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Entanglement and dipolar quantum Hall physics in tensor Chern-Simons theory

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Summary

- Background and motivation
- Action and symmetries
- Path-integrals: WF preparation, edge states
- Extended Operators
- Dipole condensation
- Ground state entanglement
- Loose ends

Background + motivation

Fruitful interplay between quantum field theory, high-energy and condensed matter physics

"Low-energy dogma:" in the IR relevant degrees of freedom described by a continuum effective field theory

Notable: TQFTs for describing gapped phases of matter

- **Responses: Hall viscosity, geometric responses** *[X.L.Qi, et. al; 2008], [Hoyos, Son; 2012], [G.Y. Cho, et. al.; 2014]…*
- **anyonic excitations** *[S.C.Zhang,et.al.; 1989],…*
- **entanglement entropies, negativities, etc.** *[M.Levin, X.G. Wen, 2006], [A.Kitaev, J.Preskill; 2006], [X.Wen, et.al., 2016],…*
- **breaking of higher-form symmetries** *[X.G. Wen; 2019]*

This dogma has been challenged by a growing understanding of "**fracton**" phases of matter

[Chamon; 2005] [Bravyi, et. al.; 2011] [Bravyi, Haah; 2013] [Haah; 2011] [Yoshida; 2013] ….

- Excitations w/ no mobility, (+perhaps some with restricted mobility)
- Enforced by novel forms of symmetry (i.e. sub-system, extensive)
- Extensive ground state degeneracy
- UV / IR mixing

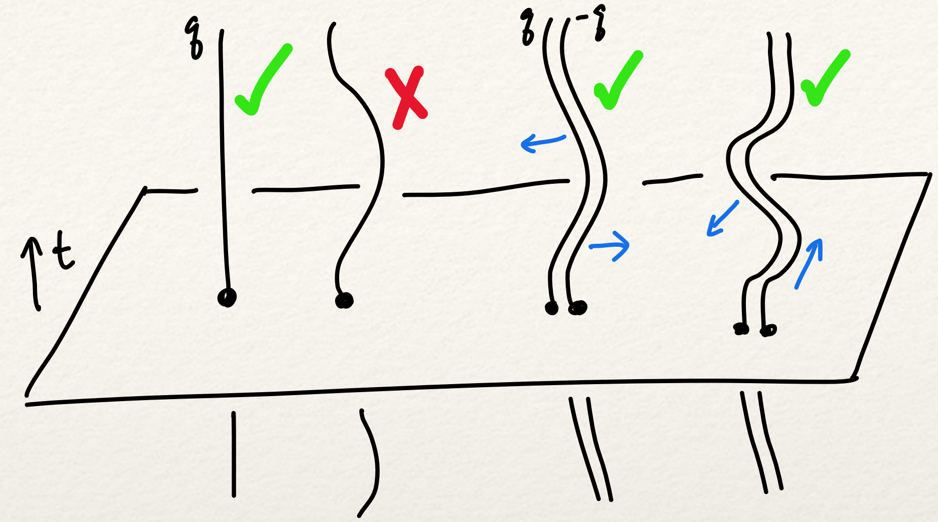
These features require new structures in IR QFT

- Discontinuous field configs *[N.Seiberg, S.H.Shao; 2020-2021]*
- Coupling to background foliations *[W.Shirley, et.al.; 2019] [K.Slagle; 2020]*
- Tensor gauge theories *[Pretko; 2017] [Prem, et.al., 2018]*

The fractonic "meat" of TGT lies in dipole conservation

$$\partial_i \partial_j E^{ij} = \rho$$

$$\partial_t \int d^D x \, \vec{x} \, \rho = 0$$



The focus of these theories: gapless phases in (3+1) dimensions

However, there is an interesting theory one can construct in (2+1)d with a Chern-Simons-like term

[M. Pretko; 2017] [A. Prem, et.al., 2018]

- Fractons via dipole + trace-quadrupole conservation
- Gapped
- Displays many characteristics similar to FQH systems:
- *"dipolar quantum Hall fluid"*

Tensor Chern-Simons

$$S_{tCS} = \frac{k}{4\pi}\int dt \int_{\mathbb{R}^2} d^2x \left(2A_0\, \varepsilon^{ij}\delta^{kl}\partial_i\partial_k\, A_{jl} - \varepsilon^{ij}\delta^{kl} A_{ik}\partial_t\, A_{jl}\right)$$

A_0 : scalar $\quad\quad$ A_{ij} : symmetric traceless tensor

Has a gauge symmetry

$$\delta A_0 = \partial_t \alpha$$

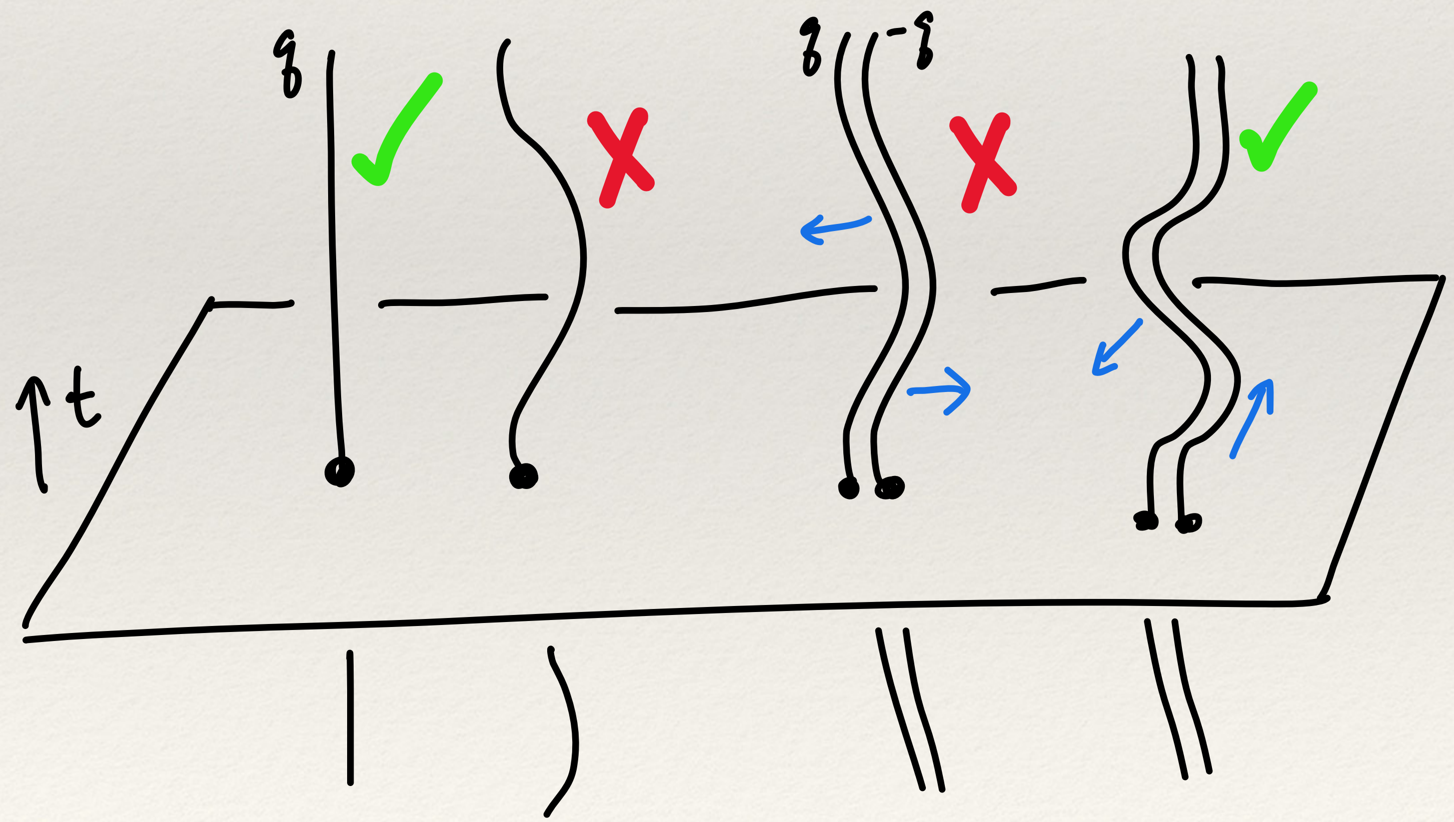
$$\delta A_{ij} = (\partial_i\partial_j - \frac{1}{2}\delta_{ij}\partial^2)\alpha \qquad \partial_t\rho - (\partial_i\partial_j - \frac{1}{2}\delta_{ij}\partial^2)J^{ij} = 0$$

Conservation of

Charge $$Q_0 = \int d^2x\, \rho$$

Dipole moment $$\vec{Q}_1 = \int d^2x\, \vec{x}\rho$$

Trace quadrupole moment $$Q_2^T = \int d^2x\, x^2\rho$$



A note on units:

$$[A_{ij}] \sim \ell^{-1} \qquad [A_0] \sim \ell^0$$

(k is dimensionless, while a possible tensor-Maxwell coupling is irrelevant)

Charge "density" $[\rho] \sim \ell^{-3}$

Gauge parameters $[\alpha] \sim \ell$

Requires an (inverse) length scale μ

Path integration

$$Z[\rho, J] = \int \frac{\mathcal{D}A_0 \mathcal{D}A_{ij}}{\mathcal{V}_g} e^{iS_{tCS} + i\int \rho A_0 + i \int A_{ij} J^{ij}}$$

A_0 appears linearly and enforces

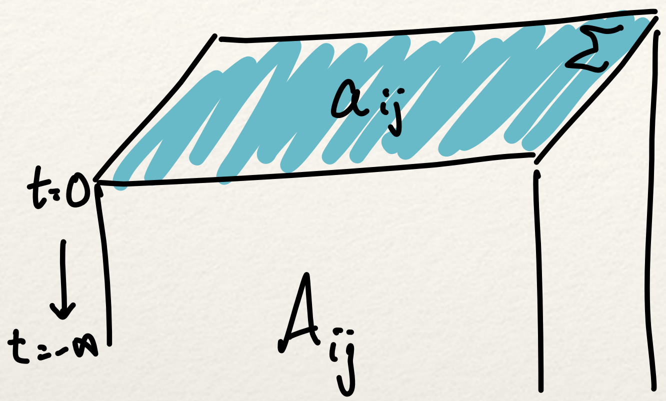
$$\varepsilon^{ij}\delta^{kl}\partial_i\partial_k A_{jl} = \frac{2\pi}{k}\rho$$

Tensor Gauss' law constraint

$$A_{ij} = A_{ij}^{(0)} + (\partial_i\partial_j - \frac{1}{2}\delta_{ij}\partial^2)\phi$$

fixed, solves constraint

Path integral 1: wavefunction



$$\psi[a] = \int \frac{\mathcal{D}A_{ij}}{\mathcal{V}_g}\bigg|_{A[\Sigma]=a} \delta[\text{t.G.L.}]e^{iS_{tCS}}$$

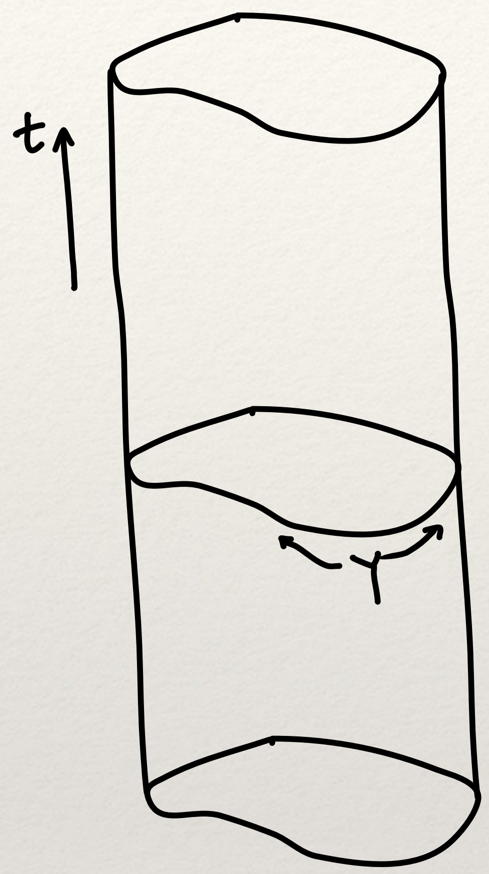
This functional gauge variant
and so not physical
Apply projector

$$\Psi[a] := \int \mathcal{D}\alpha\,(\mathcal{U}_\alpha \circ \psi)[a] = \int \mathcal{D}\alpha\,\psi[a^{(\alpha)}]$$

$$\Psi[a] \propto \delta[\sum_i q^i]\delta[\sum_i q^i(\vec{x}-\vec{x}_i)]\delta[\sum_i q^i(x-x_i)^2]$$

Unique state on $\mathbb{R}^2$ once total *charge, dipole moment, trace quad moment* sum to zero.

Path integral 2: edge modes

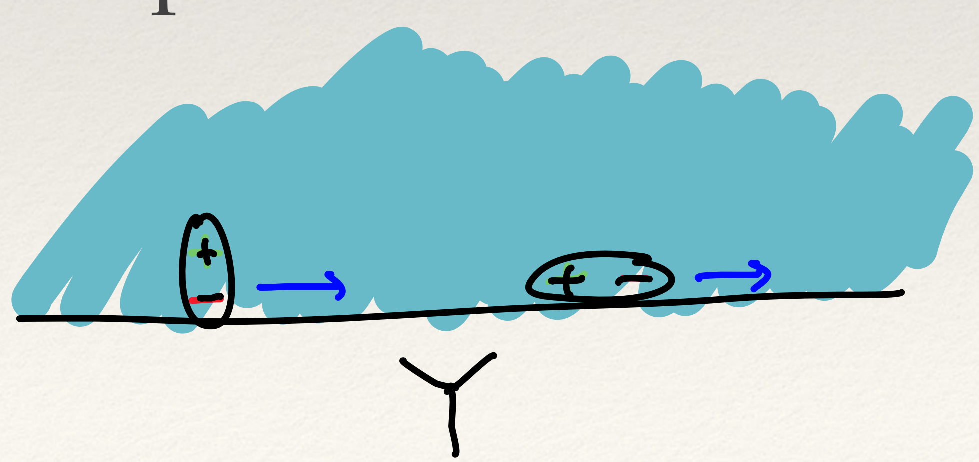


Writing $A_{ij} = (\partial_i \partial_j - \frac{1}{2}\delta_{ij})\phi$

Bulk action is a total derivative

$$S_\partial = \frac{k}{4\pi}\int_{\mathbb{R}\times Y} dt\, \delta^{ij}\partial_t(\partial_i\phi)\boldsymbol{d}(\partial_j\phi)$$

Useful physical picture: chiral boundary dipoles



Let the coordinate around Y be s and let n be a normal coordinate.

$$\xi(s) := \partial_n \phi|_Y$$

We can think of the edge theories as *chiral scalar* + *"chiral Lifshitz" scalar*

$$S = \frac{k}{4\pi} \int dt\, ds \left(\partial_t \xi \partial_s \xi - \partial_t \phi \partial_s^3 \phi \right)$$

To keep in mind: one way to think of the entanglement entropy ~ the correlations between edge modes on opposite sides of an entangling cut.

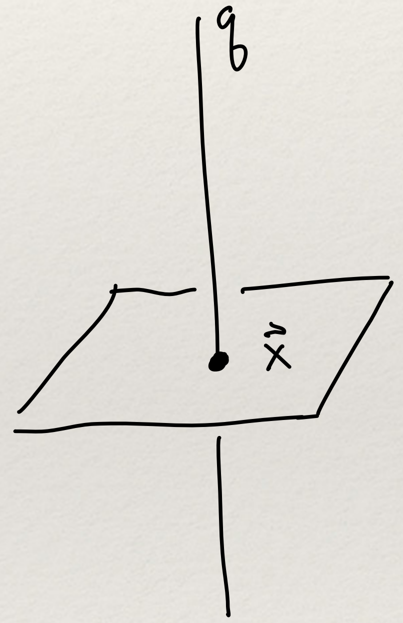
Extended operators

There are no local gauge-inv operators.
However there are extended gauge invariant operators.

Firstly, *line operators*

Charge defect operator

$$\mathbf{L}_q(\vec{\mathbf{x}}) = \exp\left(i\mu q \int dt\, A_0(t, \vec{\mathbf{x}})\right)$$

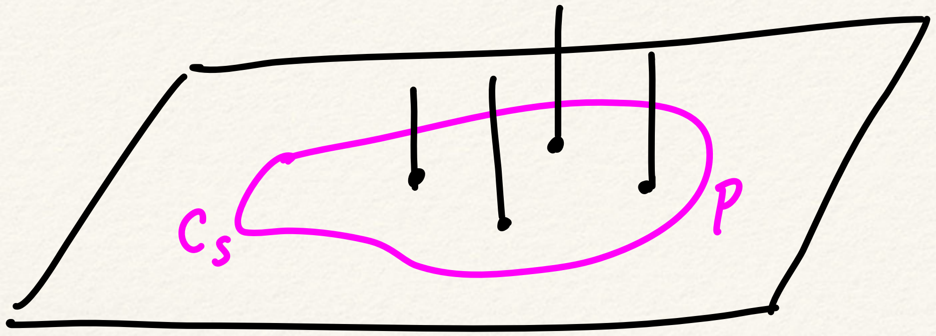


Modifies tensor-Gauss constraint

$$\varepsilon^{ij}\delta^{kl}\partial_i\partial_k A_{jl} = \mu\frac{2\pi q}{k}\delta^2(\vec{x} - \vec{\mathbf{x}})$$

Monopole string operator

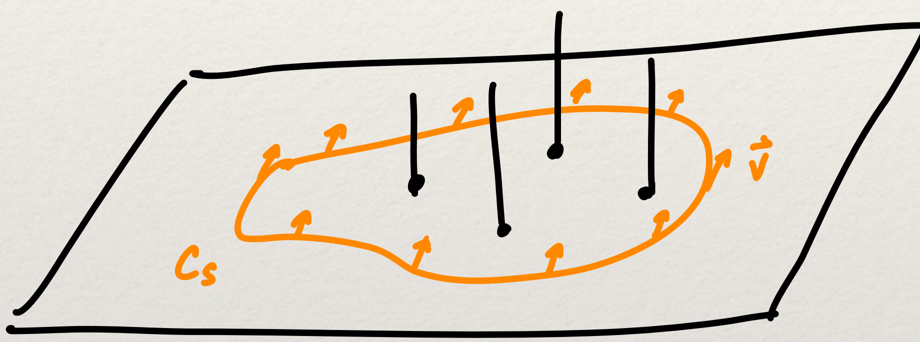
$\mathbf{M}_p[\mathcal{C}_s]$



$$\left\langle \mathbf{M}_p[\mathcal{C}_s] \right\rangle = \exp\left(i\frac{\pi}{k} \sum_{a\in\Sigma_{\mathcal{C}_s}} p q^a \right)$$

Dipole string operator

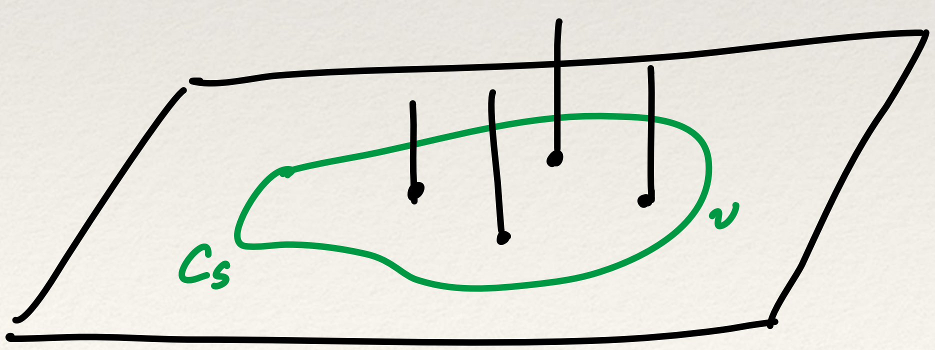
$\mathbf{D}_{\vec{\mathbf{v}}}[\mathcal{C}_s]$



$$\left\langle \mathbf{D}_{\vec{\mathbf{v}}}[\mathcal{C}_s] \right\rangle = \exp\left(-i\frac{2\pi}{k}\mu \sum_{a\in\Sigma_{\mathcal{C}_s}} q^a\, \vec{\mathbf{v}}\cdot\vec{\mathbf{x}}_a \right)$$

Trace-quadrupole string operator

$\mathbf{T}_\nu[\mathcal{C}_s]$



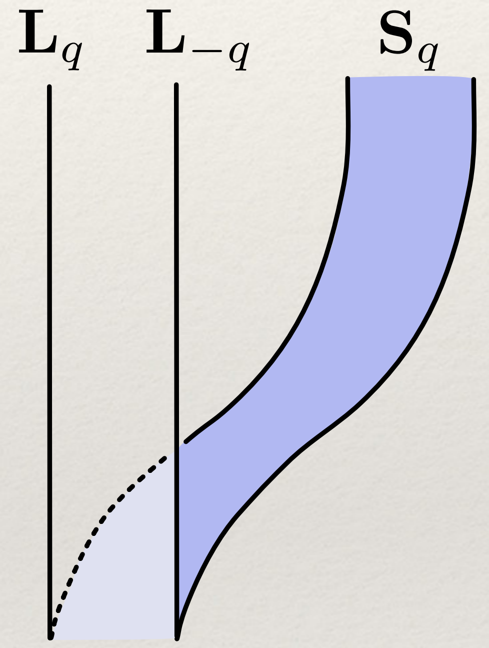
$$\left\langle \mathbf{T}_\nu[\mathcal{C}_s] \right\rangle = \exp\left(-i\frac{\pi}{k}\mu^2\nu \sum_{a\in\Sigma_{\mathcal{C}_s}} q^a\, \mathbf{x}_a^2 \right)$$

Return to charge defect op.

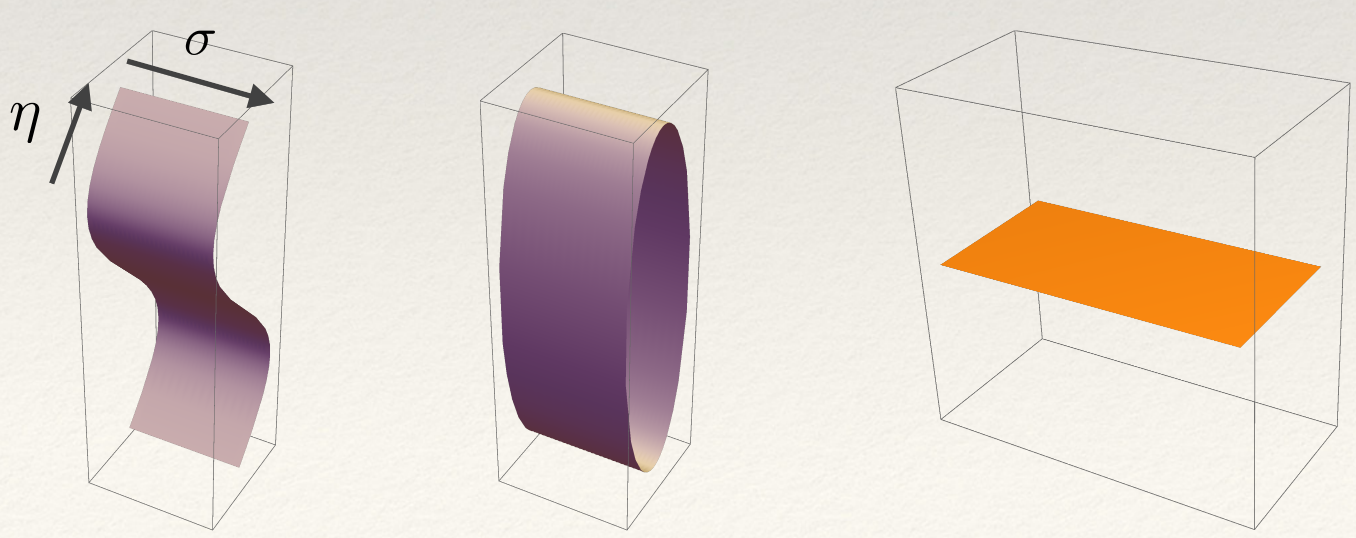
Locked along time contour ⟷ fractonic physics

Consider two defects $\mathbf{L}_q(\vec{\mathbf{x}}_1)\mathbf{L}_{-q}(\vec{\mathbf{x}}_2)$

Can use A_{ij} to deform in direction *orthogonal* to dipole moment



Restricted mobility =
local deformability of *strip operators*



$$\delta_{ij}\frac{\partial x^i}{\partial \eta}\frac{\partial x^j}{\partial \sigma} = 0$$

U(1) vs. $\mathbb{R}$: charge, level, and dipole quantization

So far: considered infinitesimal gauge tx's.
Can we treat symmetry as U(1)?

Need inverse length scale: $g_\alpha = \exp\left(i\mu\alpha\right)$

What is the role of this length scale?

Invariance under large gauge tx's quantizes defect charges,

$$q \in \mathbb{Z}$$

Can we quantize the level, other charges (string ops)?

Some inspiration from XY-plaquette model*

[A. Paramekanti, et.al., 2002] *[H.He, et.al.; 2020]*

$$H = \sum_{\hat{r}} \left(\pi_{\hat{r}} \pi_{\hat{r}} - K \cos(\hat{\Delta}_{xy} \phi_{\hat{r}}) - \frac{K}{2} \cos(\hat{\Delta}_{xx} \phi_{\hat{r}}) - \frac{K}{2} \cos(\hat{\Delta}_{yy} \phi_{\hat{r}}) \right)$$

Global *compact* dipolar shift symmetry

$$\phi \to \phi + \Lambda_x^{(1)} \hat{r}_x + \Lambda_y^{(1)} \hat{r}_y \qquad \Lambda_i^{(1)} \sim \Lambda_i^{(1)} + 2\pi$$

The continuum Hamiltonian

$$H = \int d^2x \left(\tilde{\pi}\tilde{\pi} + \frac{K}{2} (\partial_i \partial_j \tilde{\phi})^2 \right) \qquad \tilde{\phi} \sim a\phi$$

also possesses a shift symmetry….

$$\tilde{\phi} \to \tilde{\phi} + \Lambda_i^{(1)} x^i \qquad x^i \sim a\hat{r}_i$$

although in the continuum limit it is not compact.

However we might be interested in organizing states by the integer charges of the lattice theory.

$$\mu = a^{-1}$$

and treat $\mu\vec{x}$ as integer valued in gauge tx's.

Invariance under large gauge tx's requires of string ops

$$\mathbf{M}_p \quad \mathbf{D}_{\vec{\mathbf{v}}} \quad \mathbf{T}_\nu \qquad\qquad p, \mathbf{v}^i, \nu \in \mathbb{Z}$$

and of the level

$$k \in \mathbb{Z}$$

Provides mechanism to quantize dipole moments in units of lattice spacing.

Dipole condensation

Large dipolar gauge tx's shift dipole moments by $k\mathbb{Z}$

Invariance: physical dipoles fall in equivalency classes of $\mathbb{Z}_k$

$$d^i := q(\mu x^i) \sim d^i + k\mathbb{Z}$$

The vacuum forms a *condensate,* allowing "long-ish dipoles" to become transparent.

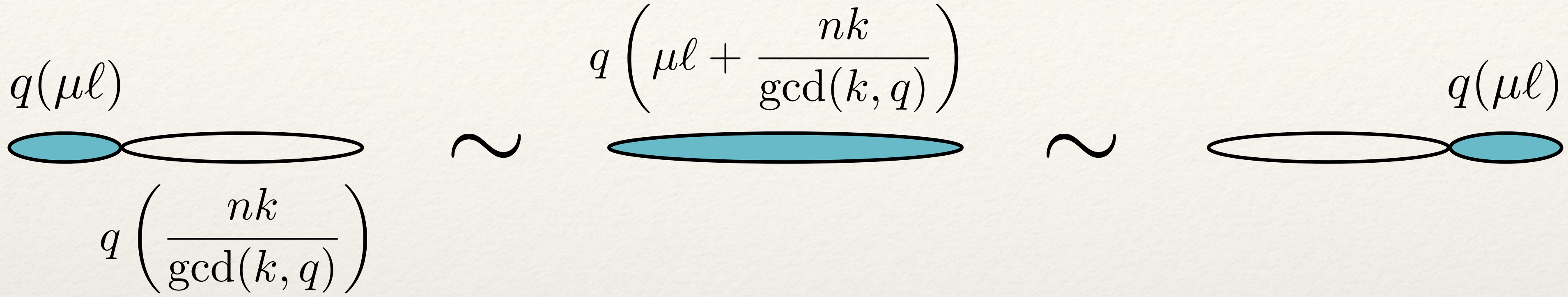
Aside: there is rich connection between tensor gauge thy & elastic thy of 2d lattices

[M.Pretko, L.Radzihovsky; 2018] [A. Gromov; 2019] [A. Gromov, P. Surowka; 2019]

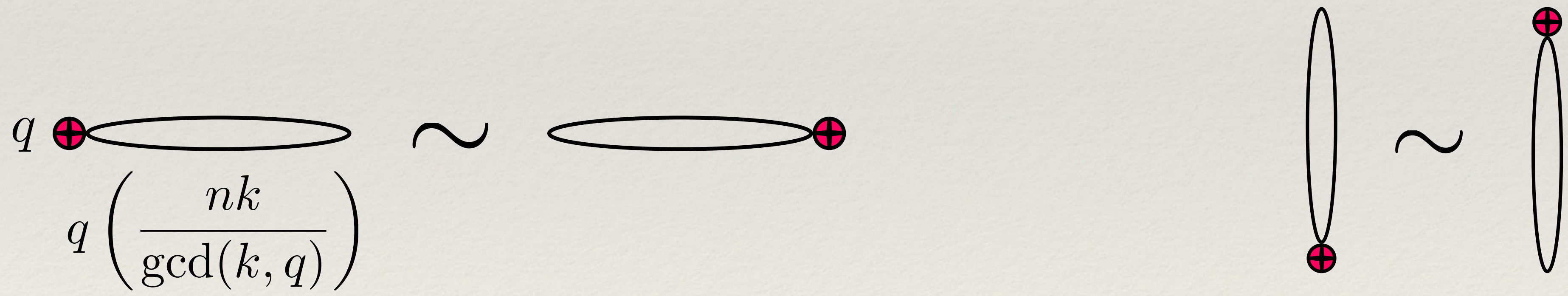
Dipole condensation ~ condensation of dislocations ~ quantum melting transition of the lattice.

[M.Pretko, L.Radzihovsky; 2018] [A. Kumar, A.C. Potter; 2019] [D.Nguyen, et.al.; 2020]

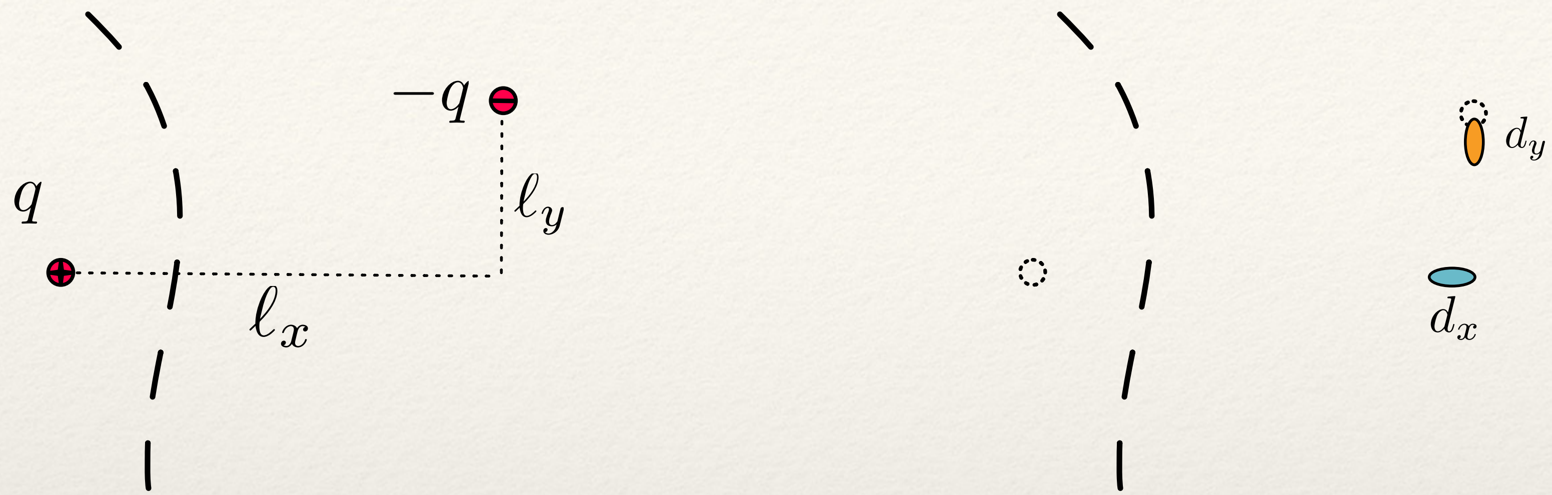
This condensate restores (macroscopic) mobility to dipoles



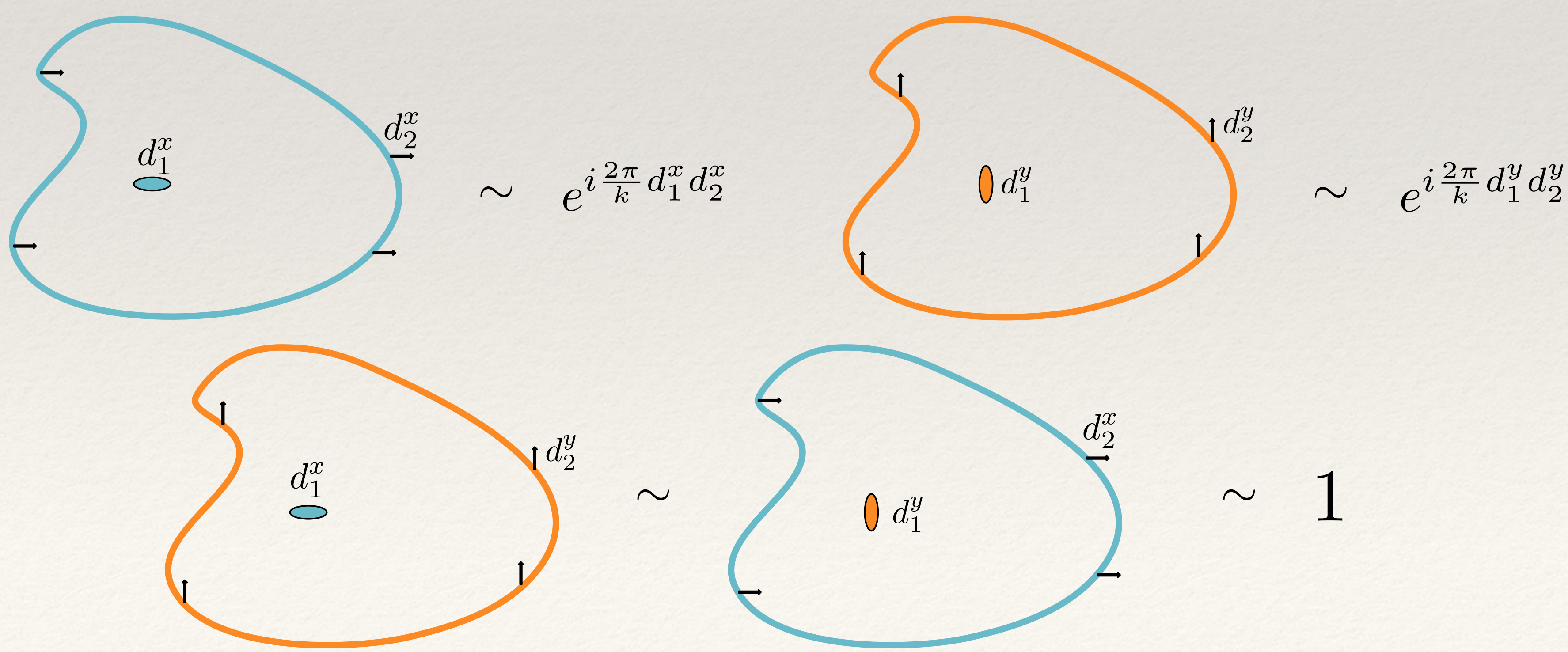
as well as charges



On $\mathbb{R}^2$ we can decompose charges into dipoles and it is useful to treat the dipoles as the fundamental objects

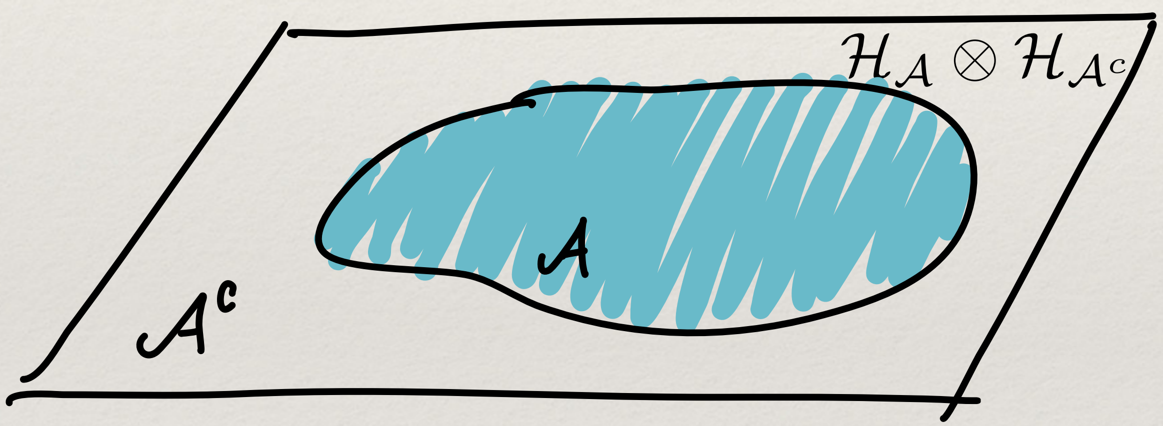


These dipoles have Abelian "anyonic" statistics determined by wrapping them with dipole string operators



Entanglement entropy

Let us push/test the FQH analogy even further by looking at the ground state *entanglement entropy*.



$$\rho_{\mathcal{A}} = \mathrm{Tr}_{\mathcal{H}_{\mathcal{A}^c}} |\psi\rangle\langle\psi|$$

$$S_{\mathcal{A}} = -\mathrm{Tr}_{\mathcal{H}_{\mathcal{A}}} \left(\rho_{\mathcal{A}} \log \rho_{\mathcal{A}}\right)$$

$$S_{\mathcal{A}} = \mathcal{C}\frac{\ell_{\mathcal{A}}}{\varepsilon} - \gamma + \ldots$$

[M.Levin, X.G. Wen; 2006]

[A. Kitaev, J. Preskill; 2006]

For an Abelian topo. phase:

$$\gamma = \frac{1}{2}\log k$$

total # of anyons

an important subtlety …

In gauge theories, $\mathcal{H} \neq \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{A}^c}$ (even in a regularized sense…)

Simple diagnosis: $\dim \mathcal{H}_{\mathbb{R}^2} = 1$ What about $\dim \mathcal{H}_{\mathcal{A}}$?

The generator of gauge tx's yields a boundary term when the parameter has support on ∂A

$$\hat{\mathcal{Q}}[\alpha] = \oint_{\partial \mathcal{A}} ds\, \hat{n}_i\, \varepsilon^{ij} \delta^{kl} \partial_k \alpha\, A_{jl} + \ldots$$

$$\left[\hat{Q}[\alpha], \hat{Q}[\beta]\right] = i \frac{4\pi}{k} \oint_{\partial A} \delta^{ij} \partial_i \alpha\, \mathbf{d}(\partial_j \beta)$$

The algebra of these charges is centrally extended and so it is not consistent to set $\hat{Q} = 0$ as a constraint.

Instead $\mathcal{H}_{\mathcal{A}}$ must carry representations of this algebra

With a little massaging, we find it is isomorphic to *two* $\mathfrak{u}(1)$ Kac-Moody algebras

$$[J^i_m, J^j_n] = \frac{2\pi}{k}(2\pi m)\delta^{ij}\delta_{m+n}$$

(Physically, these stem from independent dipolar edge modes)

The zero modes, J^i_0 , are in fact what appear in the exponent of dipole string operator and so the eigenvalues are the *total dipole moment* (in the i^{th} direction) contained in $\mathcal{A}$

Importantly these representations are *infinite dimensional.*

$$1 \neq \infty \times \infty$$

The fix: embed $\mathcal{H} \hookrightarrow \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{A}^c}$ and take trace within this larger, *extended Hilbert space*

Physical ground state, $|\psi\rangle$, maps to a $|\tilde{\psi}\rangle$ that we will identify by imposing gauge invariance by hand.

$$\left(J^i_{m,\mathcal{A}} \otimes 1_{\mathcal{A}^c} + 1_{\mathcal{A}} \otimes J^i_{-m,\mathcal{A}^c}\right)|\tilde{\psi}\rangle = 0$$

Once dipole moment in $\mathcal{A}$ has been specified, solution is unique: *Ishibashi state(s)*

$$|\tilde{\psi}\rangle = |d^x\rangle\rangle \otimes |d^y\rangle\rangle$$

Bulk ent of tCS = LR ent of dipole Ishibashi states

This reduces to a known calculation.

[X. Wen, S. Matsuura, S. Ryu; 2016]

One subtlety:
Ishibashi states have divergent norm that require regularization

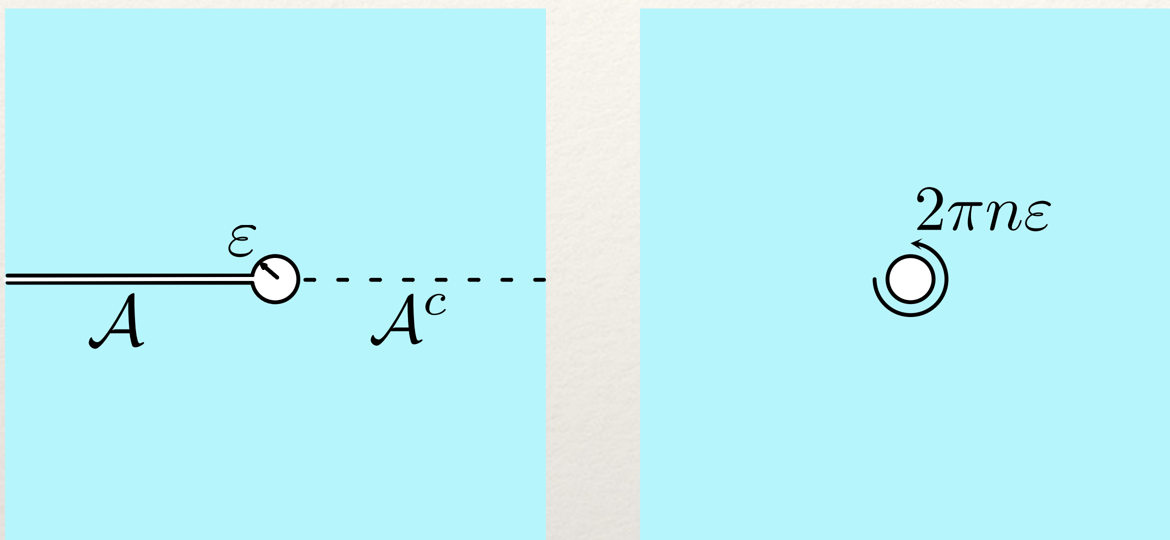
$$|\tilde{\psi}\rangle \to e^{-\epsilon \mathbf{H}}|\tilde{\psi}\rangle$$

$$S_n \sim \lim_{\epsilon \to 0} \frac{\prod_{j=x,y} \vartheta\left(ik\frac{n\epsilon}{\ell}, i\frac{n\epsilon}{\ell}d^j\right)\eta\left(i\frac{n\epsilon}{\ell}\right)^{-1}}{(n \to 1)^n}$$

η : *Dedekind eta function:* from $m \neq 0$ oscillators

ϑ : *Jacobi theta function:* from summing over $k\mathbb{Z}$ dipoles equivalent by large gauge tx

Another perspective:



$$\prod_{j=x,y} \vartheta\left(ik\frac{n\epsilon}{\ell}, i\frac{n\epsilon}{\ell}d^j\right)\eta\left(i\frac{n\epsilon}{\ell}\right)^{-1}$$

arises as the *thermal partition function* of the edge theory with inverse temperature

$$\beta \sim \frac{n\epsilon}{\ell}$$

The Jacobi theta arises from *winding modes* of $\partial_i\phi$ that arise when the symmetry group is *compact*

Taking the $n \to 1$ and $\epsilon \to 0$ limits we find

$$S_{\mathcal{A}} = \frac{k}{24\pi}\frac{\ell}{\epsilon} - \log k + \ldots$$

The subleading constant is consistent with *two separate* Abelian topological orders each with *k* anyons.

What if $\mathbb{R}$ instead of U(1)?

- No charge, level, or dipole quantizations
- Importantly, vacuum =/= "condensate of long-ish dipoles"…
- Charges remain fractonic and dipoles retain restricted mobility
- *"fractonic insulator"*

What is the entanglement entropy of such a phase?

$$S_{\mathcal{A}} = \mathcal{C}_1 \frac{\ell}{\epsilon} - \log\left(\frac{\ell}{\epsilon}\right) + \mathcal{C}_2 + \cdots$$

The coefficient of the log, 2 x (1/2), is universal but coarse: only tells us that there are two polarizations to A_{ij}

Other loose ends

- How robust is this quantum Hall analogue?
  - Unlike Abelian CS, expect that the Hilbert space of tCS is sensitive to curvature. *[A. Gromov; 2019] [K. Slagle, et.al.; 2019]*
  - Can be embedded into a theory of *"chiral elasticity."* Relevant gauge symmetry ~ *Area preserving diffeos* *[A. Gromov; 2019]*
  - Interesting connections between APDs, qH physics, TGT *[A. Cappelli, et.al.;1992] [Y.H. Du, et.al.; 2021]*
- Higher-D and gapless tensor gauge theories?
  - Two contributions to ent. entropy:
    - Edge modes
    - Bulk (gapless) modes.

- Entanglement signatures of gapped fracton phases
  - Growing body of work for stabilizer fracton codes

    [B.Shi, Y.M. Liu; 2018] *[H.Ma, et.al.; 2018]* *[W.Shirley, et.al.; 2018]*

  - Field theory desc. of fracton phases is mature enough to start investigating these types of questions.
  - Good first start: foliated field theories. *[K.Slagle; 2020]*

Thank you.

?s