi}(\phi_j(0) - \phi_{j+1}(0)) + \chi/3\right]$$

  C. Chamon, M. Oshikawa, I Affleck, PRL 91 206403 (2003), J. Stat. Mech. 0602:P02008 (2006)
- Classification of the FPs described in terms of a pertinent splitting matrix

Fixed points of the $N = 3$ junction (I)

- The Disconnected Fixed Ppoint DFP

$$[\rho_{\mathrm{DFP}}]_{j,j'} = [\hat{\rho}_{\mathrm{DFP}}]_{j,j'} = \delta_{j,j'} \Rightarrow G_{j,j'} = K_{j,j'} = 0 \, , \, \forall j, j'$$

- The $\chi_{\pm}$ fixed points

$$\rho^{g}_{\chi_{\pm}} = \frac{2}{3+g^2} \begin{pmatrix} -\frac{1-g^2}{2} & 1 \pm g & 1 \mp g \\ 1 \mp g & -\frac{1-g^2}{2} & 1 \pm g \\ 1 \pm g & 1 \mp g & -\frac{1-g^2}{2} \end{pmatrix} \, , \, \hat{\rho}_{\chi_{\pm}} = \rho^{g=1}_{\chi_{\pm}}$$

- Lorenz ratio renormalization

$$\mathcal{L}^{g}_{j,j'} = \left[\frac{2(1+g^2\delta_{j,j'}) + 2g\epsilon_{j,j'}}{3+g^2} \right] L_0 \Rightarrow \mathcal{L}^{g=1}_{j,j'} = L_0$$

- No violation of the WFL in the presence of the FL reservoirs

Fixed points of the $N = 3$ junction (II)

- The complete description of the M fixed point is not known, but it has been shown that, when the junction is connected to FL reservoirs, it is fully described in terms of a single-particle **S**-matrix [A. Rahmani *et. al.*, PRB 85, 045120 (2012)] $\Rightarrow$ no violation of the WFL. (Nothing is known about the M' FP)
- The $D_{P,N}$ fixed points

$$[\rho_D]_{j,j'} = [\hat{\rho}_D]_{j,j'} = \frac{2}{3} - \delta_{j,j'}$$

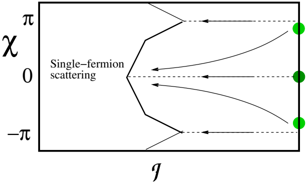
$$G_{j,j'} = \frac{e^2 g}{\pi} \left(\frac{1}{3} - \delta_{j,j'} \right) , \quad K_{j,j'} = \frac{\pi k_B^2 T}{9} \left(\frac{1}{3} - \delta_{j,j'} \right)$$

- Combined violation of the WFL

$$\mathcal{L}_{j,j'} = \frac{2}{3} \frac{1}{g} L_0$$

Violation of the WFL in the noninteracting limit

- The Lorenz ratio is renormalized by an interaction dependent factor $\frac{1}{g}$ and by the additional factor $\frac{2}{3}$. The former disappears in the noninteracting limit and/or if the junction is connected to FL reservoirs. The latter is universal and persists in any case
- The $D_{P,N}$ fixed points host a remarkable violation of the WFL that is independent of the interaction in the leads. However, they are stable only at unphysically large values of the (attractive) bulk interaction and, in any case, they disappear in the presence of FL reservoirs $\Rightarrow$ " alternative" mechanism to stabilize them



Kondo effect and WFL

- We do not consider “nonuniversal” effects related to Umklapp scattering and/or electron-electron interaction in the leads, inelastic electron-phonon scattering, *et cetera*

Kane et. al., PRL 76, 3192 (1997), Garg et. al., PRL 103, 096402 (1009), Lavasani et. al., PRB 99 (2019)

- Holds at low temperatures for:
  1. Quantum dots (low-T ⇒ Kondo effect)

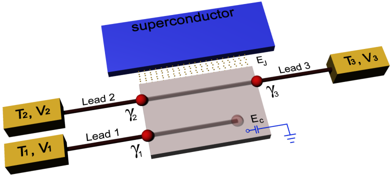
     Boese et al. EPL 56 (2001) 576; Kubala et al. PRL 100 (2008) 066801; Costi et al. PRB 81 (2010) 235127
  2. Multichannel Kondo $SU(2)_n$

     van Dalum et al. PRB 102 (2020) 041111(R), Karki et al., PRB 102 (2020)
  3. $SU(N)$ Kondo (Coqblin-Schrieffer) and relatives

$$H_t = -J \sum_{a \neq b} d_a^\dagger d_b \psi_a^\dagger(0) \psi_b(0) \ , \ \{d_a, d_b^\dagger\} = \delta_{a,b}$$

Mora PRB 80 (2009); Carmi et al. PRL 106, 106401 (2011); Lopez et al. PRB 87 (2013) 035135

Topological Kondo effect



- Spinless quantum wires deposited over a superconducting island with charging energy E_c. The Majorana modes $\gamma_1, \gamma_2, \gamma_3$ are directly connected to the leads, γ_4 is not ($\{\gamma_a, \gamma_b\} = 2\delta_{a,b}$)

$$H_{\text{Island}} = E_C[\hat{N}_c + \hat{n}_M - V_g]^2 \;, \; H_t = -t \sum_{j=1}^{3} \gamma_j \psi_j(0) + \text{h.c.}$$

Topological Kondo model Hamiltonian

- "Charging" regime ($E_C \gg t, k_B T$) and integer $n_g \Rightarrow$ twofold degenerate island state $\Rightarrow$ effective (topological) Kondo model

$$H_{T,K} = -J_K \sum_{j=1}^{3} \gamma_j \gamma_{j+1} \psi_j^\dagger(0)\psi_{j+1}(0) + \mathrm{h.c.}$$

- Bulk interaction in the wires $\Rightarrow$ bosonic junction model Hamiltonian

$$H_{T,K} \to H_{T,B} = 2J_K \sum_{j=1}^{3} [\gamma_j \Gamma_j][\gamma_{j+1}\Gamma_{j+1}] \cos\left[\sqrt{\pi}(\phi_j(0) - \phi_{j+1}(0)\right]$$

$$\phi_j(x) = g^{-\frac{1}{2}}\{\phi_{R,j} + \phi_{L,j}\} \ , \ \theta_j = g^{\frac{1}{2}}\{\phi_{R,j} - \phi_{L,j}\}$$

- The "fermion number operators" $-i\gamma_j\Gamma_j$ commute with everything. They are set to 1 *once and forever* $\Rightarrow$ Majorana-Klein hybridization

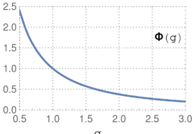
B. Béri, PRL110, 216803 (2013), A. Altland et. al., PRL110, 196401 (2013)

Phase diagram (I)

- "Putative" fixed points: $J_K = 0$ (DFP) and $J_K \to \infty$ (SFP)
- At the DFP $[\rho_{\mathrm{DFP}}]_{j,j'} = \delta_{j,j'} \Rightarrow G_{j,j'} = K_{j,j'} = 0 \Rightarrow$ finite conductances at $J_K > 0$
- $H_{T,B}$ has scaling dimension $\Delta_{\mathrm{DFP}} = g^{-1} \Rightarrow$ DSF stable for $g < 1$
- In terms of $\tilde{J}_K(T) = (2\pi k_B T)^{\frac{1}{g}-1}$

$$G_{j,j'}(T) = \frac{2\pi e^2 \Gamma^2(1/g)}{\Gamma(2/g)} (3\delta_{j,j'} - 1)\tilde{J}_K^2(T) \quad , \quad K_{j,j'} = \Phi(g) L_0 T G_{j,j'}$$

$$\Phi(g) = \frac{3\Gamma\left(2/g\right)}{g\pi\Gamma^4\left(1/g\right)} \int dzdw \frac{z_1 \left|\Gamma\left(\frac{1}{2g} + i\left(z - w\right)\right)\Gamma\left(\frac{1}{2g} + iw\right)\right|^2}{\sinh(\pi z)}$$



Phase diagram (II)

- $\Phi(g = 1) = 1 \Rightarrow$ no violation of the WFL when the system is connected to FL reservoirs
- $H_{T,B}$ is independent of $\Phi = \frac{1}{\sqrt{3}} \sum_{j=1}^{3} \phi_j \Leftrightarrow$ total charge conservation
- To recover the putative SFP, we simply assume that, as $J_K \to \infty$, the boundary potential is minimized by pinning the (relative) fields at $x = 0 \Rightarrow D_P$-like FP

$$[\rho_{\rm SFP}]_{j,j'} = [\rho_D]_{j,j'} = \frac{2}{3} - \delta_{j,j'}$$

- Violation of the WFL even with noninteracting leads ($g = 1$)
- Leading (most relevant) boundary perturbation at the SFP. $\tilde{H}_{B,T}$: linear combination of *instanton* operators between the minima of $H_{B,T}$.

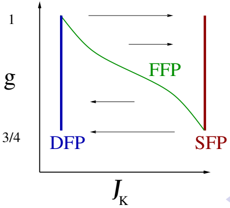
Phase diagram (III)

- By means of the *dual mapping*

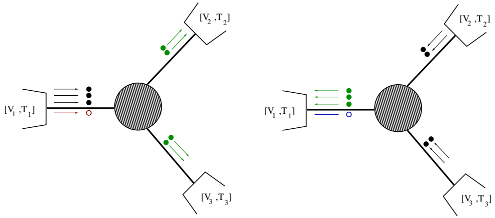
$$g \leftrightarrow \frac{3}{4g} \;,\; \frac{1}{\sqrt{2}}[\phi_1 - \phi_2] \leftrightarrow \frac{1}{\sqrt{6}}[\theta_1 + \theta_2 - 2\theta_3]$$
$$\frac{1}{\sqrt{6}}[\phi_1 + \phi_2 - 2\phi_3] \leftrightarrow -\frac{1}{\sqrt{2}}[\theta_1 - \theta_2] \quad (1)$$

one finds that $\tilde{H}_{B,T}$ has scaling dimension $\frac{4g}{3}$

- Conclusion: both the DFP and the SFP are stable for $\frac{3}{4} < g < 1$. The SFP hosts a robust violation of the WFL that *is not expected to be affected by connecting the junction to FL reservoirs*



Multiparticle scattering and violation of the WFL



- Regardless of whether the leads are interacting, or not, the SFP dynamics *cannot* be described in terms of single-particle processes only
- Multiparticle scattering involves particles and holes all together: contributions from particles and holes moving in the same directions to the charge/heat current respectively subtract from each other/add up. The opposite happens if particles and holes move in opposite directions

Multiparticle scattering, Majorana modes and heat-charge separation

- The remarkable *heat-charge separation* is one of the main features of the SFP. It leads to a *robust* and *universal* renormalization of the Lorenz ratio
- Robustness and universality are both a consequence of the stability of the SFP for $g = 1$. This is *typical of the topological Kondo effect*, whose mere existence is determined by the emergence of the localized Majorana modes at the junction
- Thus, we eventually conclude that Majorana modes $\Rightarrow$ (robust and universal) breakdown of the WFL

Conclusions and perspectives

- Junction of noninteracting quantum wires, charge conservation and single-particle **S**-matrix ⇒ WFL
- Junction of interacting quantum wires and/or multiparticle scattering at the junction ⇒ breakdown of the WFL
- Majorana modes stabilize the SFP in the Topological Kondo model
- Multiparticle scattering ⇒ breakdown of the WFL in the Topological Kondo model *even with noninteracting leads*
- Majorana fermions ⇒ breakdown of the WFL
- [Perspective]: Junctions of spin chains: no Majorana modes, but Klein factors that are expected to play a similar role in affecting the validity of the WFL

N. Crampé et. al. NPB871, 526 (2013), A. M. Tsvelik, PRL110, 147202, (2013), D. Giuliano el. al., NPB944, 11464