Infrared phases of $2d$ QCD.

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Based on [2108.02202] with J. Gomis and M. Yu.

Background.

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- Key question in theoretical physics: understanding the dynamics of strongly coupled QFTs. Perturbation theory is inadequate, hard to say anything at all.
- Example: the strong force is asymptotically free.
  - at short distances interactions are weak and Feynman diagrams capture all the physics.
  - at long distances the theory is strongly coupled and this is where the interesting questions arise.
    - phenomenologically: virtually all experiments are low energy from the point of view of the strong force.
    - theoretically: long standing questions (χSB, confinement, etc.) are about the vacuum.
- Conclusion: what we really want to understand is the macroscopic limit of a QCD theory.

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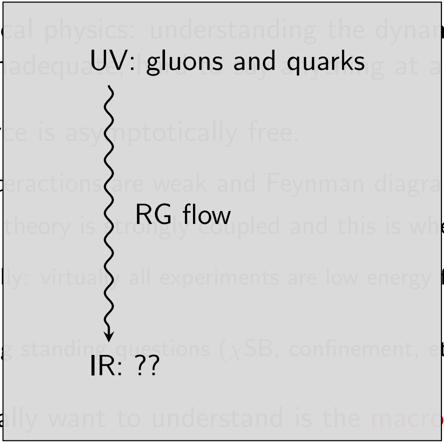

Background.

- Two things can happen. The infrared theory is gapless, or it is gapped.
- Gapless means that there are massless degrees of freedom at low energies. Gapped means everything is massive.
- The macroscopic behaviour of the theory crucially depends on this. If gapped, “trivial” dynamics at energies below the gap. If gapless, interesting phenomena at any energy. Example: electromagnetism is a long-distance interaction because the photon is massless.
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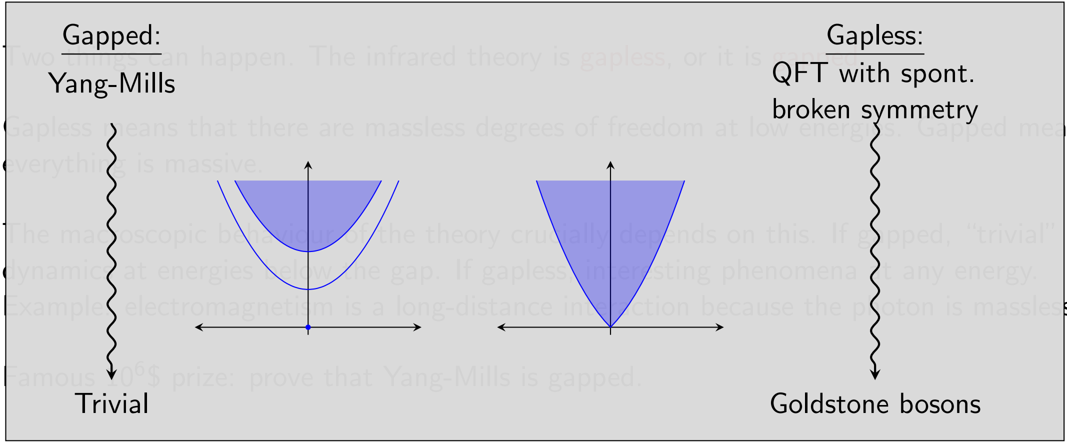

Background.

- A gapped QFT need not be completely trivial in the infrared. There are no propagating degrees of freedom but there might be multiple vacua with topological degrees of freedom. Generically, a gapped theory becomes a topological quantum field theory (TQFT) at long distances.

- A gapless QFT need not be just Goldstone particles. There might be massless degrees of freedom that do not come from broken symmetries. Generically, a gapless theory becomes a conformal field theory (CFT) at long distances.

- For a given QCD theory, we wish to know whether the theory is gapped or gapless. If gapped, we want to know which TQFT governs its macroscopic limit. If gapless, we need to figure out the corresponding CFT.

Background.

- In $3+1d$ there are several systems where the answer to these questions is at least somewhat understood.
  - $\mathcal{N}=0$: large-N, lattice, etc.
  - $\mathcal{N}=1$: Seiberg duality.
  - $\mathcal{N}=2$: Seiberg-Witten.
- In $2+1d$ also several examples (with and without SUSY).
- In $1+1d$: today.

$2d$ QCD.

$2d$ QCD.

- Lagrangian of $1+1d$ QCD:

$$\mathcal{L} = g^{-2}\,\mathrm{tr}\,F^2 + \bar{\psi} D \psi, \qquad D_\mu := \partial_\mu + iA^a_\mu t^a_R$$

  - g is the coupling constant (dimensions of mass).
  - $F = \mathrm{d}A + [A, A]$ is the field strength of a gauge group G.
  - ψ is a spinor transforming according to representation R generated by matrices t_R.

- Variations and extensions. Mass terms $m\bar{\psi}\psi$, scalar fields ϕ, chiral quarks (R_ℓ, R_r), etc.

2d QCD.

- Our main question: given QCD with (G, R), is the theory gapped? And what are the low-energy effective degrees of freedom?

- Before we tackle the general case let me give you two examples to illustrate what the final answer looks like:

$$\mathit{Spin}(7) + \mathbf{7} \quad \rightsquigarrow \quad \text{Gapped, Two vacuum states}$$

$$\mathit{Spin}(7) + \mathbf{8} \quad \rightsquigarrow \quad \text{Gapless, Ising model}$$

2d QCD.

- How do we determine the macroscopic dynamics for general (G, R)?
- Recall that g has positive mass dimension, so the gluon kinetic term is classically irrelevant. The key assumption is that this is also true quantum-mechanically. The intuition is that 2d gluons do not propagate so they do not affect the low-energy dynamics.

  [Kutasov, Schwimmer, Komargodski, Ohmori, Roumpedakis, Seifnashri, Cherman, Jacobson, Tanizaki, Ünsal...]
- Refinement of this idea: the Hamiltonian of QCD (after gauge-fixing and with appropriate ordering prescription) contains a mass term for the gluons.
- Conclusion: the low-energy dynamics is entirely captured by the fermionic term

$$\mathcal{L}_{\text{eff}} = \bar{\psi} D \psi$$

$2d$ QCD.

- This is useful because $\mathcal{L}_{\text{eff}} = \bar{\psi} D \psi$ turns out to be a rational CFT, so we have powerful techniques at our disposal.

- What is the chiral algebra of $\mathcal{L}_{\text{eff}}$? First, we write the free fermion term as a Wess-Zumino-Witten model:

$$\bar{\psi} \partial \psi \equiv SO(\dim R)_1$$

[Witten]

- Next, we gauge the symmetry $G \subset SO(\dim R)$. This yields a gauged WZW model:

$$\mathcal{L}_{\text{eff}} \equiv \frac{SO(\dim R)_1}{G_{I(R)}}$$

The quotient denotes the gauging by G, whose level $I(R)$ is given by the Dynkin embedding index of $G \subset SO(\dim R)$.

2d QCD.

- We can use the effective description

$$\mathcal{L}_{\text{eff}} \equiv \frac{SO(\dim R)_1}{G_{I(R)}}$$

to determine whether QCD is gapped or not.

- A unitary CFT is trivial if and only if its energy-momentum tensor is zero. Hence, QCD is gapped if and only if $T_{SO/G} \equiv 0$.

- Vanishing EM tensor defines a conformal embedding of CFTs. The set of gapped QCD theories is in bijection with conformal embeddings of the form $G_{I(R)} \subseteq SO(\dim R)_1$.

2*d* QCD.

- Luckily for us, conformal embeddings of WZW models have been fully classified. [Goddard, Kent, Olive, Nahm, Schellekens, Warner, Bais, Bouwknegt, Arcuri, Gomes...]
- By looking at the classification of conformal embeddings we get the list of gapped QCD theories. These are:
  - Any G and adjoint fermions $R = \mathrm{adj}$.
  - $G = S(U(N) \times U(M))$, $G = SO(N) \times SO(M)$, $G = Sp(N) \times Sp(M)$, and bifundamental fermions $R = (\square, \square)$.
  - $G = U(N)$, $G = SO(N)$, $G = Sp(N)$, and rank-2 fermions $R = \square\square, \begin{smallmatrix}\square\\\square\end{smallmatrix}$.
  - Some isolated theories (e.g., $G = Spin(9)$ and fermions in the spinor representation).
  - Combinations of the above.
- Any theory not in this list is gapless.

2d QCD.

- Important remark: the low-energy effective description of QCD as a gauged WZW model was derived under the assumption of the gluon kinetic term being irrelevant. While quite reasonable (and provable in some cases), it may turn out to be false.

- That being said, one can still use the gauged WZW model to determine some aspects of QCD, even if the assumption is wrong:
  - For example, the classification of gapped theories is still correct. The idea is that $T_{SO/G}$ can be shown to create massless states in the full QCD theory, so the spectrum can be gapped only if $T_{SO/G} \equiv 0$, whose solutions are classified as above.
  - The symmetries and anomalies (both perturbative and non-perturbative) of QCD are also fully captured by the gauged WZW model.
  - Even the non-invertible symmetries (see Ohmori's talk) of QCD are captured by the gauged WZW model.

2d QCD.

QCD with (G, R)

$$\rightsquigarrow$$

Gauged WZW

$$\frac{SO(\dim R)_1}{G_{I(R)}}$$

QCD is gapped

$$\Longleftrightarrow$$

Gauged WZW model is gapped

$$\Longleftrightarrow$$

$$T_{SO/G} \equiv 0$$

$$\Longleftrightarrow$$

$G_{I(R)}$ conformally embeds into $SO(\dim R)_1$

$$\Longleftrightarrow$$

$(G, R) \in$ list above

Infrared dynamics.

Infrared dynamics.

- We have determined that the macroscopic degrees of freedom of QCD with content (G, R) are captured by the gauged WZW model

$$\mathcal{L}_{\text{eff}} = \frac{SO(\dim R)_1}{G_{I(R)}}$$

- This effective theory describes the infrared CFT for gapless theories and the infrared TQFT for gapped ones. In other words, it contains the information about the massless particles, if any, together with their interactions; and also the topological degrees of freedom (multiple vacua), if any.

- In order to extract these degrees of freedom more explicitly we need to understand gauged WZW models in more detail.

Infrared dynamics.

- There exists a very explicit algebraic description of gauged WZW models that goes under the name of the Goddard-Kent-Olive coset construction. It deals with rational CFTs presented as the quotient of two other rational CFTs:

$$\frac{A}{B}$$

- The chiral degrees of freedom of the quotient of A/B are the so-called branching functions $\chi_a^b(q)$, defined as the coefficients of the expansion

$$\chi_a(q) = \sum_b \chi_a^b(q)\chi_b(q)$$

[Goddard, Kent, Olive, Schnitzer, Karabali, Park, Yang, Gawedzki, Kupiainen...]

where χ_a, χ_b are the chiral characters of A and B, respectively, and q is the modular parameter.

Infrared dynamics.

- If the theory is gapped, χ_a^b is an integer (q-independent) and it counts the number of vacua.

- If the theory is gapless, χ_a^b depends on q and the the coefficient a_n in the expansion

$$\chi_a^b(q) \sim a_0 + a_1 q + a_2 q^2 + a_3 q^3 + \cdots$$

  counts the number of states at "energy" $L_0 = n$.

- One can derive the full modular data of the coset, together with other properties of interest, such as scaling dimensions, OPEs, quantum numbers under the various symmetries, etc.

- For a general QCD theory (G, R), the macroscopic degrees of freedom are determined by the branching functions χ_a^b of the embedding of $G_{I(R)}$ into $SO(\dim R)_1$.

Examples.

Examples.

- A model that has received a lot of attention recently is adjoint QCD, i.e., with a quark in the adjoint representation $R = \mathrm{adj}$. Very close to being SUSY (but is not).
- This is in the list above, so gapped (known since the 90s). [Kutasov, Klebanov, Dalley...]
- Infrared described by the coset

$$\frac{SO(\dim G)_1}{G_h}$$

- The coset describes a TQFT. The branching rules of the coset were studied in the 80s. For example, there are $2^{\mathrm{rank}(G)}$ discrete vacua. [Kac, Wakimoto...]
- One can extract lots of interesting physics from this coset (see Ohmori's talk and their paper [Komargodski, Ohmori, Roumpedakis, Seifnashri]).

Examples.

- Take $SU(2)$ gauge theory. The $R = \mathbf{3}, \mathbf{5}$ representations are in the list, so gapped.
- $SU(2)$ plus a quark in the $\mathbf{7}$ is not in the list, hence the theory is gapless.
- These massless particles are described by the coset

$$\frac{SO(7)_1}{SU(2)_{28}}$$

- It turns out that this coset is equivalent to the fermionic tricritical Ising model ($c = 7/10$ in the minimal series). The branching functions $\chi_a^b(q)$ are Virasoro (super)characters.
- Infrared operators:

$$\text{bosons: } \phi_{(0,0)},\ \phi_{(1/10,1/10)},\ \phi_{(3/5,3/5)},\ \phi_{(3/2,3/2)}$$

$$\text{fermions: } \phi_{(0,3/2)},\ \phi_{(3/2,0)},\ \phi_{(1/10,3/5)},\ \phi_{(3/5,1/10)}$$

whose properties are well-understood (scaling dimensions, OPEs, etc.)

Examples.

- 't Hooft model, i.e., G plus N_F quarks in the fundamental representation.
- Generically, not in the list. Exception: $G = SO(N)$ or $G = U(N)$, and $N_F = 1$. The rest are all gapless.
- The infrared degrees of freedom are described by the coset

$$\frac{SO(\nu N_F N)_1}{G_{N_F}}, \qquad \nu = \begin{cases} 1 & \text{orthogonal} \\ 2 & \text{unitary} \\ 4 & \text{symplectic} \end{cases}$$

- By level-rank duality, this coset can be written as an ungauged WZW model:

$$\frac{SO(\nu N_F N)_1}{G_{N_F}} \equiv H_N, \qquad H = \begin{cases} SO(N_F) & \text{orthogonal} \\ U(N_F) & \text{unitary} \\ Sp(N_F) & \text{symplectic} \end{cases}$$

- Note that H is the flavor symmetry of the gauge theory. The infrared CFT H_N is in fact the minimal CFT that matches the 't Hooft anomalies for H.

Examples.

$G + \mathrm{adj}$

Gapped
$2^{\mathrm{rank}(G)}$ vacua

$SU(2) + \mathbf{7}$

Gapless
Tricritical Ising CFT

$G + N_F \,\square$

Gapless
H_N WZW model
for flavor symmetry

Summary.

Summary.

- Our goal was to understand the low-energy dynamics of QCD in $1+1d$, where the theory is strongly coupled and macroscopic phenomena emerge.
- We gave a classification of the gauge groups and quark content that result in a gapped theory. Any other theory not in the list is necessarily gapless.
- We identified the low-energy degrees of freedom of an arbitrary theory, either as a CFT for gapless theories or a TQFT for gapped ones.
- This low-energy effective theory was given in the form of a gauged WZW model SO/G, which is a rational CFT.
- Occasionally, this coset SO/G is equivalent to other rational CFTs that are more common, such as minimal models or ungauged Wess-Zumino-Witten models.
- Dynamical properties about QCD can be extracted from this rational CFT (such as critical exponents), and vice versa.

Fin.

Extra slides.

Full classification of gapped theories.

$\mathfrak{g}$	R
$\forall\mathfrak{g}$	adjoint
$\mathfrak{so}(N)$	$\square$
$\mathfrak{u}(N)$	$\square$
$\mathfrak{so}(N)$	$\square\!\square$
$\mathfrak{sp}(N)$	$\begin{array}{c}\square\\[-4pt]\square\end{array}$
$\mathfrak{u}(N)$	$\begin{array}{c}\square\\[-4pt]\square\end{array}$
$\mathfrak{u}(N)$	$\square\!\square$
$\mathfrak{su}(M)+\mathfrak{su}(N)+\mathfrak{u}(1)$	$(\square,\square)$
$\mathfrak{so}(M)+\mathfrak{so}(N)$	$(\square,\square)$
$\mathfrak{sp}(M)+\mathfrak{sp}(N)$	$(\square,\square)$

$\mathfrak{g}$	R
$\mathfrak{su}(2)$	$\mathbf{5}$
$\mathfrak{so}(9)$	$\mathbf{16}$
F_4	$\mathbf{26}$
$\mathfrak{sp}(4)$	$\mathbf{42}$
$\mathfrak{su}(8)$	$\mathbf{70}$
$\mathfrak{so}(16)$	$\mathbf{128}$
$\mathfrak{so}(10)+\mathfrak{u}(1)$	$\mathbf{16}$
$E_6+\mathfrak{u}(1)$	$\mathbf{27}$
$\mathfrak{su}(2)+\mathfrak{su}(2)$	$(\mathbf{2},\mathbf{4})$
$\mathfrak{su}(2)+\mathfrak{sp}(3)$	$(\mathbf{2},\mathbf{14})$
$\mathfrak{su}(2)+\mathfrak{su}(6)$	$(\mathbf{2},\mathbf{20})$
$\mathfrak{su}(2)+\mathfrak{so}(12)$	$(\mathbf{2},\mathbf{32})$
$\mathfrak{su}(2)+E_7$	$(\mathbf{2},\mathbf{56})$

Symmetries and anomalies in $2d$.

- Weyl and Majorana conditions are compatible. Minimal fermion: chiral and real ψ_ℓ, ψ_r.
- Continuous flavor symmetry: if we have N_F flavors, $\mathfrak{h} = \mathfrak{h}_\ell \oplus \mathfrak{h}_r$, where
  - $\psi_{\ell/r}$ is real, $\mathfrak{h}_{\ell/r} = \mathfrak{so}(N_F)$,
  - $\psi_{\ell/r}$ is pseudo-real, $\mathfrak{h}_{\ell/r} = \mathfrak{sp}(N_F)$,
  - $\psi_{\ell/r}$ is complex, either $\mathfrak{h}_{\ell/r} = \mathfrak{u}(N_F)$ or $\mathfrak{su}(N_F)$ (depending on ABJ flavor-gauge anomaly).
- 't Hooft anomaly for currents (either flavor $\mathfrak{h}$ or gauge $\mathfrak{g}$):

$$\partial \cdot \langle j^a j^b \rangle \propto \mathrm{tr}(t_R^a t_R^b) \equiv I(R)\delta^{ab}$$

- Recall: in $3+1d$ the anomaly is

$$\partial \cdot \langle j^a j^b j^c \rangle \propto \mathrm{tr}(t_R^a \{t_R^b, t_R^c\}) \equiv A(R) d^{abc}$$

  The anomaly is non-zero if and only if $\mathfrak{h} = \mathfrak{su}(N_F)$ and R is complex. Also $A(R) = -A(\bar{R})$. By contrast, in $1+1d$, $I(R) \neq 0$ for any group and any R, and $I(R) = I(\bar{R})$.

Symmetries and anomalies in $2d$.

- In $1+1d$, continuous symmetries cannot break spontaneously. There are no Goldstone bosons.
- For any d, a continuous symmetry that has an 't Hooft anomaly implies that there are massless particles in the spectrum. In $d=2$, any chiral symmetry is anomalous (cf. $I(R)\neq 0$). Hence, any $2d$ QFT with a chiral symmetry is gapless.
- If $\mathfrak{h}$ is a chiral symmetry, there are no Goldstone bosons but there are Wess-Zumino-Witten currents. Idea: $\mathfrak{h}$ cannot break, hence it is present in the infrared. But the infrared is conformal, and unitary CFTs with flavor symmetry $\mathfrak{h}$ always contain an $\mathfrak{h}_k$ subsector for some level k.
- Conclusion: in $1+1d$, a continuous chiral symmetry $\mathfrak{h}$ automatically implies gaplessness, and the massless particles include (among others) an $\mathfrak{h}_k$ CFT.
- Example: 't Hooft model with N_F fundamental quarks flows in the infrared to H_N WZW CFT, plus perhaps a flavorless CFT, where $H=U(N_F), SO(N_F), Sp(N_F)$.