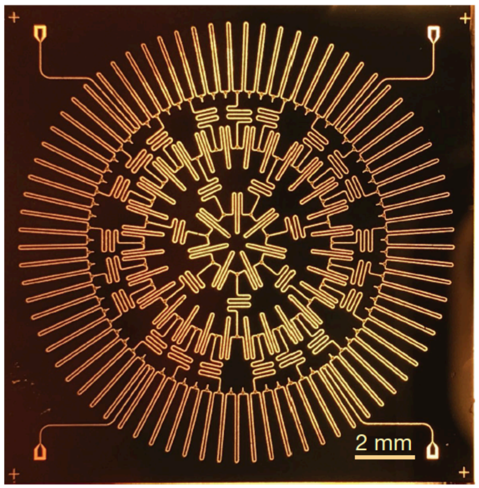
Hyperbolic band theory



Joseph Maciejko
University of Alberta

Topological Properties of Gauge Theories & their Applications to HEP and CMP
Galileo Galilei Institute for Theoretical Physics
September 16, 2021

Steven Rayan
(U. Saskatchewan)

JM and S. Rayan, Sci. Adv. 7, eabe9170 (2021)
JM and S. Rayan, arXiv:2108.09314

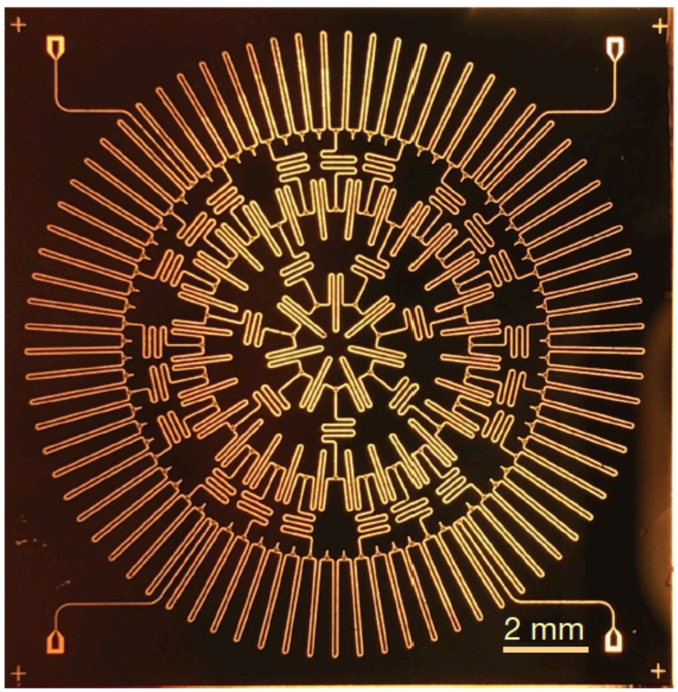
ARTICLE

https://doi.org/10.1038/s41586-019-1348-3

Hyperbolic lattices in circuit quantum electrodynamics

Alicia J. Kollár[1,2,3]*, Mattias Fitzpatrick[1] & Andrew A. Houck[1]

4 JULY 2019 | VOL 571 | NATURE | 45



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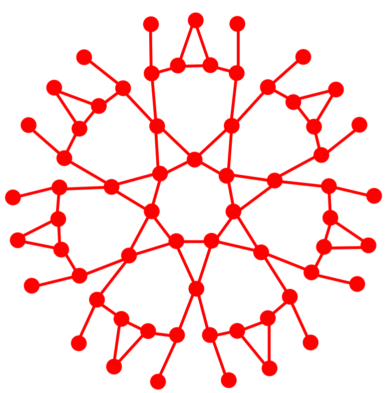

$$H_{\mathrm{TB}} = \omega_0 \sum_i a_i^\dagger a_i - t \sum_{<i,j>} (a_i^\dagger a_j + a_j^\dagger a_i)$$

ARTICLE

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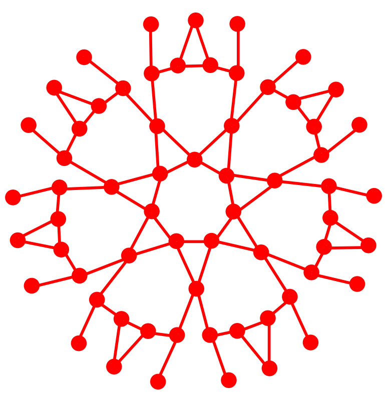

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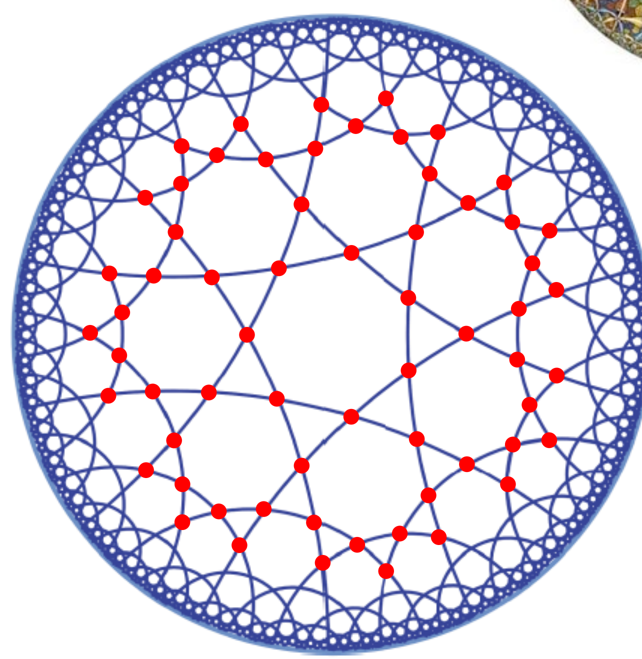
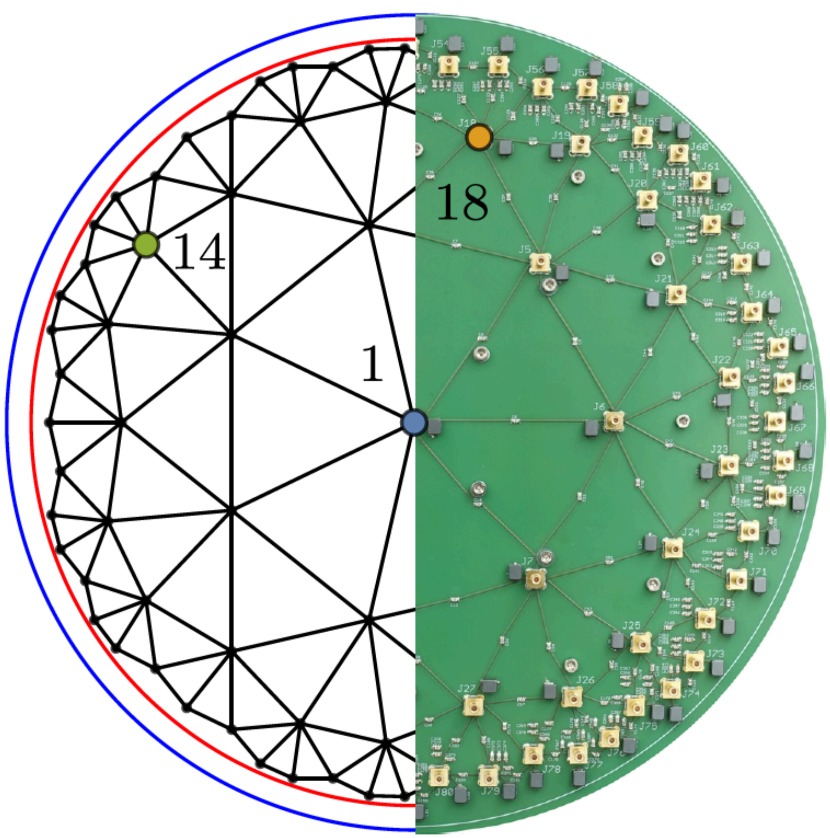


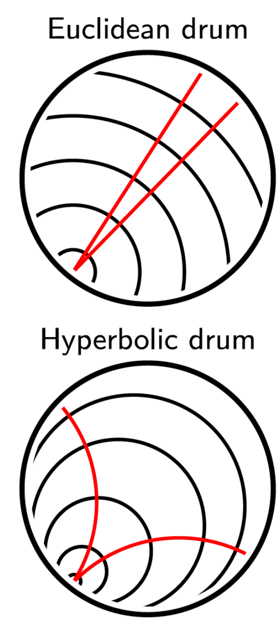
$$H_{\mathrm{TB}} = \omega_0 \sum_i a_i^\dagger a_i - t \sum_{<i,j>} (a_i^\dagger a_j + a_j^\dagger a_i)$$

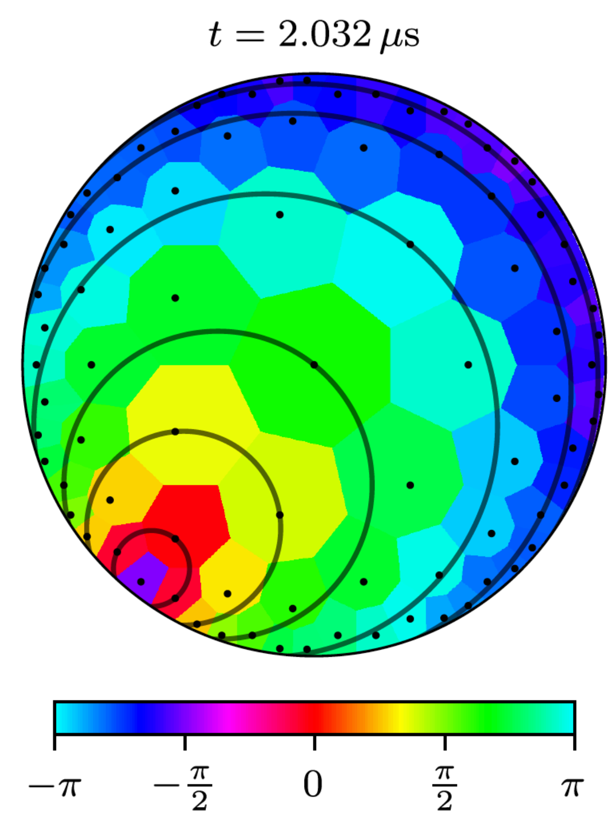
Electric-circuit realization of a hyperbolic drum

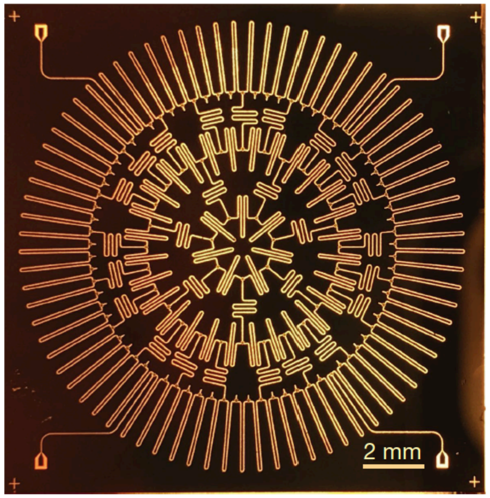
Patrick M. Lenggenhager [1,2,3,*] Alexander Stegmaier [4,*] Lavi K. Upreti [4] Tobias Hofmann,[4] Tobias Helbig [4] Achim Vollhardt,[2] Martin Greiter,[4] Ching Hua Lee,[5] Stefan Imhof,[6] Hauke Brand,[6] Tobias Kießling,[6] Igor Boettcher [7,8] Titus Neupert [2,†] Ronny Thomale [4,†] and Tomáš Bzdušek [1,2,†]

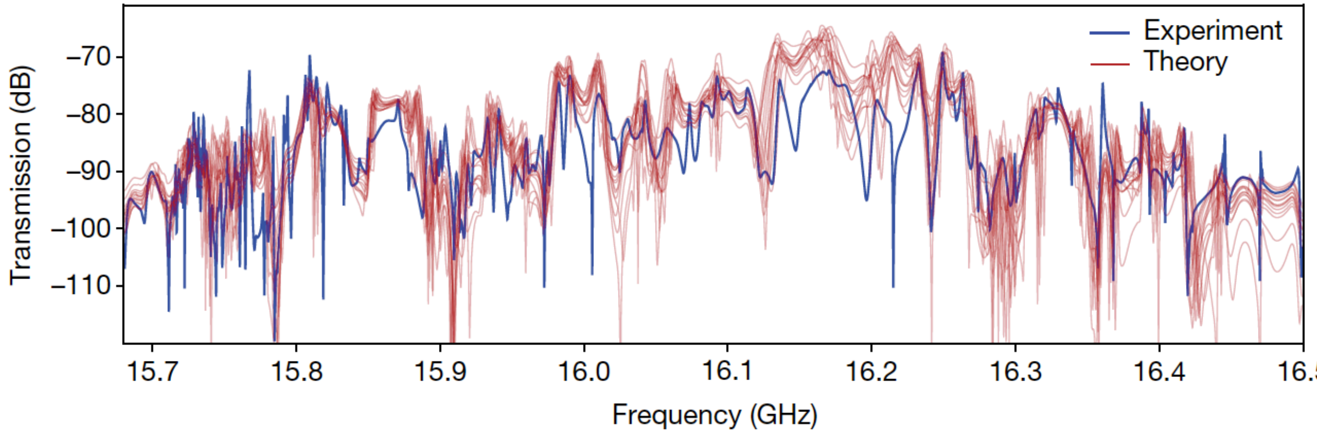
arXiv:2109.01148

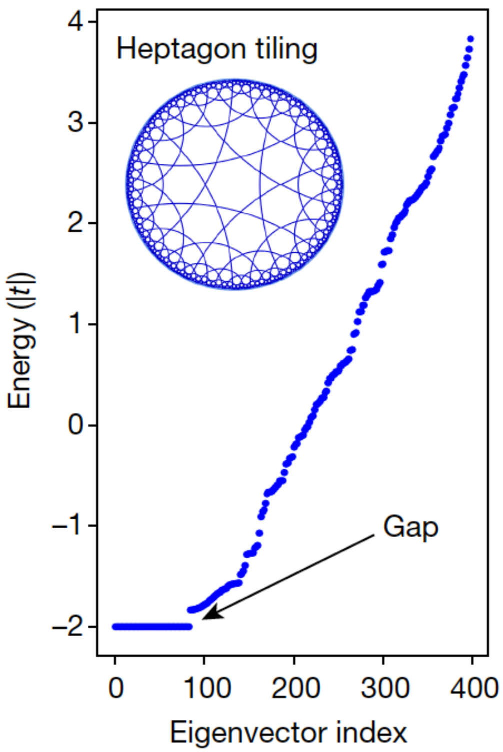






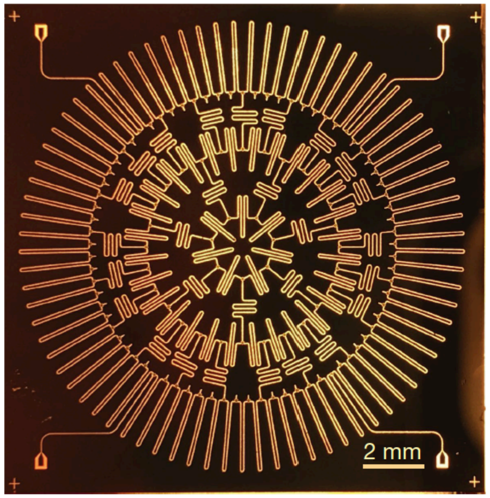


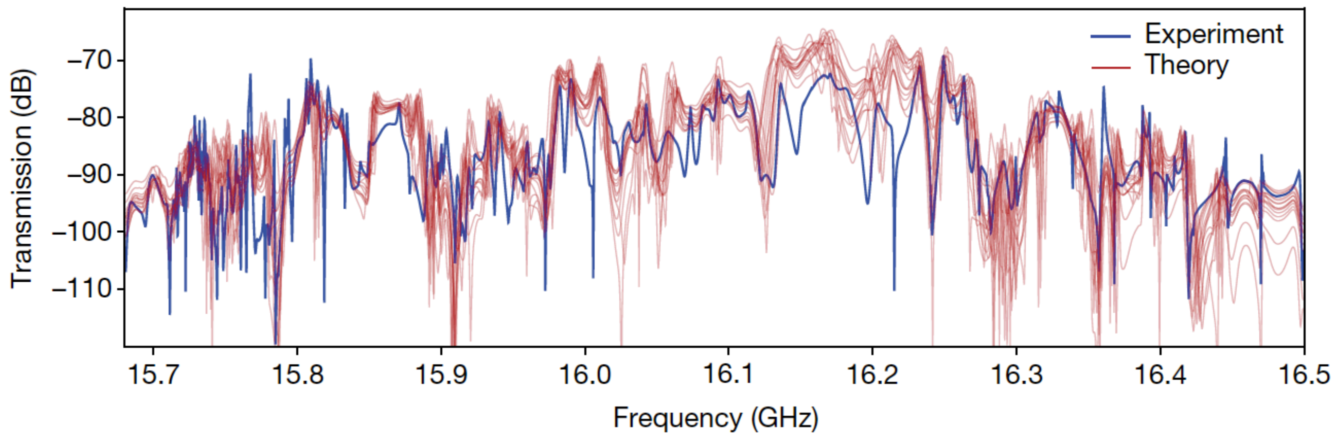


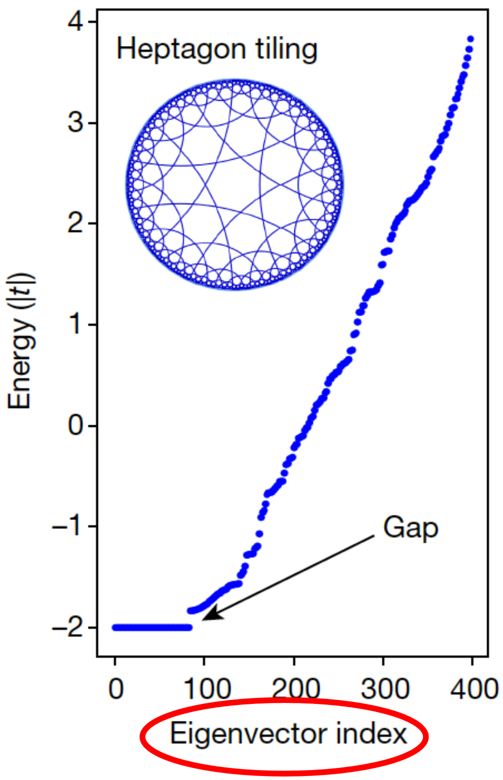


Curved-space tight-binding models. No hyperbolic equivalent of Bloch theory currently exists, and there is no known general procedure for calculating band structures in either the nearly-free-electron or tight-binding limits. Specialized methods are known for the cases of trees[49,50] but fail if there are any closed loops, except in the special case of Cayley graphs of the free products of cyclic groups[32]. The only universal method is numerical diagonalization of the hopping Hamiltonian. This is a brute-force method which yields a list of eigenvectors and eigenvalues, but no classification of eigenstates by a momentum quantum number.

Kollár, Fitzpatrick, Houck, Nature 571, 45 (2019)







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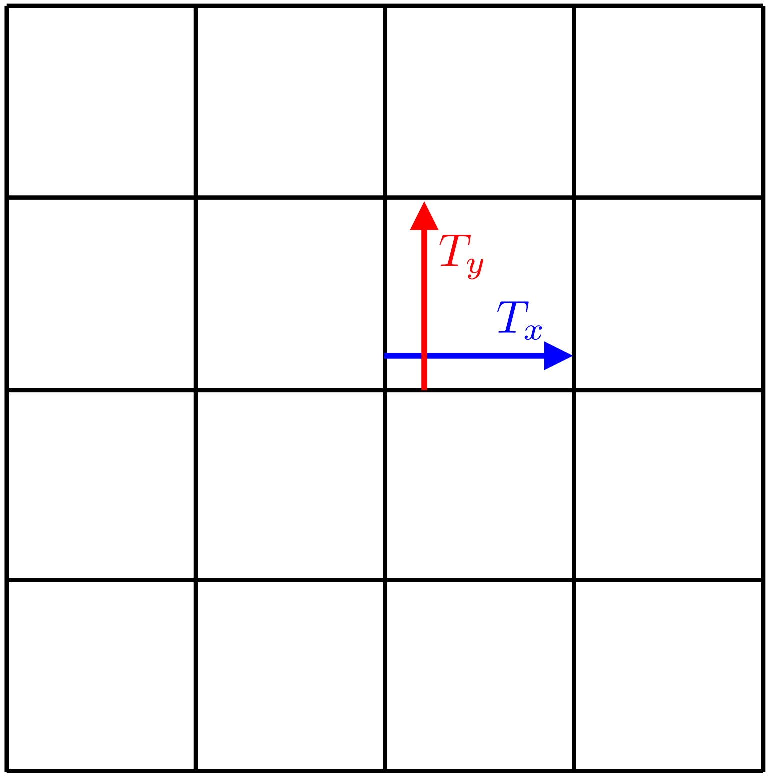
Kollár, Fitzpatrick, Houck, Nature 571, 45 (2019)

Outline

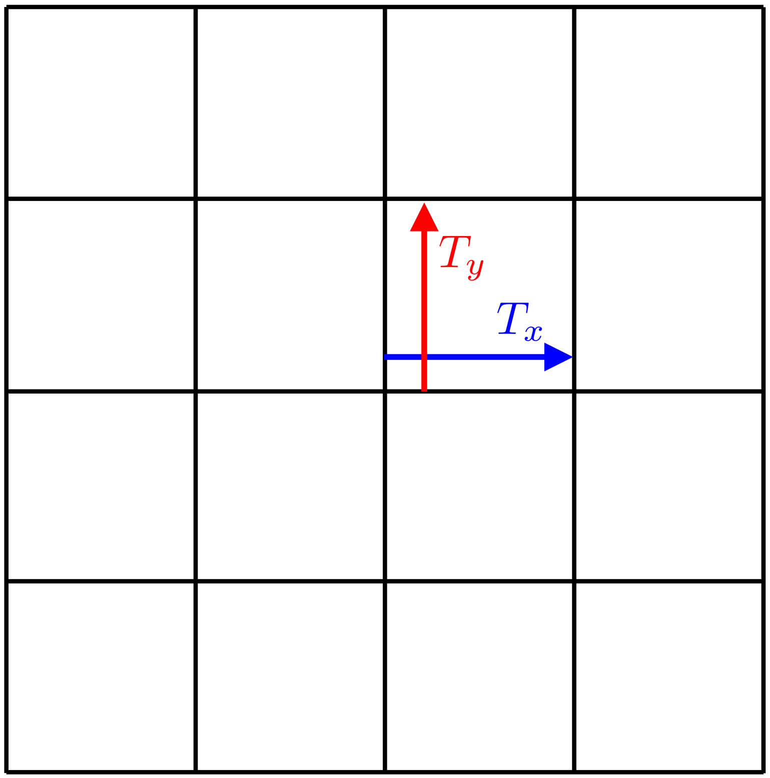
- Euclidean band theory
- Hyperbolic geometry & Fuchsian groups
- The {8,8} lattice
- Periodic boundary conditions & automorphic Bloch theorems
- Conclusion

Euclidean lattices



$$H = \frac{\boldsymbol{p}^2}{2m} + V(x, y)$$

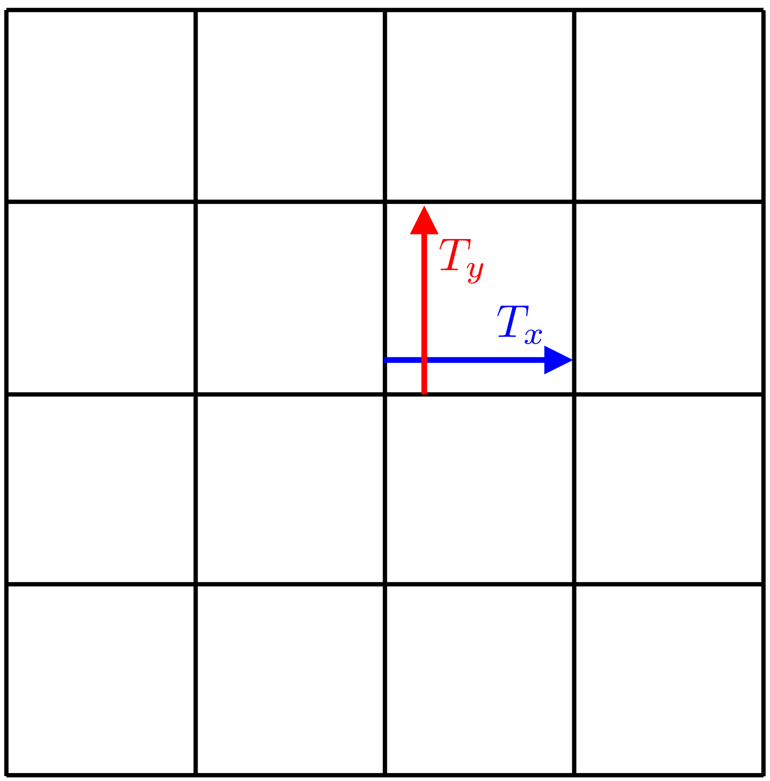
Euclidean Bloch condition



$$H\psi = E\psi$$

$$\psi(x+1, y) = e^{ik_x}\psi(x, y)$$
$$\psi(x, y+1) = e^{ik_y}\psi(x, y)$$

Brillouin zone torus



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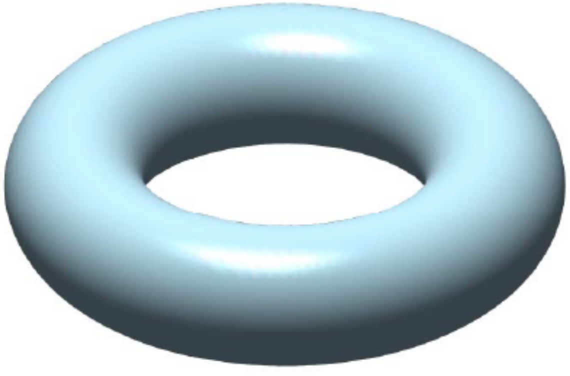
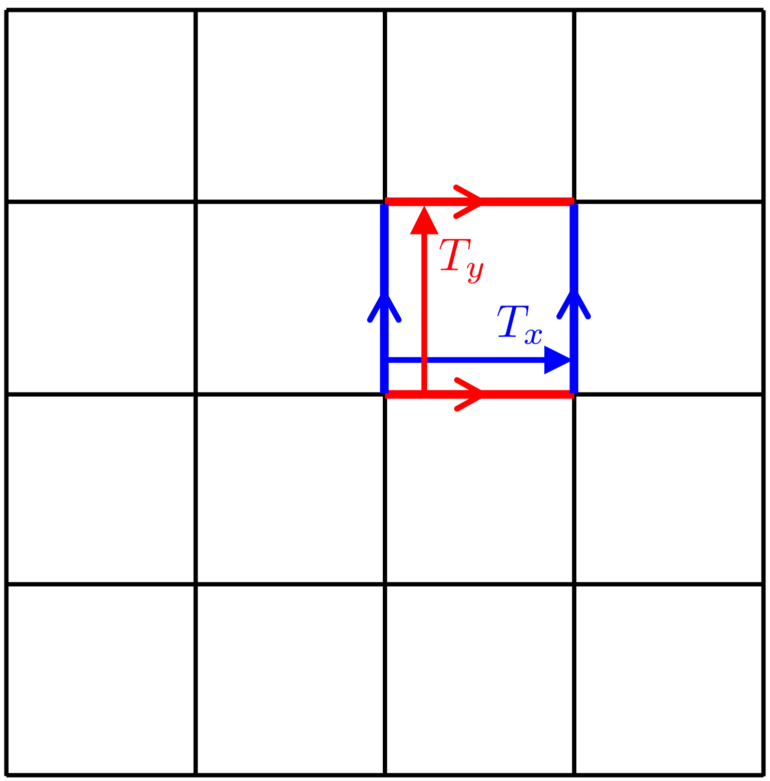
$$\psi(x+1, y) = e^{ik_x}\psi(x, y)$$
$$\psi(x, y+1) = e^{ik_y}\psi(x, y)$$

$$k_x \sim k_x + 2\pi$$
$$k_y \sim k_y + 2\pi$$

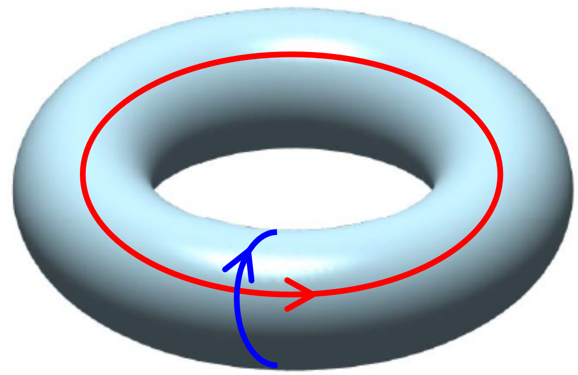


Compactified unit cell

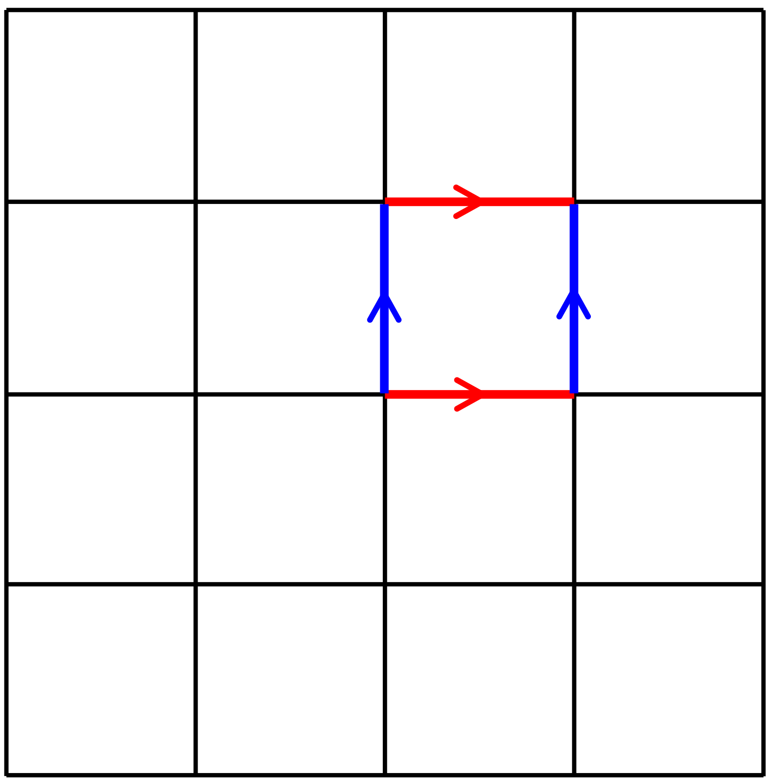
$$\mathbb{R}^2/\mathbb{Z}^2 \cong T^2$$



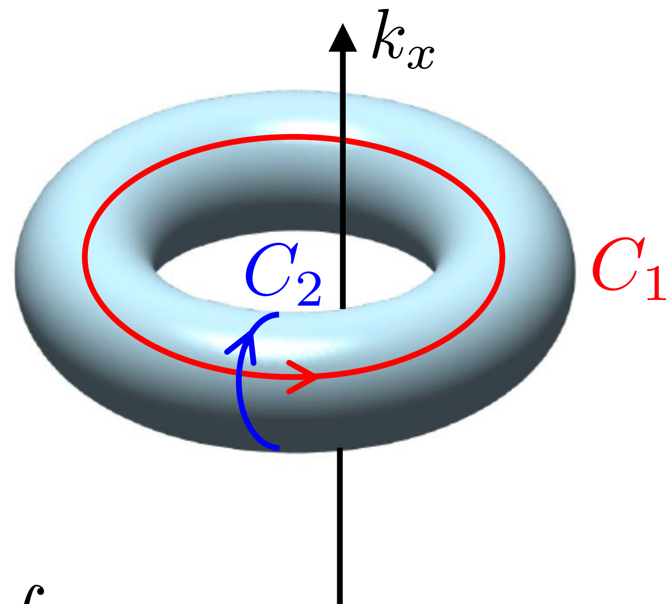
$\cong$



Aharonov-Bohm fluxes

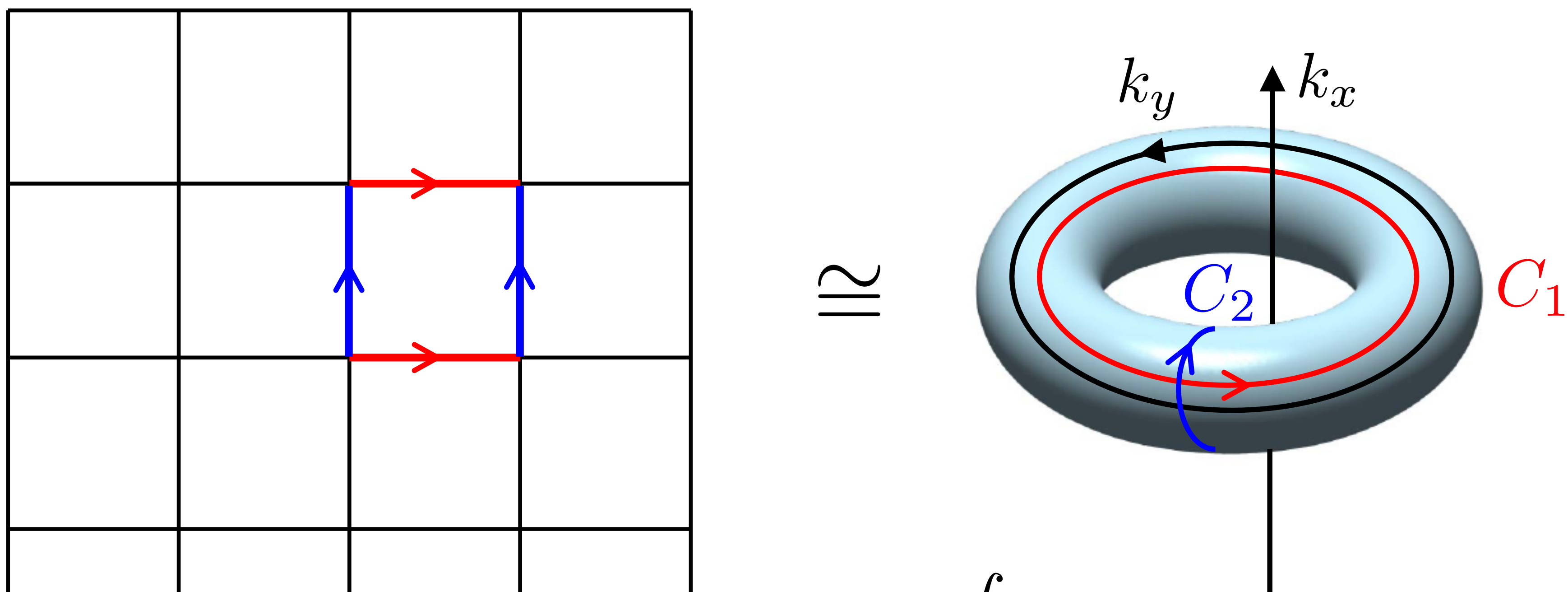


$\cong$



$$k_x = \oint_{C_1} A$$

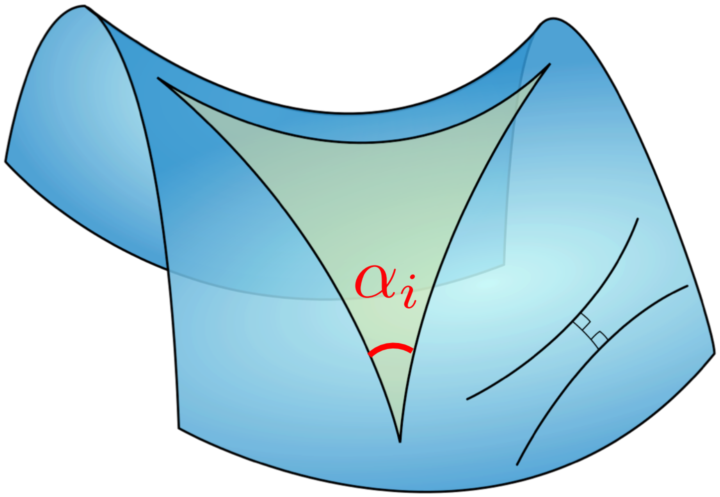
Aharonov-Bohm fluxes



$$k_x = \oint_{C_1} A$$

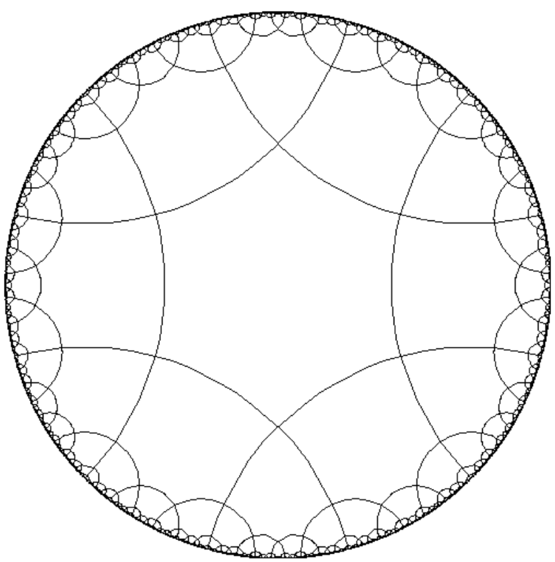
$$k_y = \oint_{C_2} A$$

Hyperbolic lattices

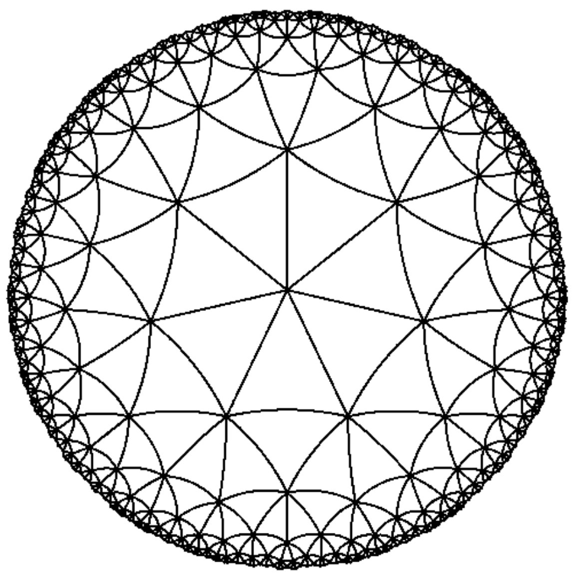


$$\sum_{i=1}^{p} \alpha_i < (p-2)\pi$$

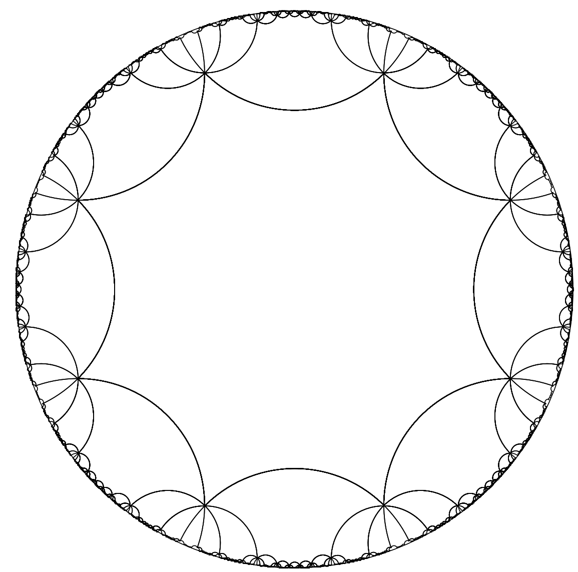
$\{6, 4\}$



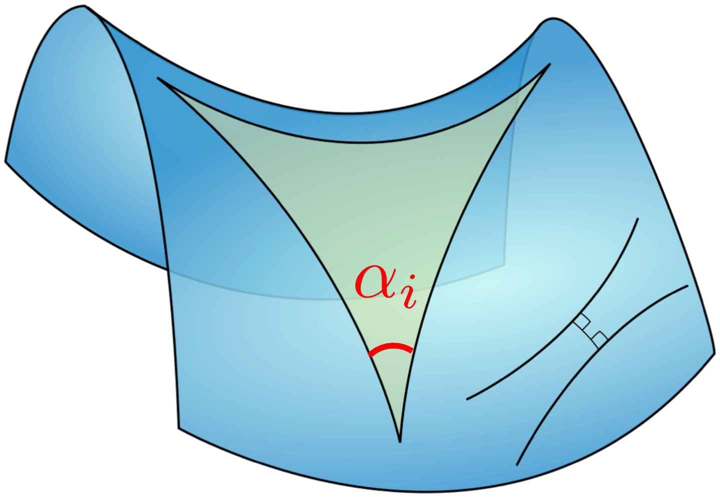
$\{3, 7\}$



$\{8, 8\}$



Hyperbolic lattices



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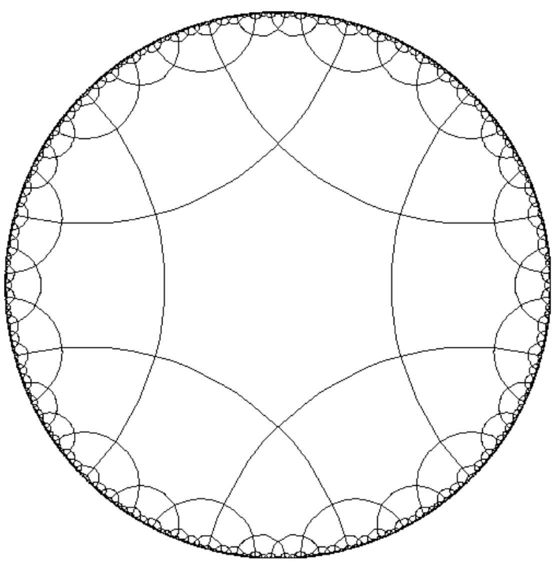

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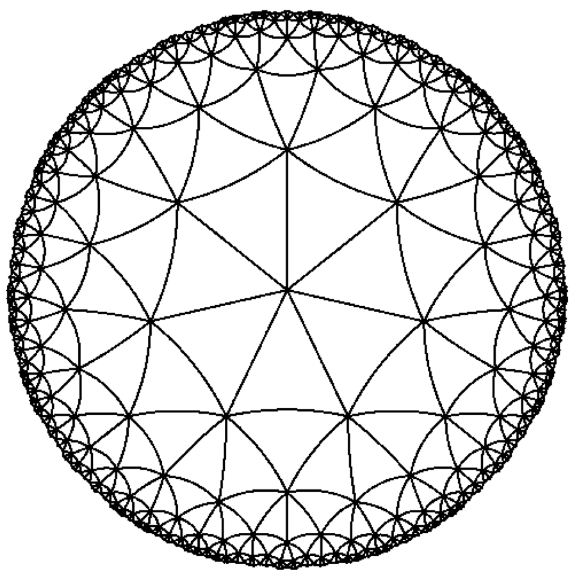

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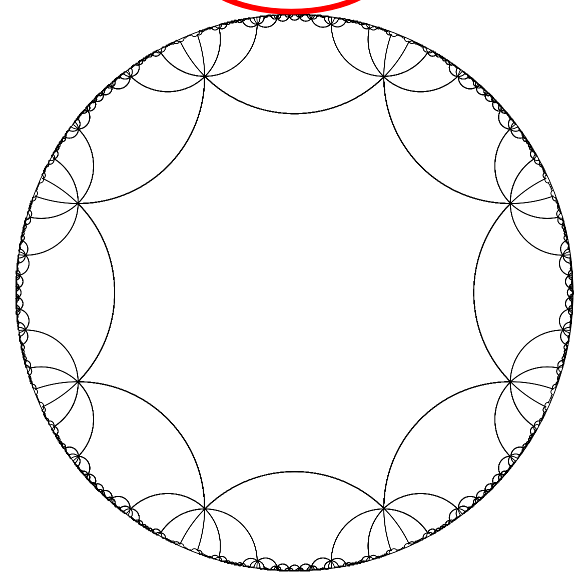
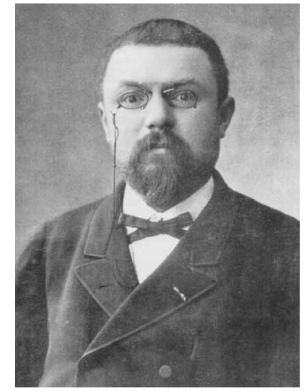
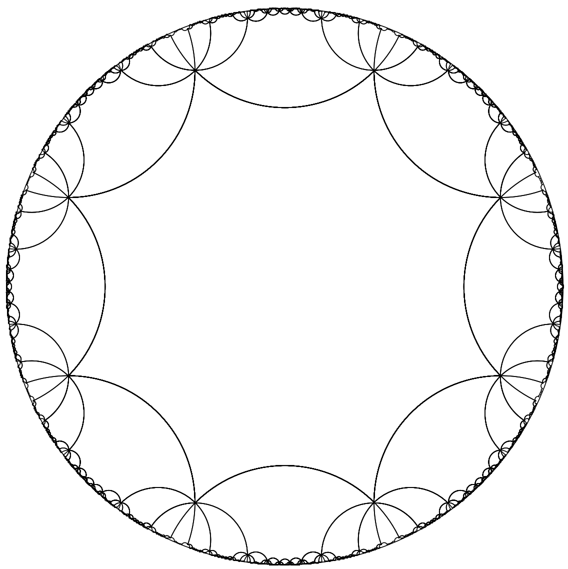


Poincaré disk



$$\mathbb{H} = \{|z| < 1\}$$



$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - |z|^2)^2}$$

Poincaré disk

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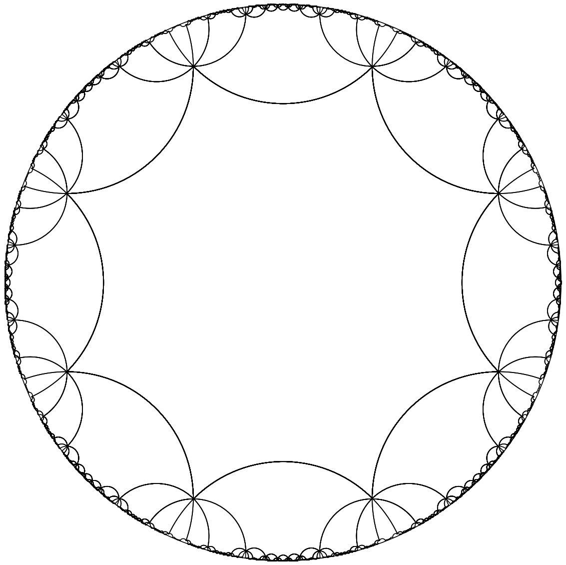

$$ds^2 = \frac{4(dx^2 + dy^2)}{(1 - |z|^2)^2}$$

PSU(1,1) ≅ PSL(2,R):

$$\gamma = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}, \det \gamma = 1$$

$$z \to \gamma(z) = \frac{\alpha z + \beta}{\beta^* z + \alpha^*}$$

Poincaré disk

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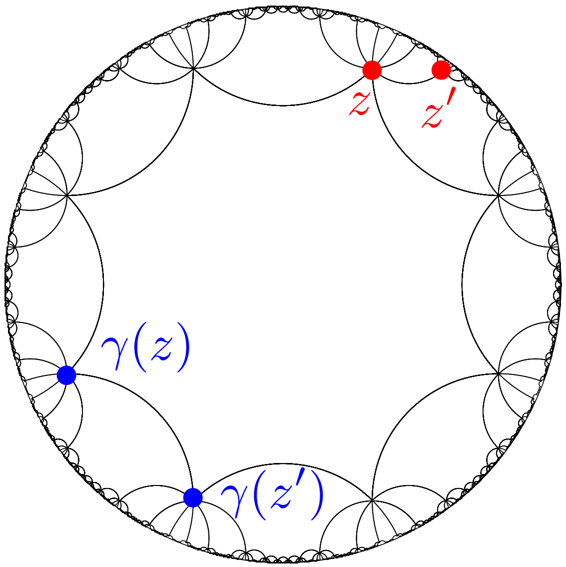

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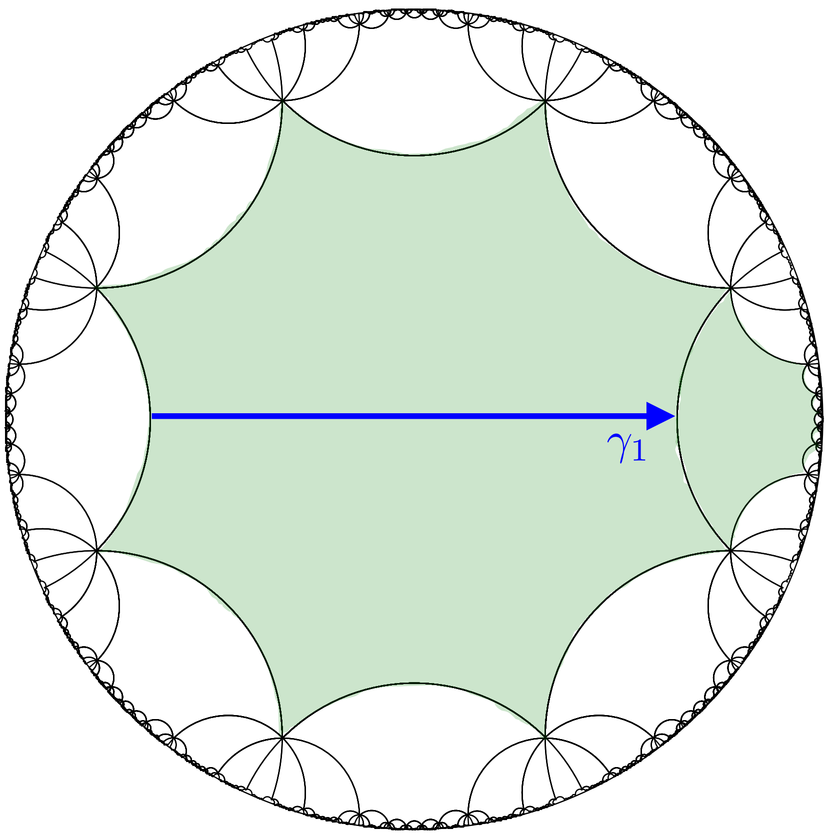
$$d(z, z') = d(\gamma(z), \gamma(z'))$$

Fuchsian groups

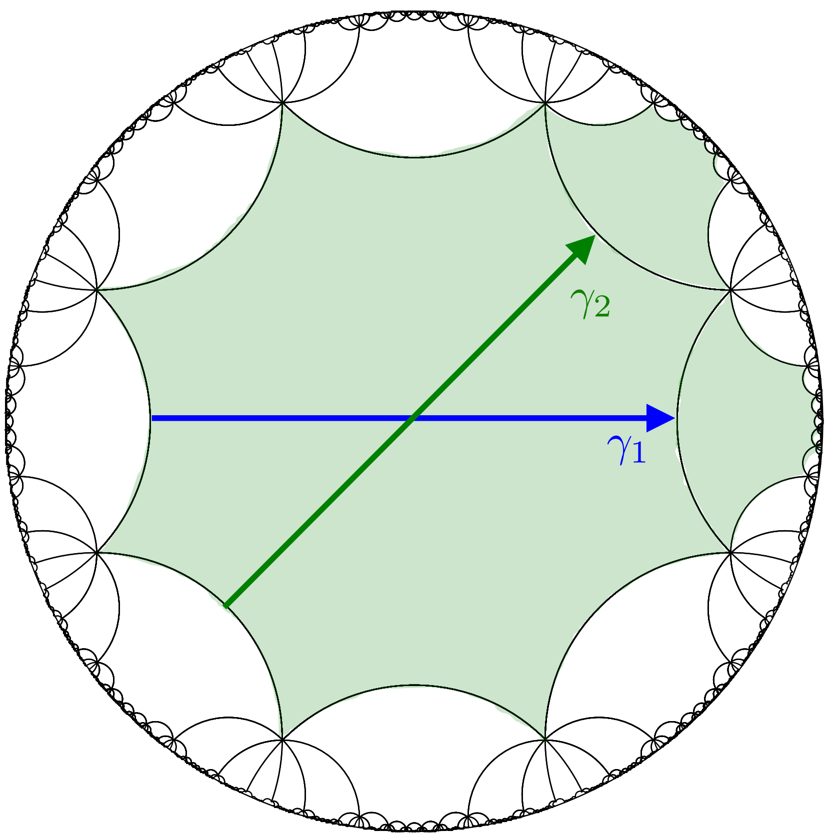


$$\Gamma \subset PSU(1,1)$$

$$\gamma_j = \begin{pmatrix} 1+\sqrt{2} & e^{i(j-1)\pi/4}\sqrt{2+2\sqrt{2}} \\ e^{-i(j-1)\pi/4}\sqrt{2+2\sqrt{2}} & 1+\sqrt{2} \end{pmatrix}, \qquad j = 1,\ldots,4$$

Balazs & Voros, Phys. Rep. 143, 109 (1986)

Fuchsian groups



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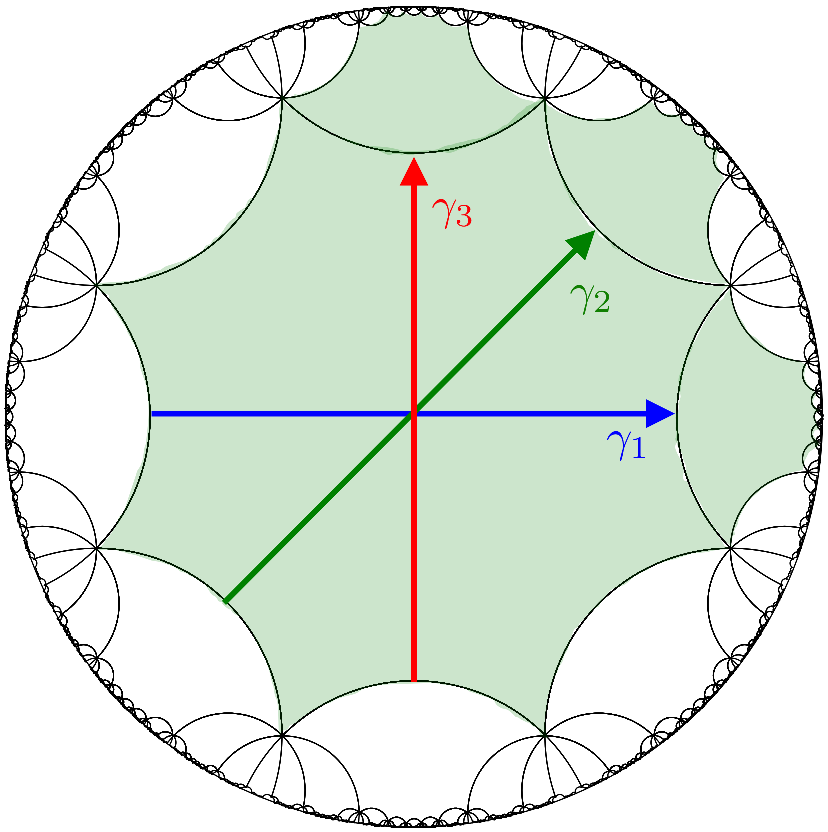

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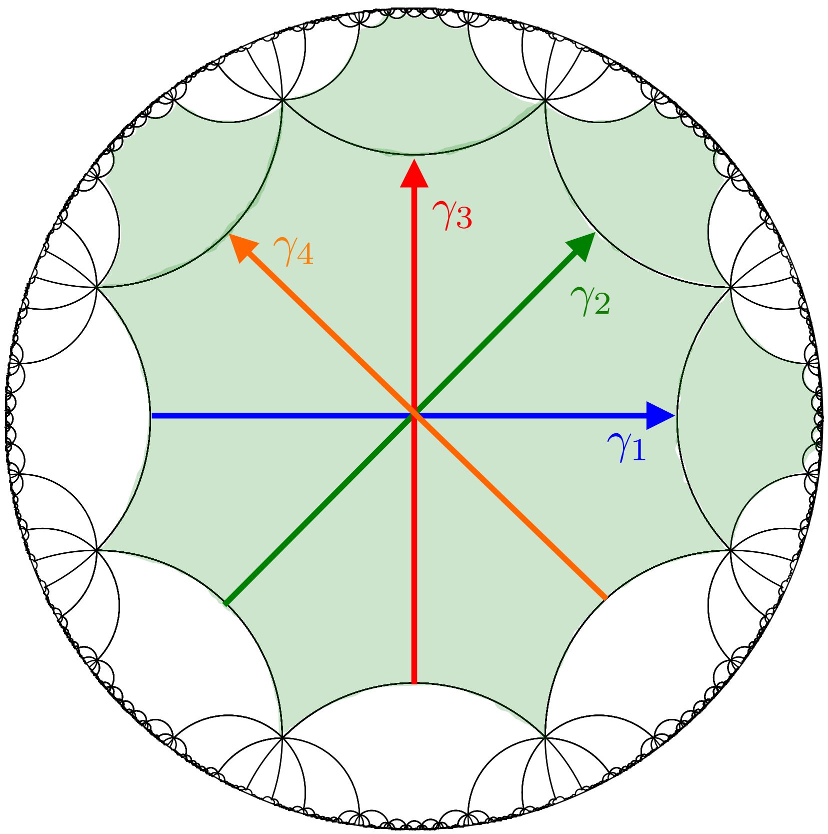

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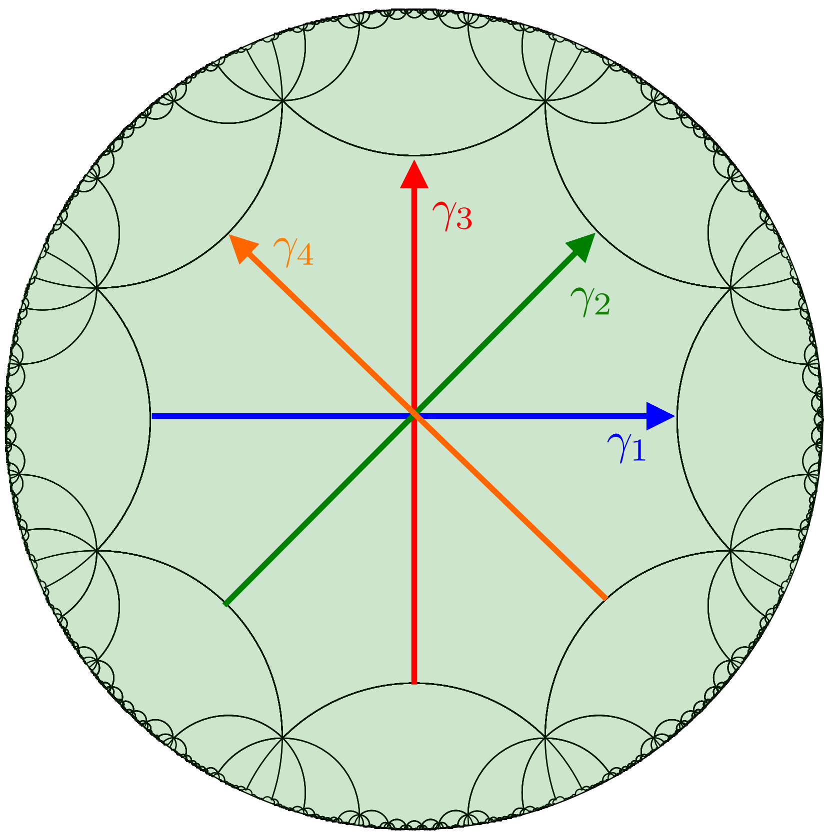

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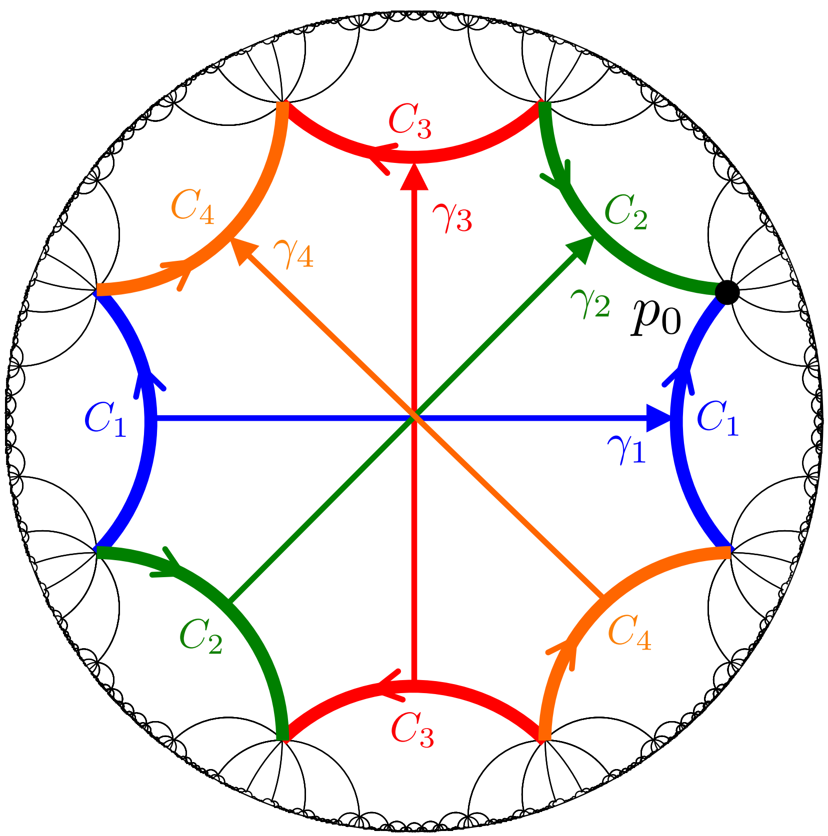
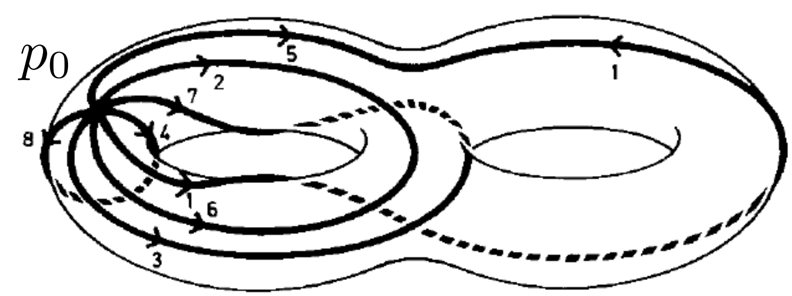
Balazs & Voros, Phys. Rep. 143, 109 (1986)

Compactified unit cell



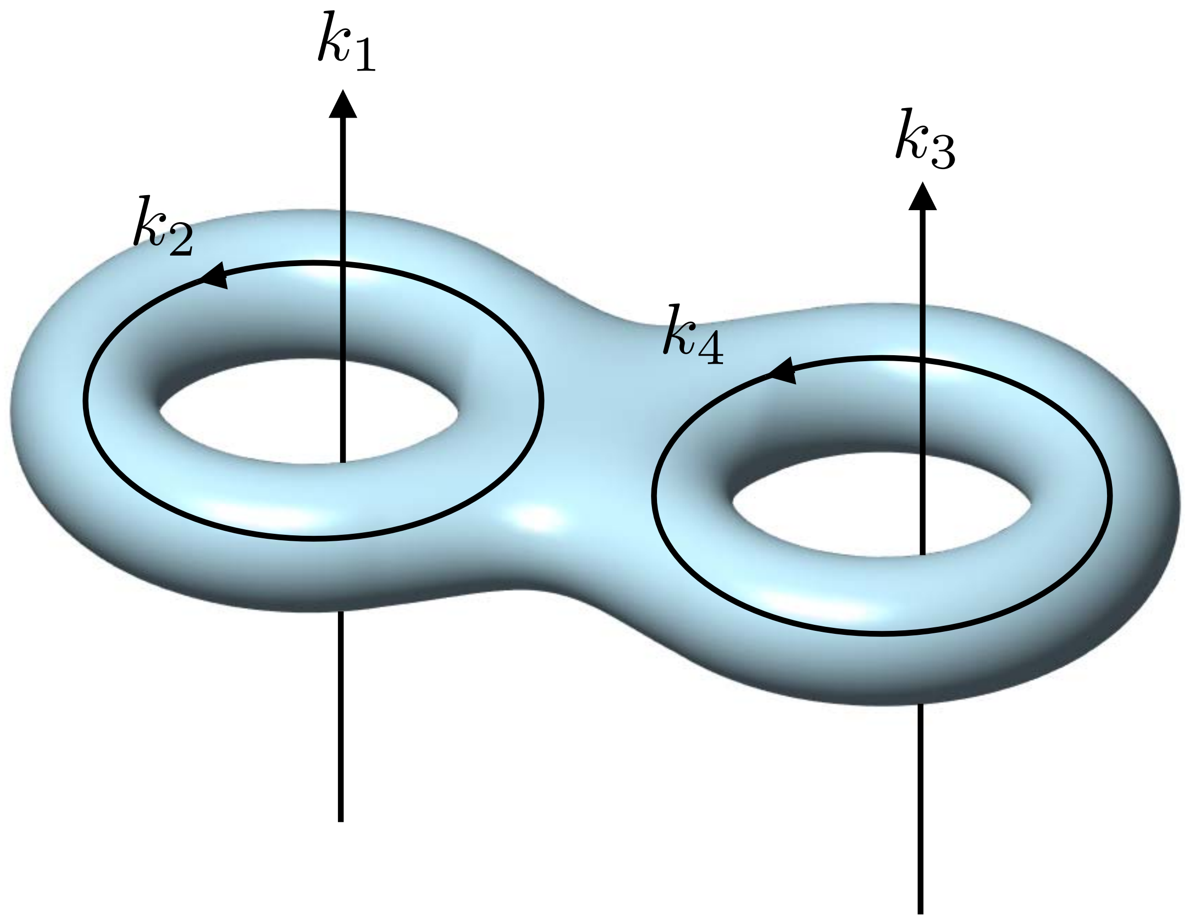
$$\mathbb{H}/\Gamma \cong \Sigma_2$$

$\cong$

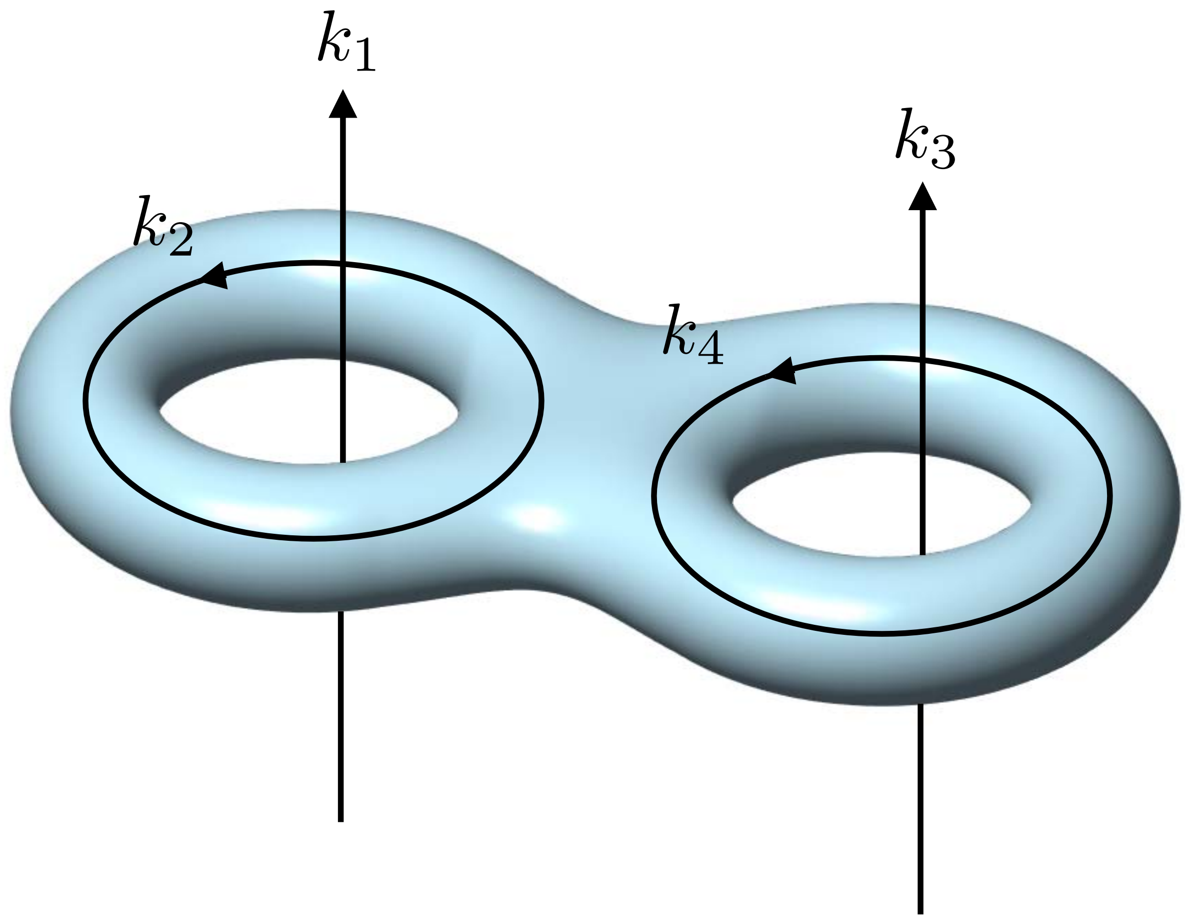


(Figure from Balazs & Voros, Phys. Rep. 143, 109 (1986))

Aharonov-Bohm fluxes

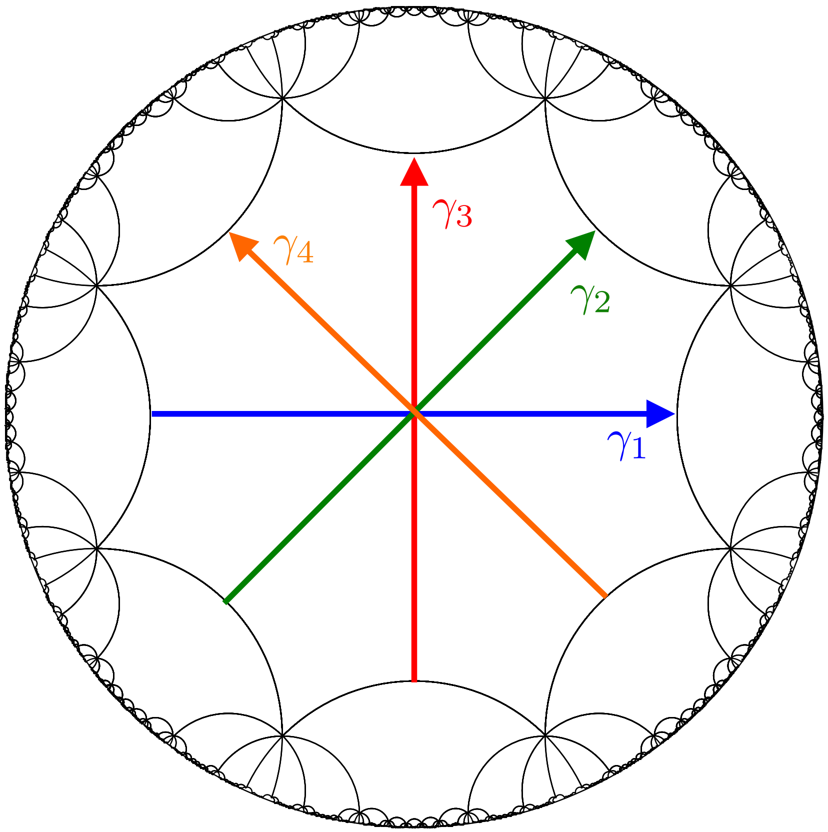


Hyperbolic crystal momentum



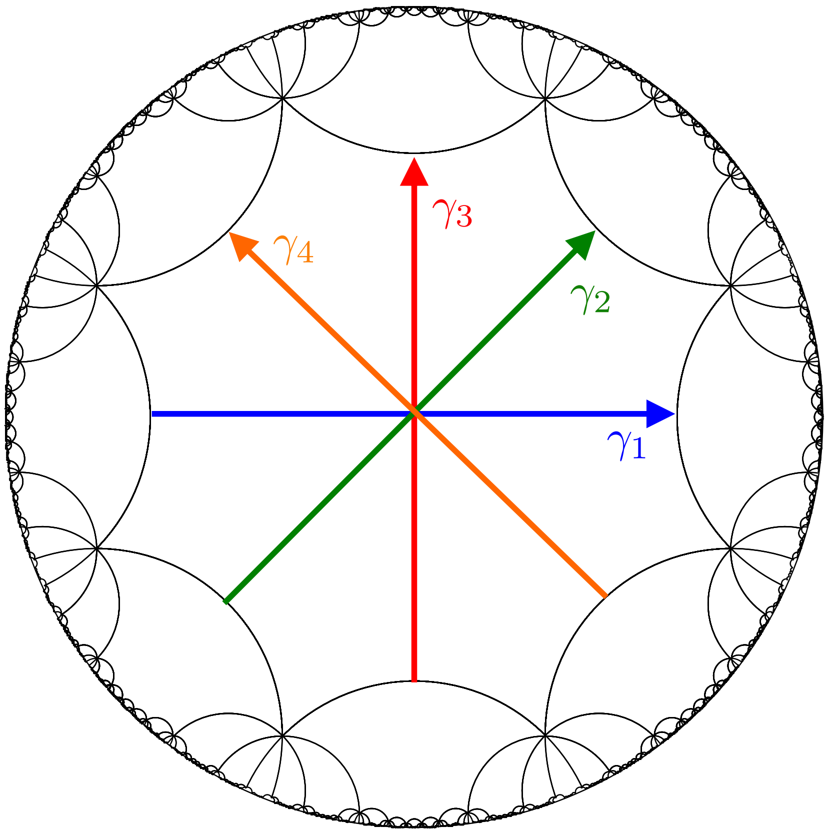
$$\boldsymbol{k} \equiv (k_1, k_2, \ldots, k_{2g-1}, k_{2g}) \in (-\pi, \pi]^{2g} \cong T^{2g} \cong \mathrm{Jac}(\Sigma_g)$$

The hyperbolic Bloch problem



$$H = -\Delta + V(z)$$

The hyperbolic Bloch problem



$$H = -\Delta + V(z)$$

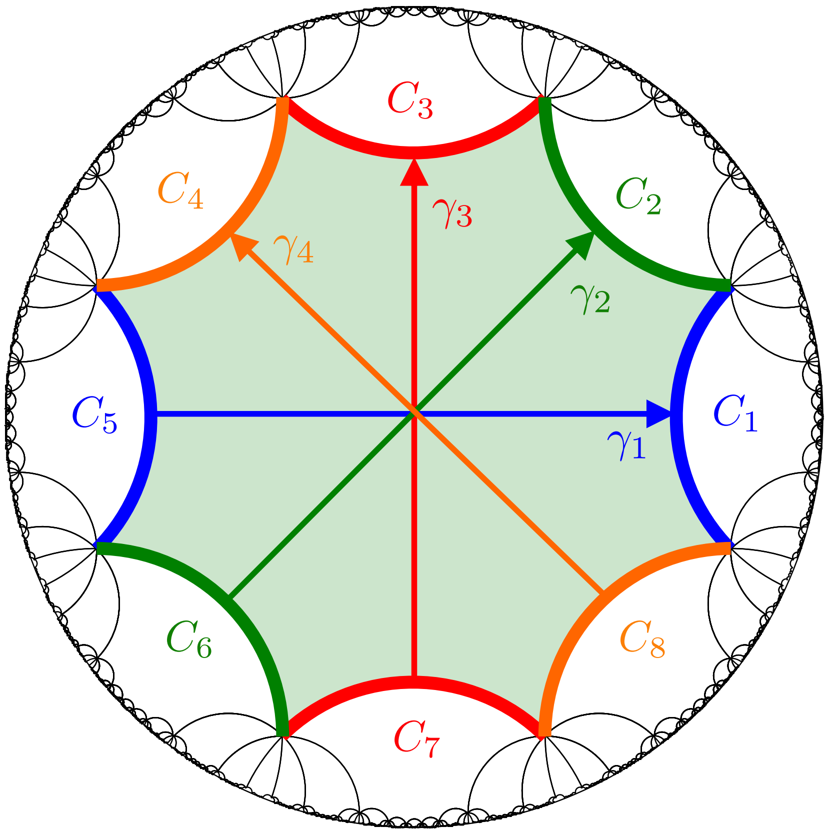
$$\Delta = \frac{1}{4}(1 - |z|^2)^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

Laplace-Beltrami operator

$$V(\gamma(z)) = V(z),\ \gamma \in \Gamma$$

“periodic” potential

Automorphic Bloch condition



$$H\psi = E\psi$$

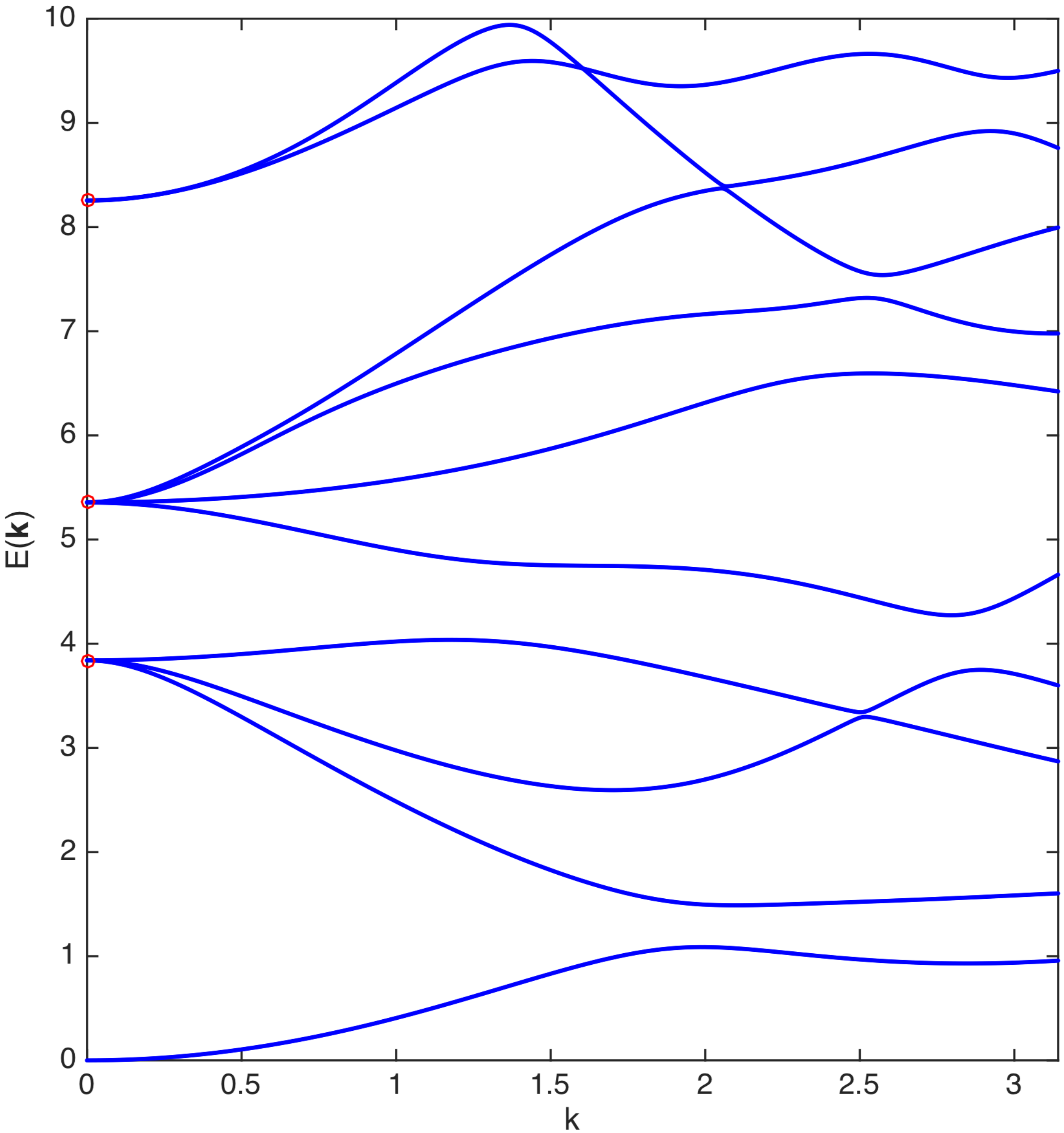
$$\gamma_j(C_{j+4}) = C_j,$$

$$\psi(C_j) = e^{ik_j}\psi(C_{j+4}),$$

$$j = 1, 2, 3, 4$$

$$\psi(\gamma(z)) = \chi(\gamma)\psi(z)$$

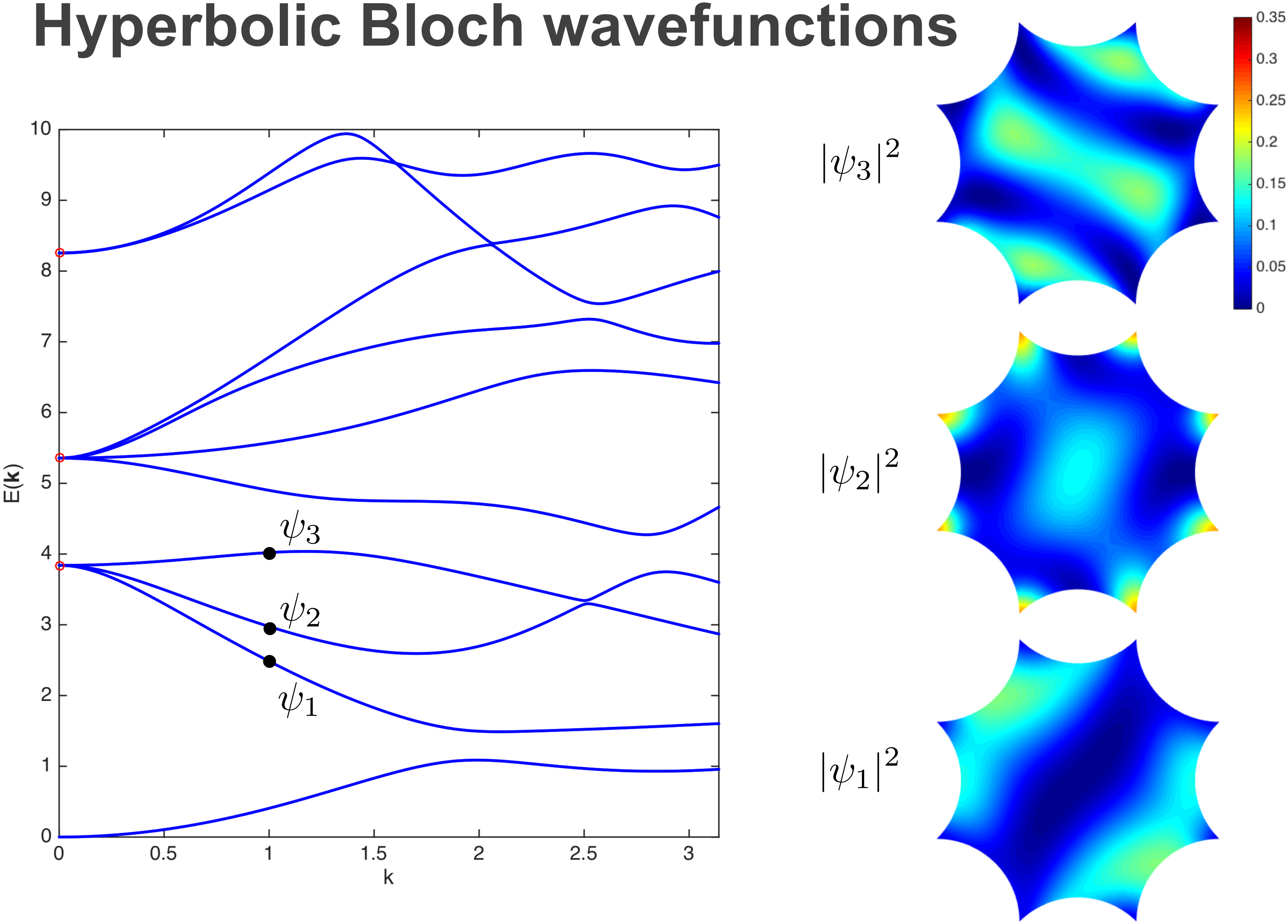
Hyperbolic bandstructure: empty lattice



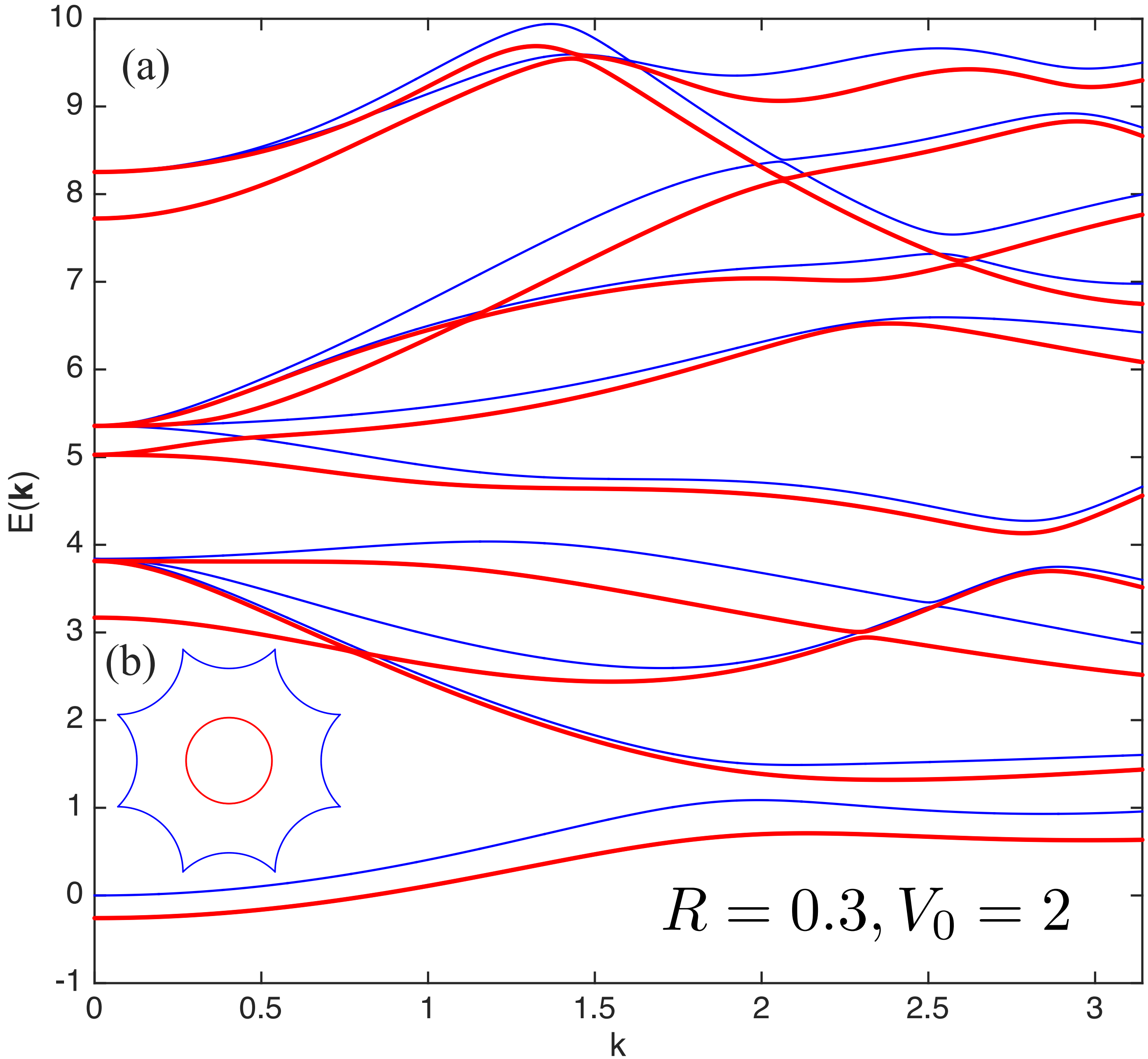
$$H = -\Delta$$

$$\boldsymbol{k} = (0.8, 0.3, 1.2, 1.7)k$$

Hyperbolic Bloch wavefunctions



Particle in an automorphic potential



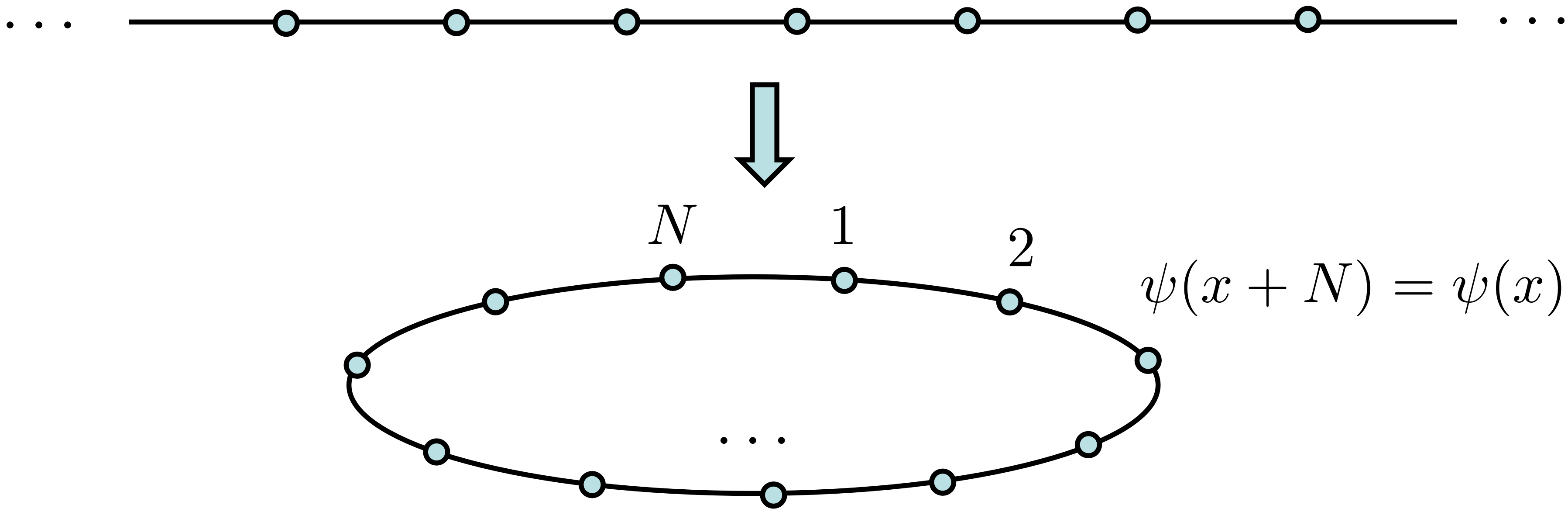
$$V(z) = \sum_{\gamma \in \Gamma} U(\gamma(z)),$$

$$U(z) = \begin{cases} -V_0, & |z| < R, \\ 0, & |z| \geqslant R. \end{cases}$$

Two issues

- Do automorphic Bloch states form a complete set (ansatz vs theorem)?
- What about finite lattices (experiment)?

Euclidean PBC

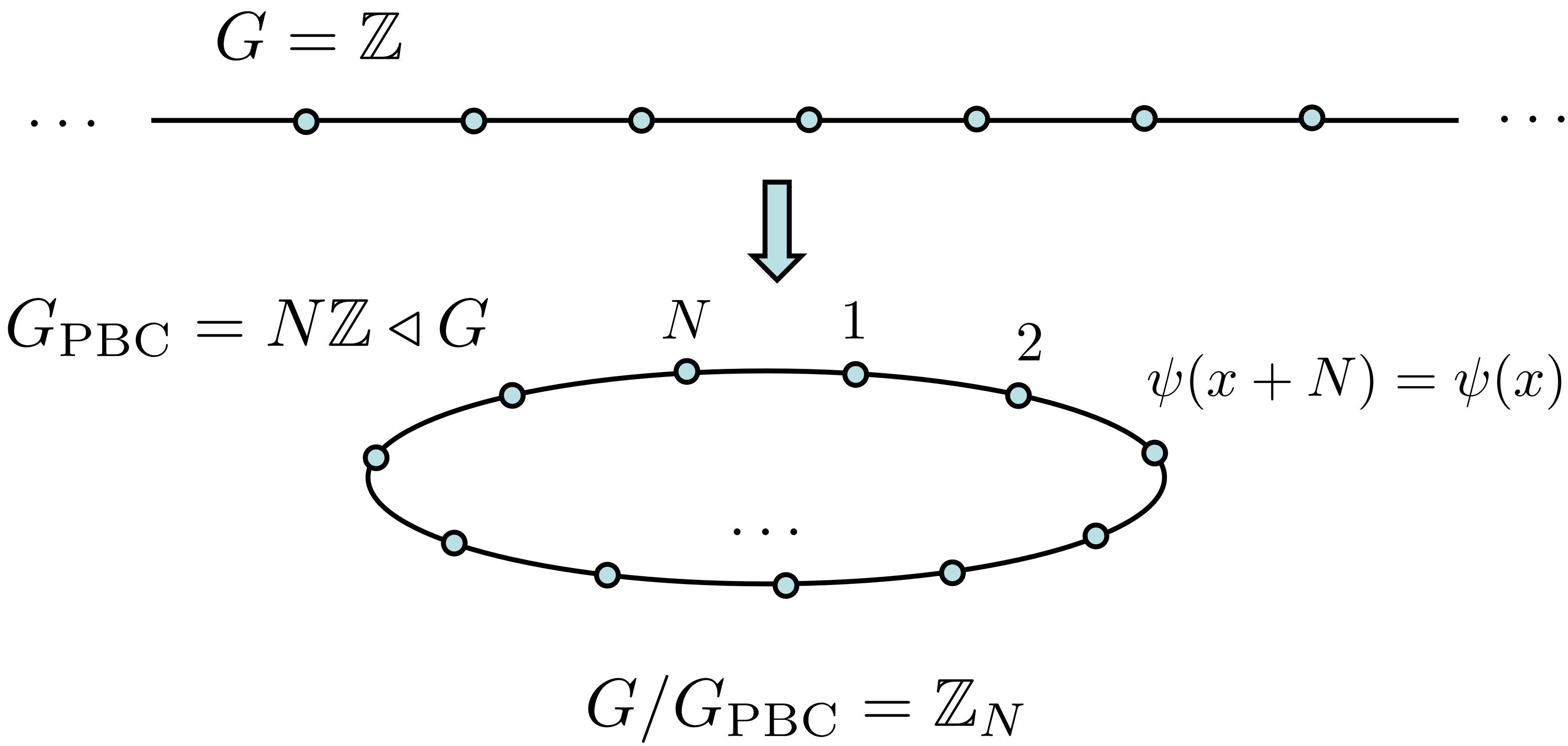


$$\psi(x+N)=\psi(x)$$

$$k=\frac{2\pi n}{N},\; n=0,1,\ldots,N-1$$

N sites = *N* Bloch states: complete set

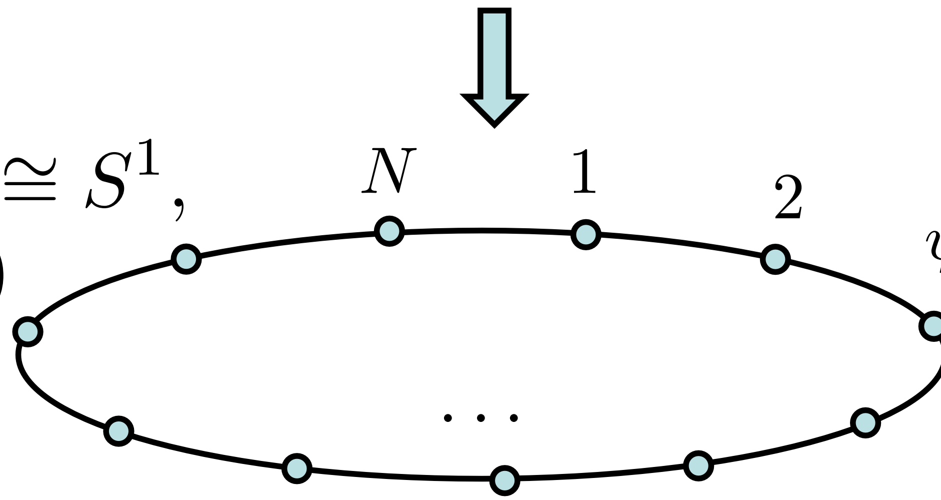
Euclidean PBC: algebraic viewpoint



N allowed *k* values = *N* unitary irreps of Z_N

Euclidean PBC: topological viewpoint

$$X = \mathbb{R}/G \cong S^1,\ G \cong \pi_1(X)$$



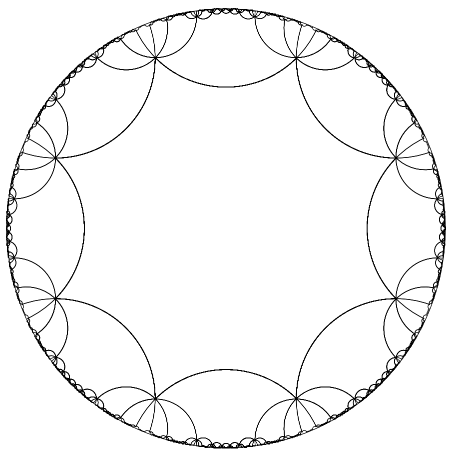
$$Y_N = \mathbb{R}/G_{\mathrm{PBC}} \cong S^1,$$
$$G_{\mathrm{PBC}} \cong \pi_1(Y_N)$$

$$\psi(x+N) = \psi(x)$$

$$X \cong Y_N/\mathbb{Z}_N,\ \mathbb{Z}_N \cong \pi_1(X)/\pi_1(Y_N)$$

Y_N = N-sheeted Galois (normal) cover of X,
with group of deck transformations Z_N

Hyperbolic PBC: algebraic viewpoint



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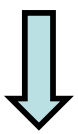


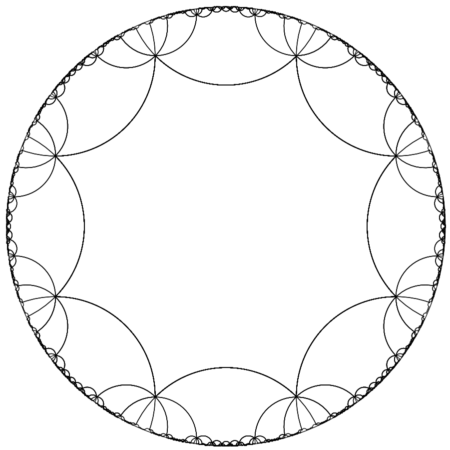
PBC cluster with *N* unit cells

$$\psi(\gamma_{\mathrm{PBC}}(z)) = \psi(z)$$

$\Gamma_{\mathrm{PBC}} \triangleleft \Gamma$: normal subgroup of index *N*

Sausset & Tarjus, J. Phys. A 40, 12873 (2007)

Hyperbolic PBC: topological viewpoint

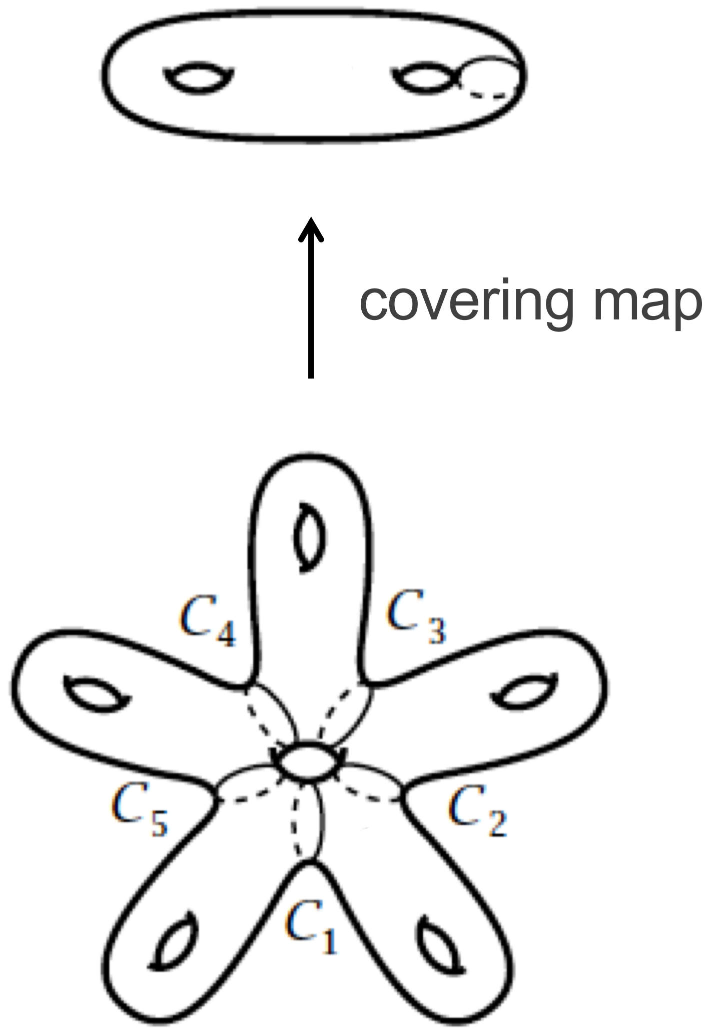


$$X = \mathbb{H}/\Gamma \cong \Sigma_g,\, \Gamma \cong \pi_1(X)$$

PBC cluster with *N* unit cells:
N-sheeted Galois cover of *X*,
with deck group $\Gamma/\Gamma_{\mathrm{PBC}}$

$$Y_N = \mathbb{H}/\Gamma_{\mathrm{PBC}},\, \Gamma_{\mathrm{PBC}} = \pi_1(Y_N)$$

Hyperbolic PBC: topological viewpoint



(e.g. N=5)

$$X \cong \Sigma_g$$

$$Y_N \cong \Sigma_h$$

$$h = N(g-1)+1$$

Riemann-Hurwitz formula

PBC vs the Bloch ansatz

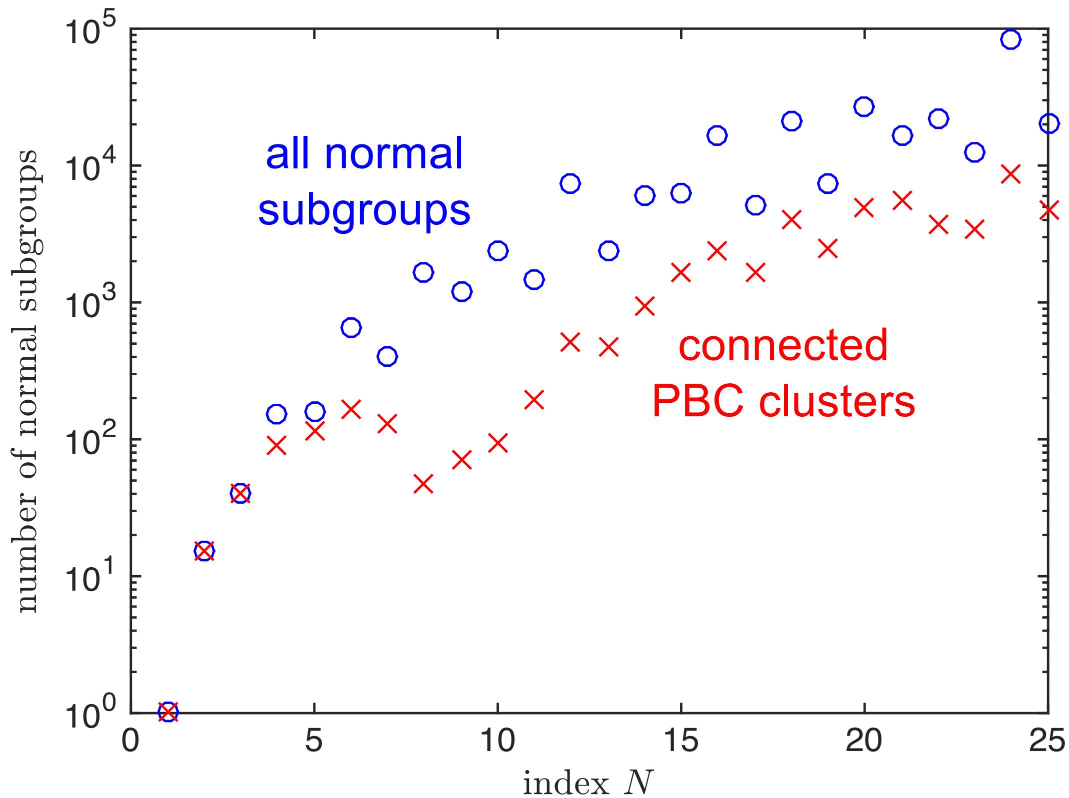
Assume $\psi(\gamma(z)) = \chi(\gamma)\psi(z)$:

$$
\begin{aligned}
\psi(\gamma\gamma_{\mathrm{PBC}}\gamma^{-1}(z)) &= \chi(\gamma\gamma_{\mathrm{PBC}}\gamma^{-1})\psi(z) \\
&= \chi(\gamma)\chi(\gamma_{\mathrm{PBC}})\chi^{-1}(\gamma)\psi(z) \\
&= \psi(z) \quad \text{since} \quad \chi(\gamma_{\mathrm{PBC}}) = 1
\end{aligned}
$$

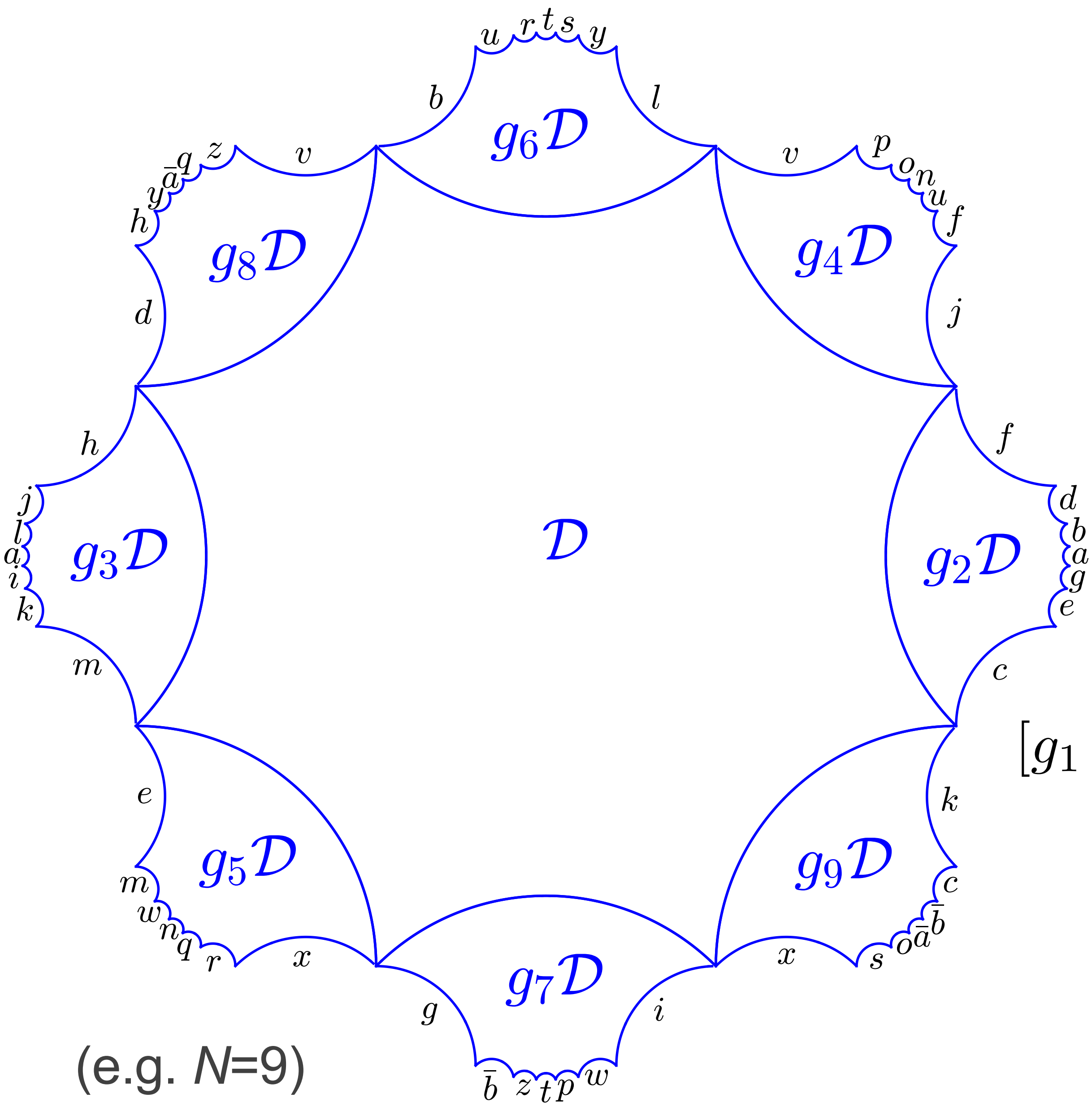
thus $\gamma\gamma_{\mathrm{PBC}}\gamma^{-1} \in \Gamma_{\mathrm{PBC}} \Longrightarrow \Gamma_{\mathrm{PBC}} \triangleleft \Gamma$

Low-index normal subgroups

- For a given *N*, many distinct subgroups $\Gamma_{\mathrm{PBC}} \triangleleft \Gamma$, although all isomorphic to $\pi_1(\Sigma_h)$
- Compute all normal subgroups of index up to *N* = 25 using Firth-Holt algorithm (GAP implementation by F. Rober)



Connected PBC clusters

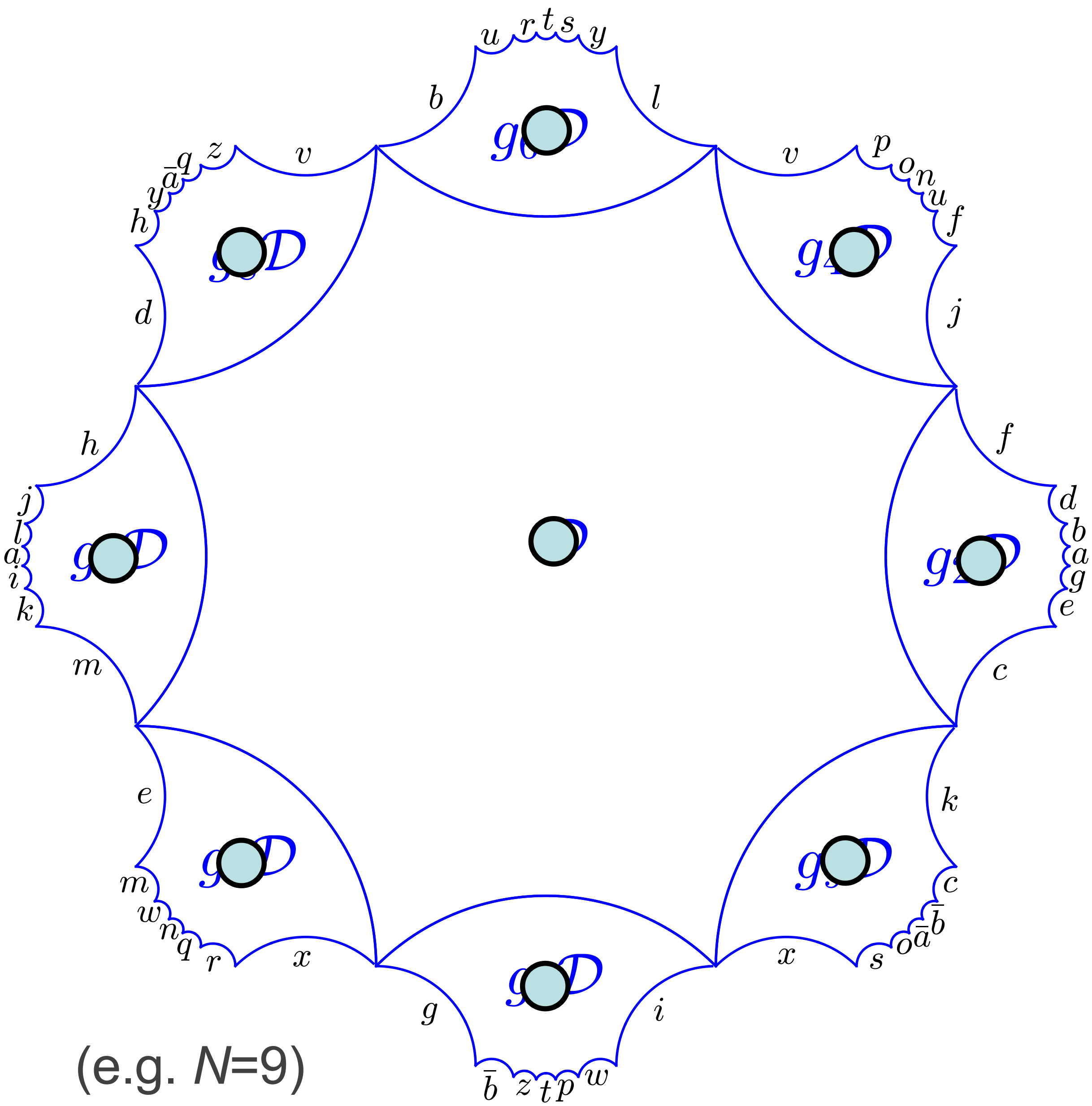


(e.g. *N*=9)

$$\mathcal{C} = \bigsqcup_{i=1}^{N} g_i \mathcal{D}$$

$$[g_1 = e], [g_2], \ldots, [g_N] \in \Gamma / \Gamma_{\mathrm{PBC}}$$

Hopping matrix

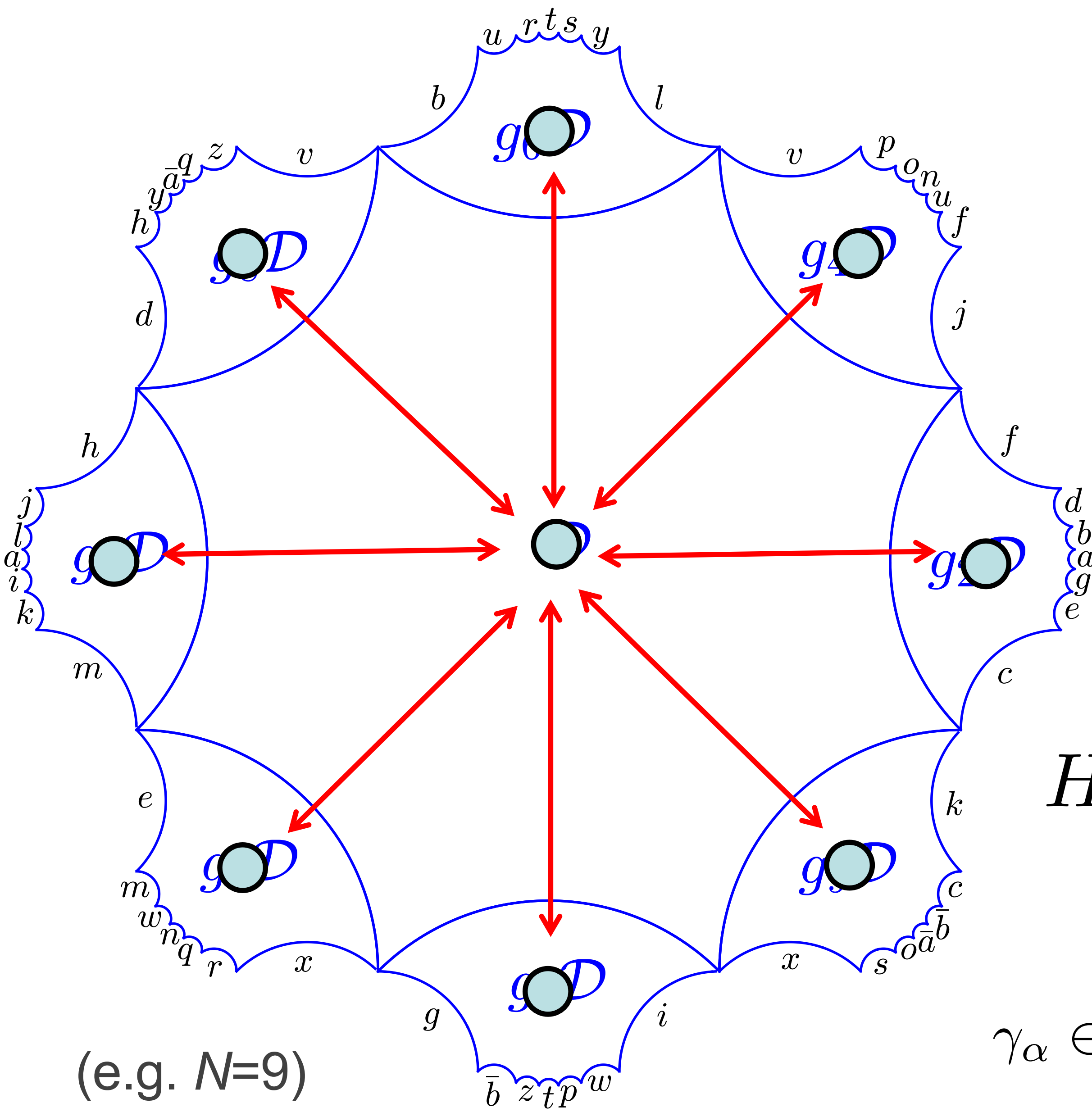


$$H_{ij} = -1$$

if i & j n.n., 0 otherwise

(e.g. N=9)

Hopping matrix



(e.g. N=9)

$$H_{ij} = -1$$

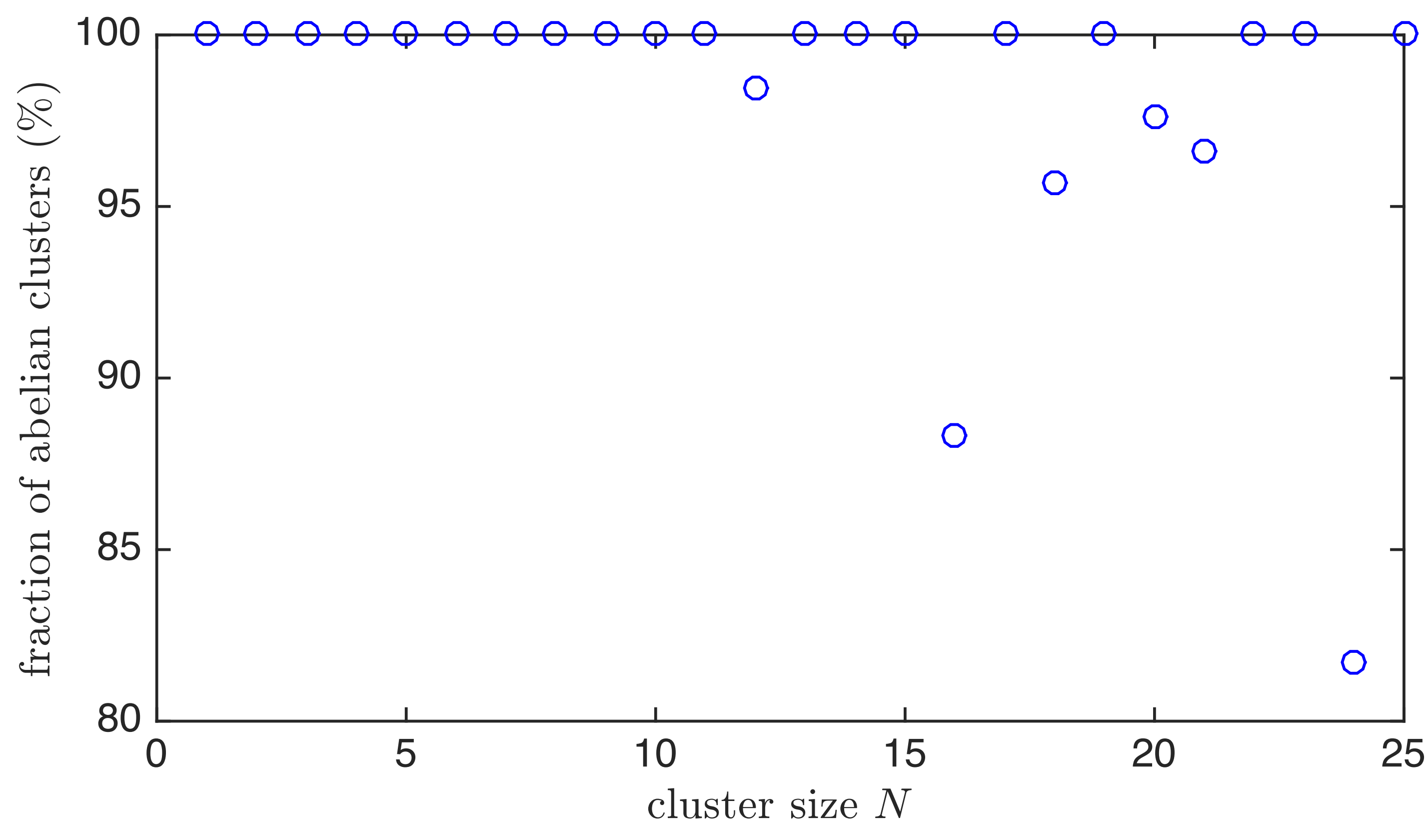
if i & j n.n., 0 otherwise

$$H_{ij} = -\sum_{\alpha} \delta_{[g_j],[g_i \gamma_\alpha]}$$

$$\gamma_\alpha \in \{\gamma_1, \ldots, \gamma_4, \gamma_1^{-1}, \ldots, \gamma_4^{-1}\}$$

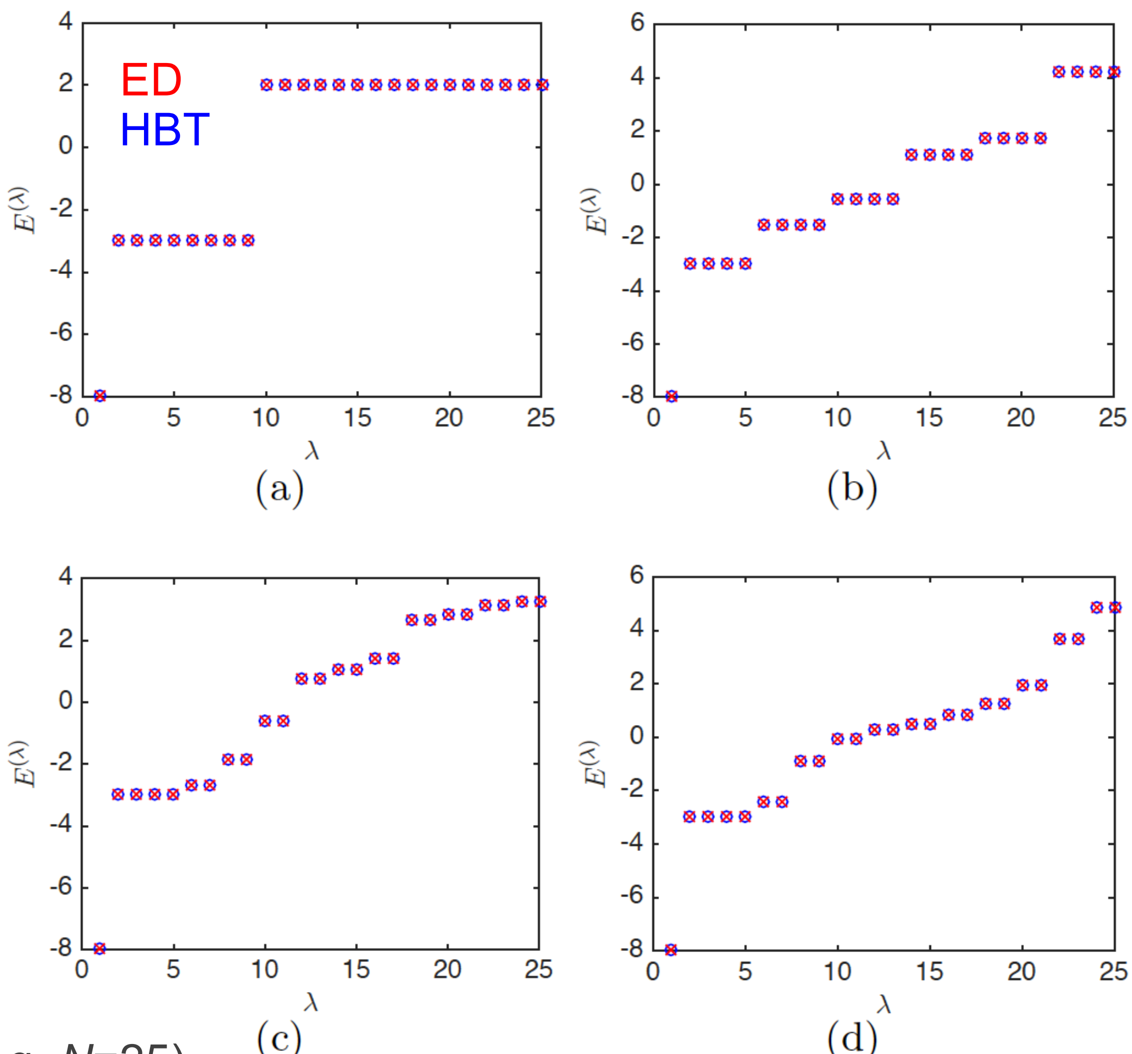
Abelian clusters

- Eigenstates of the clusters fall into irreps of the residual translation group $\Gamma/\Gamma_{\mathrm{PBC}}$ = finite group of order N
- For large fraction of clusters, this group is abelian!



Abelian Bloch theorem

$$\psi^{(\lambda)}(g_k^{-1}(z_i)) = \chi^{(\lambda)}([g_k])\psi^{(\lambda)}(z_i), \qquad [g_k] \in \Gamma/\Gamma_{\mathrm{PBC}}$$



$$E^{(\lambda)} = -2\sum_{j=1}^{4} \cos k_j^{(\lambda)},$$

$$\lambda = 1, \ldots, N$$

$$k_j^{(\lambda)} \in \frac{2\pi\mathbb{Z}}{N}$$

⇕

Jac(Σ_g)≅T^{2g} is discretized

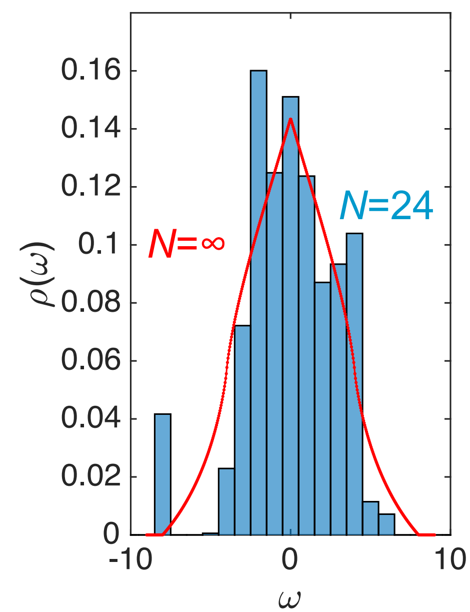
(e.g. N=25)

Maximal abelian cluster

- Finite abelian clusters are subsets of an *infinite* abelian cluster $Y_\infty = \mathbb{H}/\Gamma_{\text{PBC}}$ with $\Gamma_{\text{PBC}} = [\Gamma, \Gamma]$:

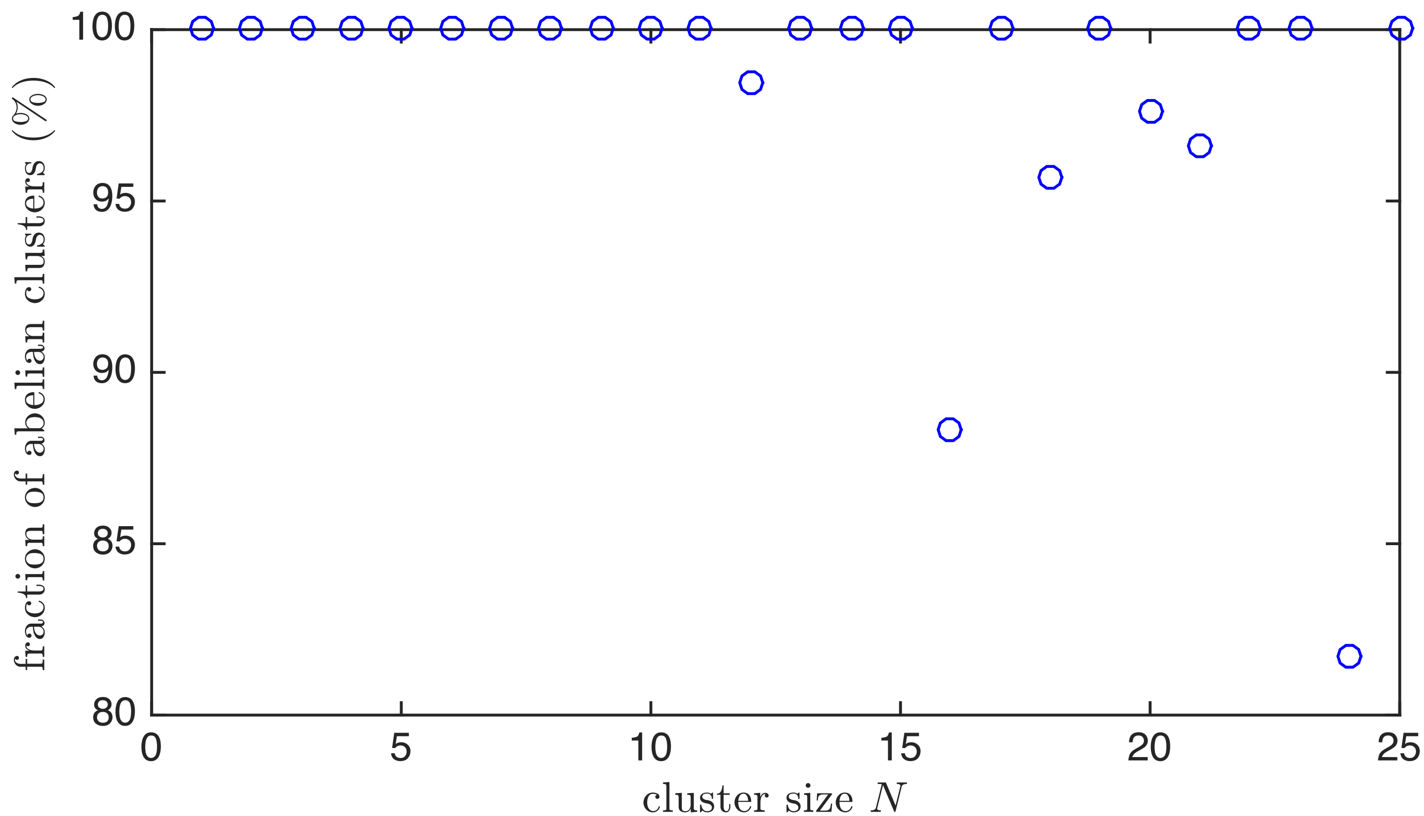
$$\Gamma/\Gamma_{\text{PBC}} \cong H_1(X, \mathbb{Z}) \cong \mathbb{Z}^{2g}$$

Euclidean lattice in 2*g* dimensions:
BZ = Jac(Σ_g) ≅ T^{2g}



Nonabelian clusters

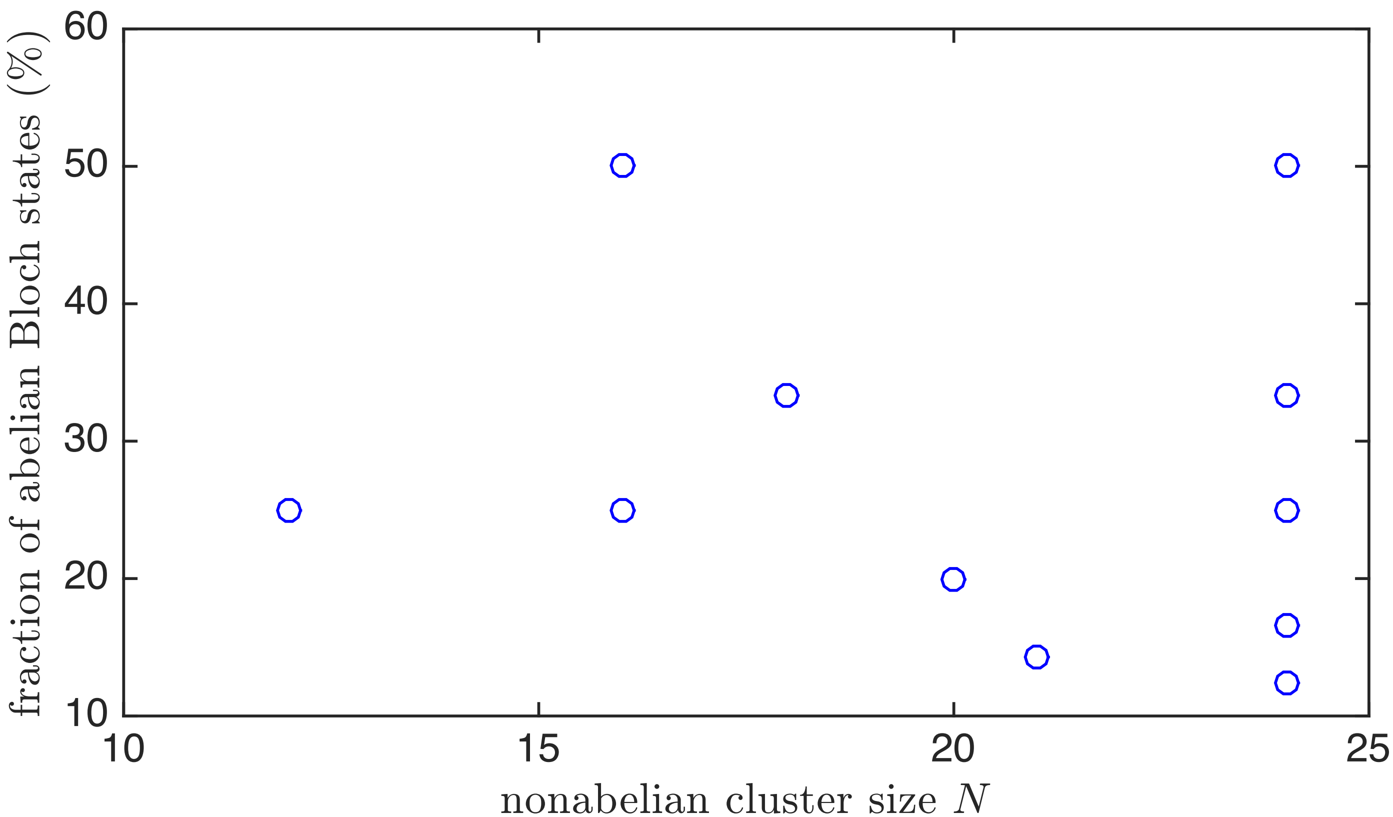
- For $N < 25$, nonabelian $\Gamma/\Gamma_{\text{PBC}}$ found only at N = 12,16,18,20,21,24



Nonabelian Bloch theorem

- Nonabelian $\Gamma/\Gamma_{\text{PBC}}$ possesses higher-dimensional unitary irreps:

$$\psi_\nu^{(\lambda)}(g_k^{-1}(z_i)) = \sum_{\mu=1}^{r_\lambda} \psi_\mu^{(\lambda)}(z_i) D_{\mu\nu}^{(\lambda)}([g_k]), \qquad [g_k] \in \Gamma/\Gamma_{\text{PBC}}$$



Nonabelian Bloch theorem

- Translation matrices belong to (reducible) regular representation: irrep of dimension r_λ appears r_λ times

- Example (N=24): 8 abelian irreps, 4 nonabelian (2D) irreps

C	1	2	3	4	5	6	7	8	9	10	11	12
n_C	1	1	1	1	2	2	2	2	3	3	3	3
$D^{(1)}$	1	1	1	1	1	1	1	1	1	1	1	1
$D^{(2)}$	1	1	1	1	1	1	1	1	-1	-1	-1	-1
$D^{(3)}$	1	-1	a	$-a$	1	-1	a	$-a$	c	$-c$	$-1/c$	$1/c$
$D^{(4)}$	1	-1	a	$-a$	1	-1	a	$-a$	$-c$	c	$1/c$	$-1/c$
$D^{(5)}$	1	-1	$-a$	a	1	-1	$-a$	a	$-1/c$	$1/c$	c	$-c$
$D^{(6)}$	1	-1	$-a$	a	1	-1	$-a$	a	$1/c$	$-1/c$	$-c$	c
$D^{(7)}$	1	1	-1	-1	1	1	-1	-1	a	a	$-a$	$-a$
$D^{(8)}$	1	1	-1	-1	1	1	-1	-1	$-a$	$-a$	a	a
$D^{(9)}$	2	2	-2	-2	-1	-1	1	1	0	0	0	0
$D^{(10)}$	2	2	2	2	-1	-1	-1	-1	0	0	0	0
$D^{(11)}$	2	-2	b	$-b$	-1	1	$-a$	a	0	0	0	0
$D^{(12)}$	2	-2	$-b$	b	-1	1	a	$-a$	0	0	0	0

Nonabelian Bloch theorem

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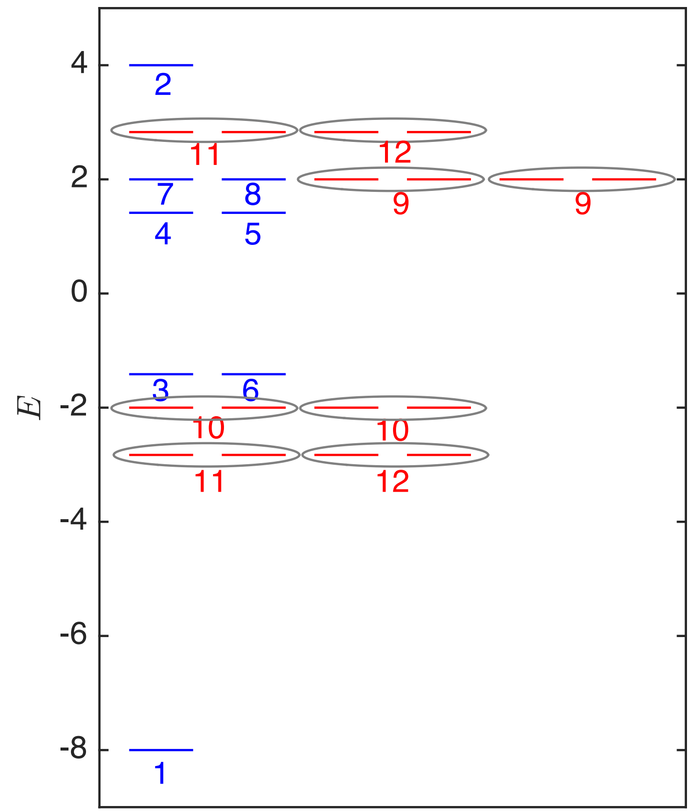
C	1	2	3	4	5	6	7	8	9	10	11	12
n_C	1	1	1	1	2	2	2	2	3	3	3	3
$D^{(1)}$	1	1	1	1	1	1	1	1	1	1	1	1
$D^{(2)}$	1	1	1	1	1	1	1	1	-1	-1	-1	-1
$D^{(3)}$	1	-1	a	$-a$	1	-1	a	$-a$	c	$-c$	$-1/c$	$1/c$
$D^{(4)}$	1	-1	a	$-a$	1	-1	a	$-a$	$-c$	c	$1/c$	$-1/c$
$D^{(5)}$	1	-1	$-a$	a	1	-1	$-a$	a	$-1/c$	$1/c$	c	$-c$
$D^{(6)}$	1	-1	$-a$	a	1	-1	$-a$	a	$1/c$	$-1/c$	$-c$	c
$D^{(7)}$	1	1	-1	-1	1	1	-1	-1	a	a	$-a$	$-a$
$D^{(8)}$	1	1	-1	-1	1	1	-1	-1	$-a$	$-a$	a	a
$D^{(9)}$	2	2	-2	-2	-1	-1	1	1	0	0	0	0
$D^{(10)}$	2	2	2	2	-1	-1	-1	-1	0	0	0	0
$D^{(11)}$	2	-2	b	$-b$	-1	1	$-a$	a	0	0	0	0
$D^{(12)}$	2	-2	$-b$	b	-1	1	a	$-a$	0	0	0	0



Nonabelian Bloch theorem

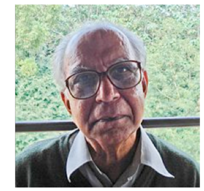
- Translation matrices belong to (reducible) regular representation: irrep of dimension r_λ appears r_λ times
- Choice of Γ_{PBC} selects U(r) irreps of Γ whose kernel includes Γ_{PBC}
- For fixed r, what is the space of all U(r) irreps of Γ?

Nonabelian Hodge theory

$$\mathrm{Irrep}(\Gamma, U(r))/U(r)$$

(topological)

Riemann-Hilbert




Narasimhan-Seshadri
(1965)

$$\frac{\text{flat } U(r) \text{ connections on } \Sigma_g}{\text{gauge group}}$$

(smooth)

Atiyah-Bott, Witten…

$$\mathcal{M}(\Sigma_g, U(r))$$

moduli space of stable holomorphic vector bundles of rank *r*

(holomorphic)

Nonabelian Hodge theory

$$\mathrm{Irrep}(\Gamma, U(r))/U(r)$$

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Atiyah-Bott, Witten…

$$\mathcal{M}(\Sigma_g, U(r))$$

moduli space of stable holomorphic vector bundles of rank *r*

(holomorphic)

- Cmplx manifold: $\dim_{\mathbb{C}} \mathcal{M}(\Sigma_g, U(r)) = r^2(g-1)+1$
- For $g = 2$, $r = 2$, $\mathcal{M} = \text{bundle of } \mathbb{CP}^3 \text{ fibers over } \mathrm{Jac}(\Sigma_2)$

Summary

- {4g,4g} hyperbolic lattices have a (nonabelian) discrete “translation” group $\Gamma \cong \pi_1(\Sigma_g)$, the Fuchsian group
- Finite (but arbitrarily large) clusters with PBC correspond to $\Gamma_{\mathrm{PBC}} \triangleleft \Gamma$
- Hyperbolic “Brillouin zones”: $\mathrm{Jac}(\Sigma_g) \cong T^{2g} \cong \mathcal{M}(\Sigma_g, U(1))$
  $\mathcal{M}(\Sigma_g, U(r)),\ r > 1$
- Automorphic Bloch theorems (abelian/nonabelian)
- Point-group symmetries: $\mathrm{Aut}(\Sigma_g)$
- Implement PBC clusters in experiment (e.g. topolectrical circuits)?

JM and S. Rayan, Sci. Adv. 7, eabe9170 (2021); arXiv:2108.09314

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