GGI review talk

On non-invertible symmetries

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§ Intro: generalizations of symmetry

§ Symmetry & higher dim TQFT

§ 1+1 d Adj QCD

§ Non-inv. sym in higher-dims. (& other prospects)

§ Intro : generalizations of "symmetry"

Continuous Symmetry $\Longleftrightarrow$ Noether charge $\Longleftrightarrow$ Topological unitary op.

e.g. $U(1)$ $\qquad Q_{\Sigma'} = \int_{\Sigma'} J \qquad U_\Sigma(e^{i\alpha}) = e^{i\alpha Q_{\Sigma'}}$

topological $\qquad \partial_\mu J^\mu = 0$

(ordinary) discrete symmetry $\Longleftrightarrow$ Top. unit. op.

e.g. $\mathbb{Z}_k$, S_n, ... $\qquad U_{\Sigma'}(g)$ for $g \in G$

e.g. "$\mathbb{Z}_2$ spin flip" defect

$$\downarrow\uparrow\uparrow \stackrel{\text{def}}{=} \uparrow\downarrow\downarrow$$

ordinary Symmetry = codim 1 invertible topological op.

$$g \in G \rightsquigarrow g^{-1} \in G \text{ s.t. } g \cdot g^{-1} = g^{-1} \cdot g = 1$$

$$g \longmapsto U(g)$$

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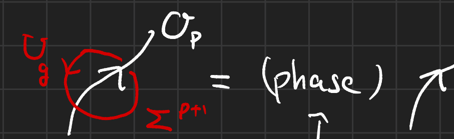
"Higher-sym" $\overset{\text{def}}{\Longleftrightarrow}$ codim $(p+1)$ invertible op. [GKSW '14]
p-form

Acting on p-dim op by linking:



p-form symmetry charge

p-form part : $G^{(p)}$: abelian group $(p \geq 1)$

"n-group" : non-trivial extension

2-group "$1 \to \underline{G^{(1)}} \to \Gamma \to G^{(0)} \to 1$" (2-grp) classified by "Postnikov class"

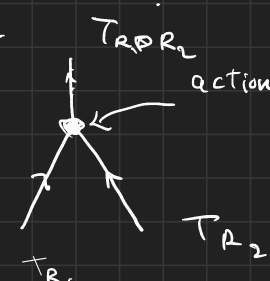
$\hat{k} \in H^3(BG^{(0)}, G^{(1)})$

Generally: Postnikov tower.

$$\begin{array}{ccc} G^{(d-1)} & \to & \\ \vdots & & \downarrow \\ G^{(2)} & \to & \vdots \\ G^{(1)} & \to & \downarrow \\ & & \downarrow \\ & & G^{(0)} \end{array}$$

physical meaning : L_1, L_2 : $G^{(1)}$-charge q_1, q_2

in gauge th.



action by an extension of $G^{(0)}$. even.

c.f. [Bhardwaj '21]

$G^{(0)} = \frac{SU(N_f)}{\mathbb{Z}_2}$ so QCD

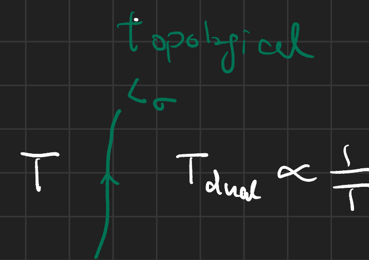
[Lee KO Tachikawa '21]

3) (possibly)

"non-invertible sym" $\overset{\text{def}}{\Longleftrightarrow}$ Any topological op.

E.g. 2d Ising model

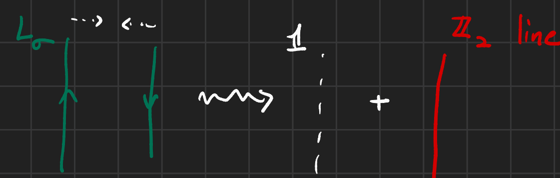
KW duality top. interface: L_σ :



[Aasen, Mong, Fendley '16]

At $T = T_{crit} = T_{dual}$, L_σ is a self-interface (Top. op. in crit Ising.

L_σ is non-invertible:



$$L_\sigma \otimes L_\sigma \cong \mathbb{1} + L_\epsilon$$

No line s.t. $L \otimes L_\sigma \cong \mathbb{1}$.

Another example:

2d diagonal RCFT: primary $\mathcal{O} \overset{1:1}{\rightsquigarrow} L_{\mathcal{O}} \in$ Verlinde lines

[Verlinde '88]

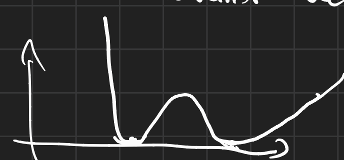
$$L_{\mathcal{O}_i} \otimes L_{\mathcal{O}_j} = \sum_k N_{ij}^k L_{\mathcal{O}_k}$$

Why do we call it "symmetry?"

① RG-flow invariant!

e.g. Tricritical Ising CFT, W, $W^2 = 1 + W$: Fibonacci line.

Relavant def with σ' $\begin{cases} \text{breaks } \mathbb{Z}_2 \\ \text{preserve } W \end{cases}$



$\rightsquigarrow$ 2 "assymmetric" vacua

[Chang Lin Shao Wang Xi]

② Gauging is sometimes possible

Ordinary Sym $\xrightarrow[\text{(co) dimension}]{\text{Higher}}$ Higher sym

group — n-group

↓ non-invertible

non-invertible sym [Johnson-Freyd]

fusion n-category. (for "finite" case)

※ Correct Math. framework for continuous non-inv. sym is not yet established.

c.f. e.g. [Thorngren Wang '21]

§ (non-inv.) Symmetry & higher dim TQFT

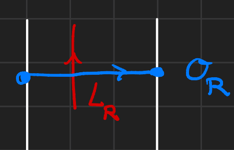
[Gaiotto Johnson-Freyd '19]
[Gaiotto Kulp '20]

diagonal WZW 1+1 d CFT G_k

for each integral rep $R \longrightarrow \begin{cases} L_R & \text{Verlinde line.} \\ \mathcal{O}_R & \text{local op.} \end{cases}$

2+1 d perspective:

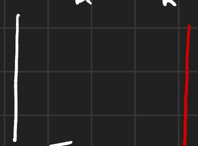
G_k CS



$\mathrm{WZW}_{\mathrm{CFT}} = \underbrace{B \qquad \bar{B}}$

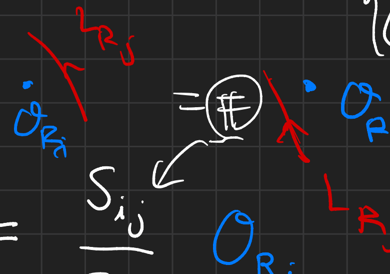
$\Downarrow$ fold

$G_k \otimes G_{-k}$



non-topological $B \boxtimes \bar{B}$ folding bdry (topological)

$$= \underbrace{\frac{S_{ij}}{S_{i0}}}_{\substack{\uparrow \\ \text{braiding in } G_k \text{ CS}}} \mathcal{O}_{R_i}$$



$\leftarrow$ Verlinde lines $= \left\{ \text{top. op. in } G_k \otimes G_{-k} \text{ preserved by (fold)} \right\}$

7) In general, if $\mathcal{T}_d$ (d-dim QFT) $=$ $\underset{B}{\bullet}\overset{\mathcal{T}^{top}_{d+1}}{\text{———}}\underset{A^{top}}{\bullet}\overset{\text{Invertible thry}}{\cdots\cdots}$ $\rightsquigarrow$ $\mathcal{T}_d$ admits top. op of $\mathcal{T}^{top}_{d+1}$ preserved by A^{top}

B: non-topological

A^{top}: top bdry

finite (non-invert.) Symmetry in QFT_d $\cong$ $(\mathcal{T}^{top}_{d+1}, A^{top})$

codim-1 top op.

- If $\exists\, I$: top interface of $\mathcal{T}^{top}_{d+1}$

We can define a new d-dim QFT by

$$\mathcal{T}_d/I := \underset{B}{\bullet}\overset{\mathcal{T}^{top}_{d+1}}{\text{———}}\underset{I}{\bullet}\text{———}\underset{A^{top}}{\bullet}\overset{\text{Inv.}}{\cdots\cdots} = \underset{B}{\bullet}\text{————}\underset{I\circ A^{top}}{\bullet}\cdots\cdots$$

Generalized (discrete) gauging!

8] Symmetry $(T^{top}, A^{top}) \xrightarrow{\text{"I-gauging"}} (T^{top}, I \circ A^{top})$

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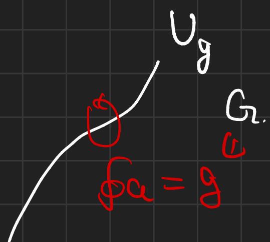
e.g. Ordinary finite sym G with anomaly ω : T^{top}_{d+1} = G-gauge th. w/ twist ω

A^{top} : Dirichlet bdry/interface (d+1-dim)



G. $\longleftarrow$ 't Hooft / twist op. (dim $d-1$) in G-gauge

$= U_g$ in T_d.

- take $H \subset G$ s.t. $\omega|_H = 0$

$\rightsquigarrow$ can gauge H in T_d.

H-gauge field on I_H

$$I_H := \text{"}\delta(a_L - a_R \in H)\text{"} = \int \frac{\mathcal{D}b}{H\text{-gauge}} \delta(a_L - a_H - b)$$

$$I_H \circ A_{\text{Dirichet}} = (a|_{bdry} \in H)$$

9)

e.g. $\mathbb{Z}_2^{(1)}$ sym in 4d

$$(T^{top}_{5d}, A^{top}) = (\mathbb{Z}_2^{(1)}\text{-gauge th, Dirichlet})$$

Top. interfaces forms $SL(2,\mathbb{Z}_2)$ [Gaiotto Kulp] [Bhardwaj Lee Tachikawa]

$\rightsquigarrow$ $SL(2,\mathbb{Z}_2)$ orbit of theories.

(analog $SL(2,\mathbb{Z})$ orbit of Witten's 3d $U(1)$ gauge theories).

§ 1+1d Adj QCD [Komargodski, KO, Roumpedakis, Seifnashri]

$SU(N)$ gauge th with adjoint ψ ($N_f = 1$, Majorana)

$$c(\psi) = c(\text{Sugawara of } \widehat{SU(N)}_N)$$

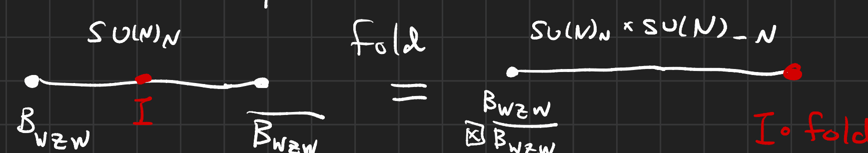
$\rightsquigarrow$ expected to be $c(\text{Adj QCD}) = 0$

i.e. gapped.

What is the IR TQFT?

- Note that ψ is an non-diagonal (spin-) RCFT with $\widehat{SU(N)}_N$

[Kapustin Saulina]: $\exists$ top interface I in $SU(N)_N$ CS s.t.

$\psi =$ 

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Gauge the diagonal of $SU(N)_N \times SU(N)_{-N}$ gauge fields on the left bdry

$\Rightarrow$ Adj QCD $=$ $SU(N)_N \times SU(N)_{-N}$

$\left(\overline{B_{WZW}} \boxtimes B_{WZW} \right) / SU(N)$ $\quad$ I $\circ$ fold

$\Big\downarrow$ RG $\qquad\qquad$ $\Big\downarrow$ bdry RG

IR TQFT $=$ B^{IR}

$\uparrow$ $SU(N)_N \times SU(N)_{-N}$ $\quad$ I $\circ$ fold

Some top. bdry.

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- only finitely many (simple) top. bdry condition for a 2+1d TQFT.

- Classification is very hard for larger N $(N \geq 6)$
  ($\simeq$ classification of RCFTs with given chiral alg.)

- $g_{YM} \to \infty$, B^{IR} = fold. # vac $= 2^N$ ← SSB of non-invertible sym.
  [Bran, Thompson, ...] (exponentially many!)

$$\mathrm{TQFT} = \left(\mathrm{Spin}(N^2-1) / SU(N) \right) / \mathbb{Z}_2^{\mathrm{Arf}}$$

- Vacua with different one-form charge $\Rightarrow$ deconfinement
  massless ψ
  $m = 0$

- mass for ψ ⇝ ~~Non-inv. sym~~ k-string tension $T_k \sim |m| \sin\left(\frac{\pi k}{N}\right) + O(m^2)$

§ About finite Non-inv. sym in higher dim

finite Non-inv. sym in QFT_d $\Longleftrightarrow$ (T^{top}_{d+1}, A^{top})

$$(T^{top}_{d+1}, A^{top}) \xrightarrow{I\text{-gauging}} (T^{top}_{d+1}, I \circ A^{top})$$

Sym. up to gauging $\Longleftrightarrow$ T^{top}_{d+1}

$\overset{+1}{d=2}$: rich zoo of $T^{top}_{\underset{2+1}{3}}$ | as many as UMTC's |

$\rightsquigarrow$ "exotic" symmetries (no gauging can make it invertible)

[Kong, Wen + ⋯] [Johnson-Freyd + Yu]

$\underset{2+1,\,3+1}{d=3,4}$: All spin-$TQFT_{s_{4,5}}$ are finite 1 or 2-group gauge theory!

$\rightsquigarrow$ non-inv. symmetries are gaugeable to invertible ones.

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Useless? Maybe not.

- Non-inv. sym might "naturally" arises

  while invertible one coming from gauging
  is more "artifitial".

  e.g. [WIP by Koide Nagoya Yamaguchi] KW duality ~~fa~~ surface in 4+0 d

- explicit weak ~~Non-inv. sym~~ might not admit model
  invertible sym description.

We need examples!

- More general lattice?
- Continuous gauge th?

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Interesting points in 1+1 d

- More on 1+1d gauge theories (e.g. Adj QCD $N_f \geq 2$)

- Truely exotic fusion category?

Haagerup cat. Not obtainable from gauging $\begin{cases} \cdot \text{ finite grp} \\ \cdot \text{ Verlinde lines} \end{cases}$

$\exists$ 1+1 d TQFT with an arbitrary fusion cat. [Thorngren Wang] [Huang, Lin + Seifnashri] '21 WIP

$\exists$ 1+1d lattice model with arbitrary fusion cat sym.

"Anyonic chain" ["golden chain" '07] Feiguin, Trebst, Ludwig, Troyer, Kitaev, Wang, Freedman

[Aasen Fendley Mong '20]

$\longrightarrow$ gapped / gapless?

numerics [Wolf '21]

[Huang Lin KO Tachikawa Tezuka] WIP

Continous (non-inv.) sym?

(TQFT, bdry) description is not suitable.

- Instead: G: conti group

Free G-gauge th.

$$\mathcal{T}_d = \lim_{L \to 0} \quad \underbrace{\mathcal{B} \text{ ——— Dirichlet}}_{L} \cdots\cdots \text{Inv.}$$

- (orbifold)
  $c=1$ CFT can admit conti. non-inv. syms [Thorngren-Wang]
  $\longleftrightarrow$ bdry conditions of 3d free ths.