Topological Terms & Diffeomorphism Anomalies in Fluid Dynamics

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Topological Properties of Gauge Theories & Their Applications to High-energy and Condensed-matter Physics

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- There has been a lot of interest recently in gauge and gravitational anomalies in fluid dynamics, partially motivated by
  - the chiral magnetic effect from high energy physics
  - quantum Hall fluids, superfluids from condensed matter physics

  Here I will focus on a related but somewhat more subtle issue.

- We use a formalism for fluids based on group theory. This is based on the following observations: (Bistrovic, Jackiw, Nair, ...(2002); Karabali & Nair (2014))
  - Point-particles correspond to unitary irreducible representations (UIRs) of the Poincaré group (+ internal symmetry groups, if needed).
  - UIRs can be obtained by quantizing the co-adjoint orbit actions of a group.
  - Coarse-grain a large number of particles (co-adjoint orbit actions) as Lagrange did for Newtonian mechanics to get fluid dynamics

- In this way one can write down an action for fluid dynamics including all possible anomalies of the Standard Model of particle physics, including chiral magnetic effects, chiral vorticity effects (Nair, Ray, Roy (2011); Monteiro, Abanov, Nair (2014)).
- For sigma models, it is possible to have a conflict between diffeomorphism invariance of the target space and of the base space (spacetime).
- This can lead to anomalies in the commutator algebra for the energy-momentum tensor.
- This can also occur for fluids, their dynamics is in terms of maps to a target space.
- The immediate motivation is the anomalous algebra found by Wiegmann and Wiegmann & Abanov for a fluid of vortices in 2+1 dimensional superfluids and in Hall effect.

- The Hamiltonian dynamics for vortices in a fluid was given by Kirchhoff in the late 1800's. (Already in the 1895 edition of Lamb's book on *Hydrodynamics*.)
- The key result is that the transverse position variables for a single vortex are Poisson-conjugate to each other.
- Wiegmann (and Wiegmann & Abanov) considered a (secondary) fluid made of a large number of vortices in an (underlying) fluid. Utilizing Kirchhoff's work, they formulated the quantum hydrodynamics of this vortex fluid in 2+1 dimensions.
- A key result of this work is an anomalous commutator algebra (with noncentral extensions) for the generators of diffeomorphisms (i.e. the momentum densities) for this vortex fluid.

- Anomalies in commutator algebra of generators are related via descent equations to Chern-Simons terms and to anomalies at the action level.
- So diffeomorphism (commutator) anomalies are puzzling since there are no gravitational anomalies in 2+1 (and 3+1) dimensions.
- We will consider the standard Lagrangian for fluid dynamics and some of the topological terms which can be added to it. We will see that:
    - One such topological term (in 2+1 dimensions) leads exactly to the anomalous fluid dynamics constructed by Wiegmann.
    - There are interesting cases in 3+1 dimensions as well. For example, a central term we find in 3d algebra can be related to some recent work on the 3d torus.
    - Another term in 3+1 dimensions may be related to a fluid of knots in a large number of vortices.

- To provide an analogy for what we are trying to do:
  - The Kac-Moody algebra may be obtained by point-splitting analysis of fermionic currents in 1+1 dimensions. Or it can be read off from the canonical quantization of the WZW action.
  - We seek a similar action-based derivation of the extended algebra.
- The vortex fluid may also be interesting in the context of recent discussions of vortex-particle duality.

The relevant reference is `arXiv:2008.11260`. (Earlier papers discussing anomalies in fluid dynamics can be traced from this.)

- A simple useful example is a sigma model in 2+1 dimensions, with target space $\mathbb{CP}^2 = SU(3)/U(2)$.
- The target space is a Kähler manifold with the Kähler two-form given by $\Omega = dA$, where

$$A = i\frac{2}{\sqrt{3}}\mathrm{Tr}(t_8 U^{-1} dU), \qquad A(Ue^{-i\sqrt{3}\,t_8\theta}) = A(U) - d\theta$$

- A is a one-form on $SU(3)$ but $\Omega = dA$ is well-defined on $\mathbb{CP}^2$.
- $\Omega \wedge \Omega$ is an element of H^4 which is nontrivial.
- Write $\Omega \wedge \Omega = d\Gamma = d(A \wedge dA)$. We take the action as

$$S = \frac{1}{2}\int G_{ab}\,\partial_\mu \varphi^a \partial^\mu \varphi^b + \int \underbrace{dt\,\Gamma_a \dot{\varphi}^a}_{k\,\Gamma}$$

- Again $\Gamma = A \wedge dA$ is not defined on $\mathbb{CP}^2$, but the equations of motion descend to $\mathbb{CP}^2$.

- The canonical momentum is $\Pi_a = G_{ab}\,\dot{\varphi}^b - \Gamma_a$. The energy-momentum tensor is given by

$$T_{\mu\nu} = G_{ab}\partial_\mu\varphi^a\partial_\nu\varphi^b - \eta_{\mu\nu}\,\frac{1}{2}(G\partial\varphi\partial\varphi)$$

- The generator of the transformation $x^i \to x^i + \xi^i$ is thus given by

$$\begin{aligned} T(\xi) &= \int (\xi^i\partial_i\varphi^a)\,G_{ab}\,\dot{\varphi}^b = \int (\xi\cdot\partial\varphi^a)\left(-i\frac{\delta}{\delta\varphi^a} + \Gamma_a\right) \\ &= -i\int (\xi\cdot\partial\varphi^a)\,D_a \end{aligned}$$

- We have a covariant derivative on the target space with $\Gamma_a\delta\varphi^a$ as the gauge potential or connection one-form.

- This gives the commutator algebra

$$\begin{aligned} [T(\xi), T(\xi')] &= i\,T([\xi,\xi']) - i\,k\int V_\xi\rfloor V_{\xi'}\rfloor(\Omega^2) \\ V_\xi &= \int (\xi\cdot\partial\varphi^a)\frac{\delta}{\delta\varphi^a} \end{aligned}$$

- The result is an anomalous commutation rule.
- However, this anomaly can be avoided. We can define

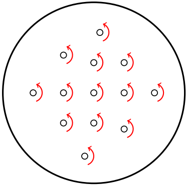
$$\mathcal{T}(\xi) = T(\xi) - \int (\xi \cdot \partial\varphi^a)\, \Gamma_a = -i \int (\xi \cdot \partial\varphi^a) \frac{\delta}{\delta\varphi^a}$$

These obey the non-anomalous commutation rules

$$[\mathcal{T}(\xi), \mathcal{T}(\xi')] = i\, \mathcal{T}([\xi, \xi'])$$

- This eliminates the extension term, but is problematic since Γ is not defined on $\mathbb{CP}^2$.
- Thus we have a conflict: Either we have anomalous commutation rules for the energy-momentum tensor on the base space, or we have an anomaly on the target space.
- We will see that a similar structure is possible for fluid dynamics.

- Rotating superfluid Helium develops a number of individual vortices which account for the total angular momentum, instead of the whole fluid rotating as a single unit.



- This state is known as a state of chiral flow. We want to analyze this by approximating a large number of vortices by another vortex fluid.

- The problem is effectively 2-dimensional, so writing $z = x + iy$, the positions of the vortices obey the equations

$$\dot{\bar{z}}_\alpha = -i\,\Omega\,\bar{z}_\alpha + i\sum_\beta{}' \frac{\gamma_\beta}{z_\alpha(t) - z_\beta(t)}$$

$\Omega =$ overall angular velocity.

- The Hamiltonian and commutation rules are (Kirchhoff)

$$H = \sum_\alpha \Big[|\Omega||z_\alpha|^2 - \gamma \sum_\beta{}' \log |z_\alpha - z_\beta|^2 \Big], \qquad [\bar{z}_\alpha, z_\beta] = \delta_{\alpha\beta}$$

- One can write down the ground state wave function for N vortices and calculate the commutators of the operator for the momentum density.

- For $N \to \infty$, this leads to the anomalous algebra (Wiegmann)

$$[P(x), P^\dagger(x')] = -\tfrac{1}{2}\,(P \times \nabla)\delta(x - x') + \underbrace{\frac{\gamma}{2}\left(2\pi\rho^2\delta(x - x') + \frac{1}{4}\nabla \cdot \left[\rho\nabla(\delta(x - x')\right]\right)}_{\text{anomalous term}}$$

(This is to be understood as acting on $|0\rangle$.)

- Also the density of vortices ρ is related to vorticity ω and the angular velocity Ω by

$$\rho = \bar{\rho} - \frac{\omega}{4\pi\gamma}, \qquad \Omega = \pi\gamma\bar{\rho} \qquad \text{(Constitutive relation)}$$

- Our aim is to construct an action which leads to this result.
- The standard way to construct an action for fluid dynamics is to use the Clebsch parametrization for the velocities in terms of three arbitrary functions θ, α, β,

$$v_i = \partial_i \theta + \alpha\, \partial_i \beta$$

- A suitable action for fluid dynamics (in terms of the Eulerian variables) is then

$$S = \int \rho\, \dot{\theta} + \rho\, \alpha\, \dot{\beta} - \left[\frac{1}{2}\, \rho\, v^2 + V(\rho) \right] = \int \left[\rho\, (\dot{\theta} + \alpha\, \dot{\beta}) + J^i (\partial_i \theta + \alpha \partial_i \beta) \right] - \left[V - \frac{J^2}{2\rho} \right]$$

- It is simpler to write this in terms of $SU(1,1)$,

$$-i \mathrm{Tr} \left(\sigma_3\, g^{-1}\, dg \right) = d\theta + \alpha\, d\beta, \qquad \alpha = \frac{\bar{u} u}{1 - \bar{u} u}$$

$$g = \frac{1}{\sqrt{1 - \bar{u} u}} \begin{pmatrix} 1 & u \\ \bar{u} & 1 \end{pmatrix} e^{-i \sigma_3 \theta / 2}, \qquad \beta = (-i/2) \ln \left(\frac{u}{\bar{u}} \right)$$

(We use $SU(2)$ if vorticity is quantized.)

- The action, with relativistic generalization, is

$$S = -i \int J^\mu \,\mathrm{Tr}\left(\sigma_3 g^{-1} \partial_\mu g\right) - F(n), \qquad n = \sqrt{J^\mu J^\nu g_{\mu\nu}}$$

- $J^0 = \rho$ and J^i can be eliminated via its equation of motion.

- $F(n)$ carries information about pressure and energy density; it depends on the fluid under consideration. In fact

$$T^{\mu\nu} = (nF')\, u^\mu u^\nu - \eta^{\mu\nu}(nF' - F), \qquad J^\nu = n u^\nu$$

- $SU(1,1)$ with its compact direction θ may seem puzzling, since there is no such compactness for the usual Clebsch parametrization.

- From the action, we find

$$[\rho(f), g(x)] = -i\, g(x)\, \frac{\sigma_3}{2} f(x), \qquad \rho(f) = \int f(x) \rho(x)$$

- This means that in the quantum theory

$$\mathcal{U}^\dagger \, g \, \mathcal{U} = g \, e^{i\pi\sigma_3} = -g$$

with $\mathcal{U} = \exp\left[-2\pi i \int \rho\right]$.

- All observables have even powers of g, so, effectively, $\mathcal{U} = 1$. This means $\int \rho = N$, consistent with underlying particulate nature of the fluid. The compact direction is a good feature.

- With $T^{\mu\nu}$ and commutation rules from S, it is easy to check that

$$[T(\xi), T(\xi')] = i\, T([\xi, \xi']), \qquad T_{i0} = \rho\,(\partial_i \theta + \alpha \partial_i \beta)$$

- The vorticity of the fluid in the Clebsch parametrization is given by

$$\omega = dv = i\,\mathrm{Tr}\left[\sigma_3 (g^{-1} dg)^2\right]$$

- Among many topological terms, the following two are most useful for us:

  1. $I_2 = \int \mathrm{Tr}(g^{-1}dg)^3 \wedge C$, $\quad C$ = 1-form in 3+1, 0-form in 2+1

  2. $I_3 = \int \mathrm{Tr}(\sigma_3 g^{-1}dg) \wedge \Omega$, $\quad \Omega$ = 3-form in 3+1, 2-form in 2+1

- We will first analyze the I_3-term. The action is now

$$S = -i\int \rho\,\mathrm{Tr}(\sigma_3 g^{-1}\partial_0 g) + ik\int \mathrm{Tr}(\sigma_3 g^{-1}dg)\wedge\Omega - \int dt\,H$$
$$= -i\int (\rho-\bar{\rho})\,\mathrm{Tr}(\sigma_3 g^{-1}\partial_0 g) - \int dt\,H$$

where $\bar{\rho} = \frac{k}{3!}\epsilon^{ijk}\Omega_{ijk}$, (3 dim), $\frac{k}{2!}\epsilon^{ij}\Omega_{ij}$ (2 dim).

- The canonical one-form is

$$\mathcal{A} = -i\int (\rho-\bar{\rho})\,\mathrm{Tr}(\sigma_3 g^{-1}\delta g)$$

- The topological term does not contribute to the $(i0)$- component of the energy-momentum tensor, $T_{\mu\nu} = 2\frac{\delta S}{\delta g^{\mu\nu}}$.

- One can easily verify that

$$[T(\xi), T(\xi')] = i\,T([\xi,\xi']) - i \underbrace{\int \left(\frac{\rho\bar{\rho}}{\rho-\bar{\rho}}\right) \xi^i\, \xi'^j (\partial_i \mathsf{v}_j - \partial_j \mathsf{v}_i)}_{\text{anomaly?}}$$

- Define $\mathcal{T}(\xi) = T(\xi) - \int \bar{\rho}\, \xi^i \mathsf{v}_i$. Then

$$[\mathcal{T}(\xi), \mathcal{T}(\xi')] = i\,\mathcal{T}([\xi,\xi'])$$

- $\bar{\rho}\,\xi^i \mathsf{v}_i$ is well-defined, so this is not a true anomaly. This is consistent with having no gravitational anomalies in 2+1 dimensions.

- But we can consider a reduction to the incompressible case via the constraint

$$\rho - \bar{\rho} - \rho_0 \approx 0, \qquad \rho_0 = \text{constant}$$

- Since ρ is conjugate to θ, $\partial_i\theta$ in v_i is not in the reduced phase space.
- The attempted redefinition involves a "gauge" direction, and so we cannot redefine $T(\xi)$. The extension is then a true anomaly.
- The situation is analogous to the case of the sigma model.
- We now make two more changes:
  - We impose the constitutive relation $\rho = \bar{\rho} - \frac{\omega}{4\pi\gamma}$.
  - We add a (cohomologically trivial) term to $T(\xi)$ to get

$$\tilde{T}(\xi) = T(\xi) - \frac{\gamma}{2}\int (\nabla \times \xi)\,\rho$$

- For the combinations $P = -\frac{1}{2}(\tilde{T}_{01} - i\tilde{T}_{02})$, $P^\dagger = -\frac{1}{2}(\tilde{T}_{01} + i\tilde{T}_{02})$, we get

$$[P(\xi), P^\dagger(\xi')] = i\int \left(\bar{\xi}'\bar{\partial}\xi\, P - \xi\partial\bar{\xi}'\, P^\dagger\right) - 2\pi\gamma \int \bar{\xi}'\xi\, \rho\bar{\rho} - \gamma \int \bar{\partial}\xi\, \partial\bar{\xi}'\, \rho$$

  This reproduces the result by Wiegmann exactly.

- So we can conclude: The action

$$\begin{aligned} S &= -i\int \rho\,\mathrm{Tr}(\sigma_3 g^{-1}\partial_0 g) + ik\int \mathrm{Tr}(\sigma_3 g^{-1}dg)\wedge\Omega - \int dt\, H \\ &\quad + \int A_0 dt\left[i\,\mathrm{Tr}[\sigma_3(g^{-1}dg)^2] + 4\pi\gamma(\rho - \bar{\rho})\right], \end{aligned}$$

  upon canonical quantization, reproduces the dynamics of the vortex fluid in superfluid Helium/ Hall effect.

- In 3+1 dimensions, we can write

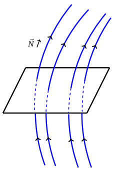
$$\omega_{ij} = \partial_i \mathsf{v}_j - \partial_j \mathsf{v}_i = \epsilon_{ijk} N^k \omega$$

N^k is a unit vector giving the orientation of the vorticity at a point.

- We also use the same constitutive relation $\rho = \bar{\rho} - \frac{\omega}{4\pi\gamma}$ as before.
- The algebra takes the form

$$[T(\xi), T(\xi')] = i\, T([\xi, \xi']) + i \int \epsilon_{ijk}\, \xi^i \xi'^j c^k, \qquad c^k = 4\pi\gamma\rho\bar{\rho} N^k$$

- For a dense collection of vortices, $|N|$ will be approximately constant, although the orientation can change from point to point; for approximately constant orientation in some region, there is a 2d vortex fluid on a surface transverse to N^k.



- If the orientation of N^k is constant in some volume, the extension term reduces to

$$\text{Extension} = c^k \int \epsilon_{ijk} \xi^i \xi'^j$$

- In the incompressible case (which is what we have), ξ, ξ' are divergence-free.
- Take space to be a 3-torus and parametrize the vector fields as

$$\xi^i = \epsilon^{iab} \alpha_a m_b \, e^{i\vec{m}\cdot\vec{x}}, \qquad \xi'^j = \epsilon^{jrs} \beta_r n_s \, e^{i\vec{n}\cdot\vec{x}}$$

- The extension term is now

$$\text{Extension} = -(\vec{\alpha} \times \vec{\beta}) \cdot \vec{n} \;\, \vec{c} \cdot \vec{n}$$

  This seems to be the same as what was found in some recent work (Rajeev, arXiv:2005.12125).

- The reduction to approximately constant N^k was to show this connection. It should be interesting to explore the general case with noncentral extension.

- The action is

$$S = -i \int \rho \, \mathrm{Tr}(\sigma_3 g^{-1} \partial_0 g) - \frac{k}{3} \int \mathrm{Tr}(g^{-1} dg)^3 \wedge C - \int dt \, H$$

- In this case we find

$$[T(\xi), T(\xi')] = i \, T([\xi, \xi']) + 8\pi i \, k \int \left(\frac{\rho \, \sigma}{2 \, \rho + k \vec{B} \cdot \vec{v}} \right) (\vec{\xi} \times \vec{\xi'}) \cdot \vec{B}, \qquad B = dC$$

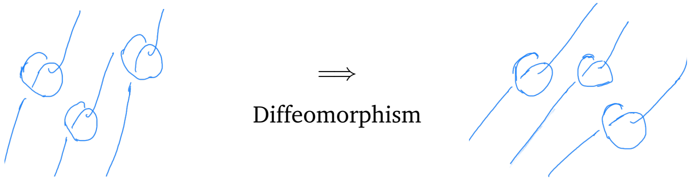
- Here σ is the density for helicity defined by

$$\mathcal{C} = \frac{1}{8\pi} \int v \cdot \omega = \frac{1}{12\pi} \int \mathrm{Tr}(g^{-1} dg)^3 \equiv \int \sigma$$

- The helicity is basically an Abelian Chern-Simons term and corresponds to knot invariants or linking numbers. So consider a collection of knots in vortices.

THE TERM I_2 IN 3+1 DIMENSIONS (*cont'd.*)

- For a tight knot in a vortex line, we can define an approximate position which can be moved around by diffeomorphisms.



- Diffeomorphisms in a dense collection of knots can define a fluid dynamics of knots in vortices.
- We end with a conjecture:

> The action given above with the I_2 topological term is the (approximate) Eulerian description of this knot-fluid.

Thank You