Some Galois Actions in Topological Quantum Field Theory

GGI Workshop on Topological Properties of Gauge Theories

Matthew Buican
Queen Mary University of London

Based on:
M.B. and R. Radhakrishnan,
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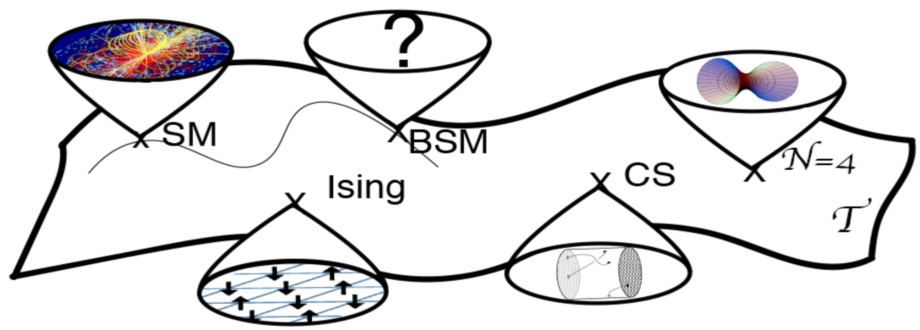
Introduction and Motivation

• **Broad goal:** Use $2+1d$ TQFT* as a simple toy model / stepping stone to understand global structure of space of QFTs, $\mathcal{T}$.



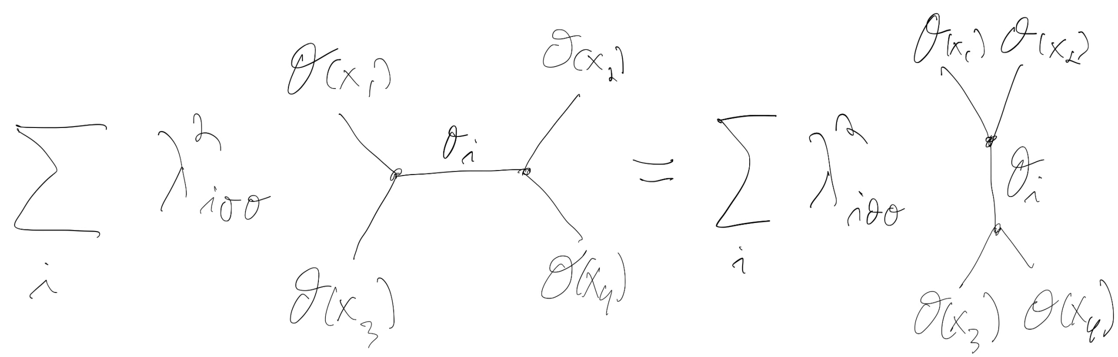
• Natural questions: e.g., which theories arise from Lagrangians? Which groups act naturally on $\mathcal{T}$?

• All these questions can be asked in the simpler arena of TQFT.

*For the rest of this talk, unless otherwise specified, TQFT := 2+1d TQFT.

Introduction and Motivation (cont...)

• A related but harder problem in the same spirit is to understand the space of CFTs via the bootstrap.



• These types of diagrams have complicated solutions in CFT.

• Above diagram reappears in TQFT as a solution to a finite system of polynomial equations $\Rightarrow$ simpler problem with natural Galois group action.

Introduction and Motivation (cont...)

• As we will see, this picture leads to a natural partitioning of the space of TQFTs, $\mathcal{T}_{\text{top}}$, into orbits of the Galois group.

• Since the orbits are discrete, there is no immediately natural notion of a "small" deformation; e.g., unitary theories sometimes go to non-unitary ones and vice versa.

• At the same time, this Galois group action has certain nice physical properties: it preserves 1-form symmetries, it preserves certain averaged link invariants / measures of topological entanglement entropy [M.B. and Radhakrishnan (2019)]; useful in classification of TQFTs [Rowel, Strong et. al] and study of gapped boundaries [Kaidi, Komargodski et. al.]

Partial Summary of our Claims

• We found many results that indicate the Galois action is, in some sense, well controlled (too many to discuss at length; I'll discuss some aspects of those with a "$*$"):

• **On symmetries:** unitary 0-form symmetries and 2-groups are preserved under the Galois action (i.e., these structures are isomorphic before and after the Galois action); up to a mild assumption, the same is true for anti-unitary 0-form symmetries and 2-groups. Allows for a refinement of symmetries associated with Galois orbits and an often easier way to understand if two theories are part of the same orbit.

Partial Summary of our Claims (cont...)

• **0-form gauging:** 0-form symmetry gauging interacts with Galois conjugation in a controlled way:

$$(q(C_G))^G = q'(C^G) \ . \tag{1}$$

• **Drinfeld center (for the experts):** Roughly, Galois conjugation "commutes" with taking the Drinfeld center:

$$\mathcal{Z}(q(C)) = q'(\mathcal{Z}(C)) \ . \tag{2}$$

• **Global structure of TQFTs*:** Product structure / primality of TQFTs preserved under Galois

Partial Summary of our Claims (cont...)

- **Subspaces of TQFTs*:** Space of discrete gauge theories closed under Galois conjugation + related statements.

- **On dualities*:** For broad classes of theories (e.g., discrete gauge theories), duality structure is preserved under Galois conjugation.

- **1-form gauging and Galois invariance*:** Galois invariance is preserved by 1-form symmetry gauging (and more general Rep(G) condensation). The story for 0-form gauging is more complicated (interesting generalization of 't Hooft anomalies).

- **Simplicity of unitary Galois invariant TQFTs*:** Theories with nteger quantum dimensions, can be constrained and (up to a conjecture) classified; Much less wild than non-unitary theories.

2+1d TQFT and MTC Review

• TQFTs on some spacetime manifold, $\mathcal{M}$, are QFTs that (up to minor subtleties) don't depend on the metric on $\mathcal{M}$. We will focus on theories that also don't depend on a spin structure.

• Will be useful to have a more algebraic (and non-Lagrangian) approach to TQFT. Reasons: **(1)** Lagrangian descriptions contain redundancies (e.g., Toric code TQFT $=$ $\mathbb{Z}_2$ discrete gauge theory $=$ Spin$(16)_1$ CS theory $=$ $U(1)\times U(1)$ CS theory) **(2)** often easier to prove general statements.

• But will also want to keep in mind Chern-Simons theory:

$$S=\frac{k}{4\pi}\int_{\mathcal{M}}\mathrm{Tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)\ , \tag{3}$$

2+1d TQFT and MTC Review (cont...)

- Basic observables of CS theory are Wilson lines:

$$W_R(\mathcal{C}) = \mathrm{Tr}_R P \exp\left(\int_C A\right) \ . \tag{4}$$

Correlators compute various topological invariants of $\mathcal{M}$.

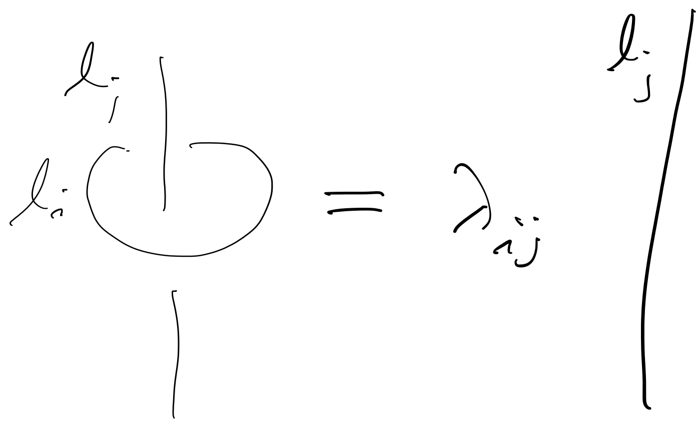
- Wilson lines topological $\Rightarrow$

$$W_{R_i} \times W_{R_j} = \sum_k N_{ij}^k W_{R_k} \ , \quad N_{ij}^k \in \mathbb{Z}_{\geq 0} \ , \tag{5}$$

For given k, there are a finite number of R_i.

2+1d TQFT and MTC Review (cont...)

• These lines also have interesting (non-degenerate) anyonic braiding



• This basic structure is encapsulated in an algebraic object called a modular tensor category (MTC).

2+1d TQFT and MTC Review (cont...)

• Start from finite abstract collection of objects / anyons (generalizations of Wilson lines), $\{\ell_i\}$, w/ comm. / assoc. fusion

$$\ell_i \times \ell_j = \sum_k N_{ij}^k \ell_k \ , \qquad N_{ij}^k \in \mathbb{Z}_{\geq 0} \ . \tag{6}$$

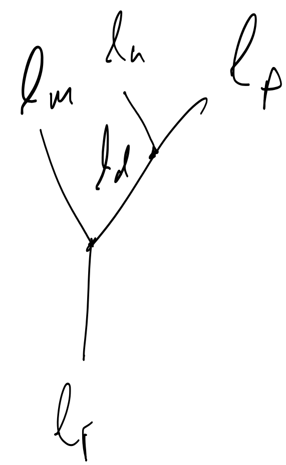
• The $N_{ij}^k = \dim(V_{ij}^k)$ are dimensions of certain Hilbert spaces called "fusion spaces."

• If $\forall \ell_{i,j}$, $\sum_k N_{ij}^k = 1$, the theory is Abelian with fusion rules those of a finite Abelian group. If not, the theory is non-Abelian. Abelian part gives the 1-form symmetry.

2+1d TQFT and MTC Review (cont...)

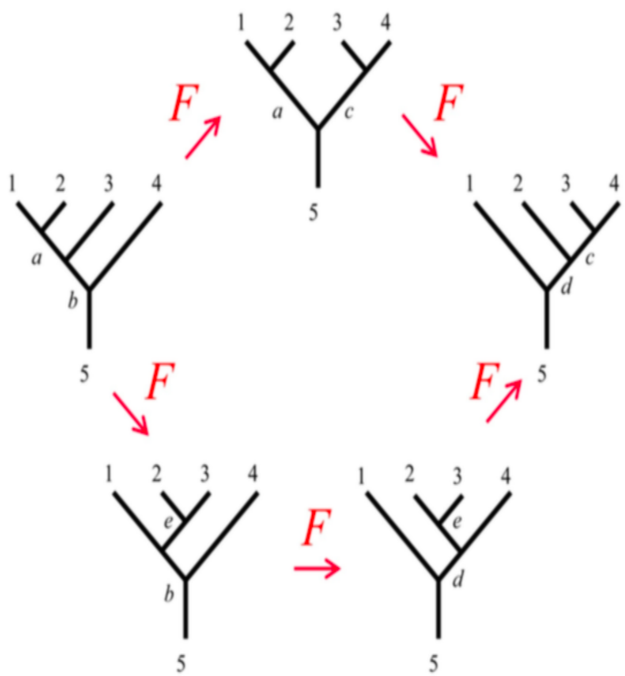
- Fusions should be associative, so

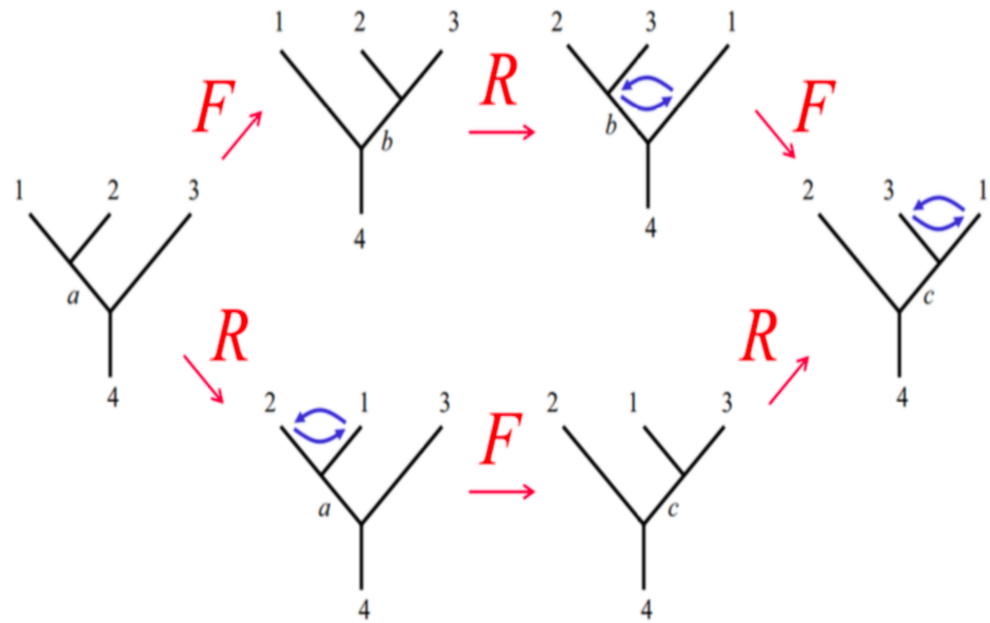
$$= \sum_d \left(F^{mnp}_r\right)^d_c$$



2+1d TQFT and MTC Review (cont...)

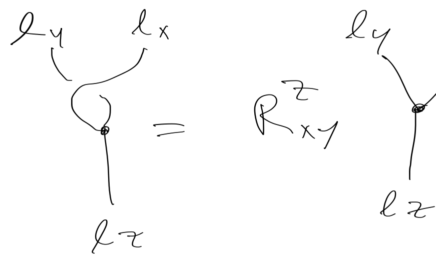
• Above is a TQFT version of CFT “bootstrap” diagram and satisfies:





2+1d TQFT and MTC Review (cont...)

• Where R is depicted as



Can also be used to define self-statistics/twists, θ_{ℓ_i}, of the anyons.

• Together F and R characterize an MTC. However, they have an ambiguity arising from choosing a basis for fusion spaces.

• After quotienting out by this ambiguity, the solutions become discrete **[Ocneanu; Etingof et. al.]**, and MTCs are rigid.

Galois Conjugation

• Galois groups arise in the context of certain special field extensions, E/N. In our context, we are mostly interested in them as arising from roots of polynomials with coefficients in N. The Galois group, Gal(E/N) then arises as the automorphisms of E/N that fix N pointwise.

• For example, $N = \mathbb{Q}$ and $E = \mathbb{Q}(i)$. This field extension arises from solutions to

$$x^2 + 1 = 0 \ , \tag{7}$$

and Gal$(E/N) \simeq \mathbb{Z}_2$ is just complex conjugation:

$$g(a + ib) = a - ib \ , \quad a, b \in \mathbb{Q} \ . \tag{8}$$

Galois Conjugation (cont...)

• Pentagon / hexagon equations are polynomials with coefficients in $\mathbb{Q}$, so natural that TQFT data takes values in algebraic number field, $\mathbb{Q}(F,R)$, with Galois group Gal($\mathcal{T}$) [Davidovich, Hagge et. al.]. **Key point:** things constructed algebraically from F,R transform nicely.

• Turns out can prove modular data is in a cyclotomic field (therefore corresponding Galois subgroup is abelian) [de Boer and Goeree (1991); Coste and Gannon (1994); $\cdots$].

• By construction, Gal($\mathcal{T}$) *preserves fusion algebra.* It therefore relates theories with isomorphic 1-form symmetries.

• For MTC data that depends on fusion spaces, Gal($\mathcal{T}$) is basis dependendent. When needed, can prove we can pick a nice basis.

Galois Conjugation (cont...)

Example 1: Semion $\simeq SU(2)_1$ CS theory

- The non-trivial fusion rules are

$$s \times s = 1 \ . \tag{9}$$

- Can take

$$F_s^{sss} = -1 \ , \quad R_1^{ss} = i \ . \tag{10}$$

- Alternatively

$$\theta_1 = 1 \ , \quad \theta_s = i \ . \tag{11}$$

- Therefore, Gal$(SU(2)_1) = \mathbb{Z}_2$ acts via complex conjugation. It takes us to $(E_7)_1$.

Galois Conjugation (cont...)

- The un-normalized S-matrix / Hopf-link takes the form

$$\tilde{S}_{ab} = \frac{\theta_{a\times b}}{\theta_a \theta_b} \ . \tag{12}$$

- Quantum dimensions are all equal to one

$$d_a = \tilde{S}_{a1} = 1 \ . \tag{13}$$

Example 2: Double semion $\simeq$ $SU(2)_1 \times (E_7)_1$ CS theory $\simeq$ twisted $\mathbb{Z}_2$ discrete gauge theory.

- Fusion rules generate $\mathbb{Z}_2 \times \mathbb{Z}_2$

$$(s,1)\times(s,1) = (1,\bar{s})\times(1,\bar{s}) = (1,1) \ , \ (s,1)\times(1,\bar{s}) = (s,\bar{s}) \ . \tag{14}$$

Galois Conjugation (cont...)

• Can take

$$\theta_{(1,1)} = 1 \ , \quad \theta_{(s,1)} = i \ , \quad \theta_{(1,\bar{s})} = -i \quad , \quad \theta_{(s,\bar{s})} = 1 \ . \qquad (15)$$

• Again Gal($SU(2)_1 \times (E_7)_1$) $= \mathbb{Z}_2$ acts via complex conjugation and exchanges

$$\theta_{(s,1)} \leftrightarrow \theta_{(1,\bar{s})} \ . \qquad (16)$$

• This action maps the theory to itself; it can be undone by the application of a time-reversal symmetry. All quantum dimensions again equal to one.

Galois Conjugation (cont...)

• Note that there is a Lagrangian subcategory*, $\text{Rep}(\mathbb{Z}_2) \simeq \{(1,1),(s,\bar{s})\}$, of Wilson lines that is itself Galois invariant. This is an important fact we will return to later.

Example 3: Fibonacci $\simeq (G_2)_1$ CS theory

• The non-trivial fusion rule has the form

$$\tau \times \tau = 1 + \tau \ . \tag{17}$$

*This is a subset of bosons, closed under fusion, with trivial mutual braiding satisfying $\dim(\text{Rep}(\mathbb{Z}_2))^2 = \dim(SU(2)_1 \times (E_7)_1)$. In this case $\tilde{S}_{\text{Rep}(\mathbb{Z}_2)} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Galois Conjugation (cont...)

- The non-trivial MTC data can be taken to be

$$F^{\tau}_{\tau\tau\tau} = \begin{pmatrix} \varphi^{-1} & \varphi^{-1/2} \\ \varphi^{-1/2} & -\varphi^{-1} \end{pmatrix} , \quad R^{1}_{\tau\tau} = \xi^{-2} ,$$
$$R^{\tau}_{\tau\tau} = \xi^{\frac{3}{2}} , \ \varphi = \frac{1}{2}(1+\sqrt{5}) = \xi^{-1} + 1 + \xi , \ \xi = \exp\left(\frac{2\pi i}{5}\right) .$$

- We have modular data

$$\tilde{S} = \begin{pmatrix} 1 & \varphi \\ \varphi & -1 \end{pmatrix} , \quad \theta_1 = 1 , \quad \theta_\tau = \exp\left(\frac{4\pi i}{5}\right) . \qquad (18)$$

- The quantum dimensions are $d_1 = \tilde{S}_{11} = 1$ and $d_\tau = \tilde{S}_{\tau 1} = \frac{1+\sqrt{5}}{2}$.

Galois Conjugation (cont...)

• For simplicity, let us restrict to a $\mathbb{Z}_5^\times$ Galois group acting on the above modular data and quantum dimensions. Acting with $4 \in \mathbb{Z}_5^\times$ takes $(\xi, \theta_\tau) \to (\xi^4, \theta_\tau^4) = (\xi^{-1}, \theta_\tau^{-1})$ and takes $d_\tau \to d_\tau$. This is a map taking $(G_2)_1 \to (F_4)_1$. OTOH, taking $\xi \to \xi^3$ takes $(\xi, \theta_\tau) \to (\xi^3, \theta_\tau^3)$ and $d_\tau \to \frac{1-\sqrt{5}}{2} < 0$. This is a non-unitary theory.

Upshot: Galois transformations can be "wild," but, as we have hinted above, there is also order.

• **General abelian picture:** have a space of theories corresponding to abelian finite groups, organized by rank, partitioned into different Galois orbits. Captured by abelian CS theory.

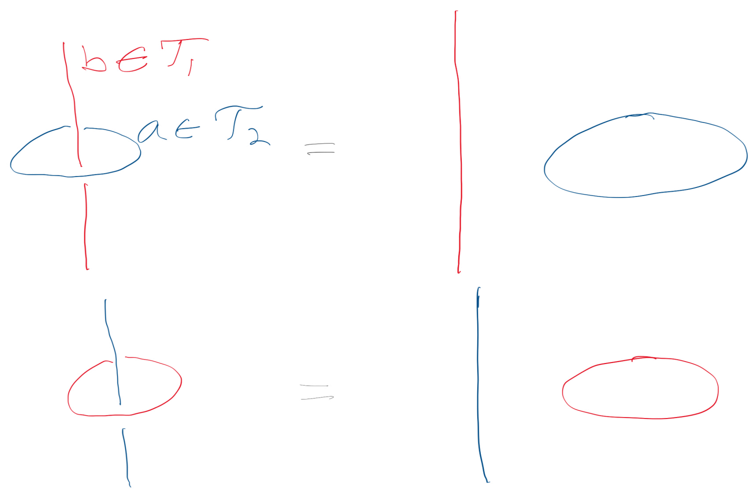
Galois Conjugation (cont...)

- **Non-abelian picture:** space of fusion categories that can be completed into MTCs, organized by rank, partitioned into different Galois orbits. Unclear how much captured by CS theory.

Galois Conjugation and Global Structure of TQFTs

• Perhaps the most basic question we can ask about the global structure of a TQFT is if it is a “product” theory or not.

• What should we mean by this? We have notions of fusion and braiding... The rough idea is that if $\mathcal{T} = \mathcal{T}_1 \times \mathcal{T}_2$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ are separately closed under fusion and transparent under braiding



Galois Conjugation and Global Structure of TQFTs (cont...)

• A theorem of [Müger (2003)] gives a necessary and sufficient condition for such factorization: $\tilde{S}_{\mathcal{T}}$ contains a non-degenerate sub-S matrix, $\tilde{S}_{\mathcal{T}_1}$.... Repeated application of this criterion yields a "prime factorization."

• In the examples, we saw that prime theories were mapped to prime theories and composite to composite under the Galois action... Is this generally true? Yes.

• This follows from the fact that the S-matrix is quadratic in the braiding:

$$\tilde{S}_{ab} = \mathrm{Tr} R_{\bar{b}a} R_{a\bar{b}} \ . \tag{19}$$

Galois Conjugation and Global Structure of TQFTs (cont...)

- So, if $\mathrm{Det}(\tilde{S}_{\mathcal{T}_1}) = \zeta \neq 0$, then, acting with Galois element $q \in \mathrm{Gal}(\mathcal{T})$ yields

$$q(\mathrm{Det}(\tilde{S}_{\mathcal{T}_1})) = (\mathrm{Det}(\tilde{S}_{q(\mathcal{T}_1)})) = q(\zeta) \neq 0 \ . \tag{20}$$

- By Müger's theorem, this means the factorization persists.

Some Galois-closed Subspaces of TQFTs

• Since Galois actions preserve the fusion rules, abelian theories are mapped to abelian theories. This space is therefore closed.

• In the examples, we saw that the twisted $\mathbb{Z}_2$ discrete gauge theory is mapped to itself under Galois action. This fact suggests at least three questions that will occupy the rest of the talk: **(1)** Are discrete gauge theories always mapped to other discrete gauge theories under Galois action? **(2)** Can we say something more general about Galois-invariant TQFTs?

• More indirectly, it will suggest a third question: **(3)** if we start with a Galois-invariant theory and gauge a (bosonic) 1-form symmetry, do we end up with something Galois invariant. This is because gauging $\mathrm{Rep}(\mathbb{Z}_2)$ takes us to a $\mathbb{Z}_2$ SPT which is also Galois invariant.

Some Galois-closed Subspaces of TQFTs (cont...)

• Regarding **(1)**: a G discrete gauge theory, $\mathcal{T}_G$, always has a set of Wilson lines that are closed under fusion (there are also magnetic fluxes and dyons). The set of such lines is generated by those labeled by irreps of G. We have

$$d_{W_{R_i}} = \tilde{S}_{W_{R_i}W_1} = |R_i| \ . \tag{21}$$

• The Wilson lines are a set of bosons with trivial mutual braiding and form a Rep(G) subcategory. This is a Lagrangian subcategory

$$(\mathrm{dim}(\mathrm{Rep}(G)))^2 := \left(\sum_i |R_i|^2\right)^2 = |G|^2 = \mathrm{dim}(\mathcal{T}_G) \ . \tag{22}$$

Some Galois-closed Subspaces of TQFTs (cont...)

• This is enough to decide something is a discrete gauge theory.

• This is roughly because 0-form gauging & Rep(G) 1-form symmetry gauging (for abelian G; or general Rep(G) "condensation" for non-abelian G) are inverses of each other and the 1-form gauging results in projecting out all lines carying flux (anything with magnetic charge braids non-trivially with at least one W_{R_i}) while the Wilson lines are identified with the vacuum.

• It is then easy to show that this space of theories is closed under Galois action. The point is that

$$d_{W_{R_i}} \in \mathbb{Z} \ , \quad \lambda(W_{R_i}, W_{R,j}) = \frac{\tilde{S}_{W_{R_i}W_{R_j}}}{\tilde{S}_{W_1 W_{R_j}}} = 1 \ , \quad \mathrm{dim}(\mathcal{T}_G) \in \mathbb{Z} \ , \tag{23}$$

and so they are unaffected by the Galois action.

Some Galois-closed Subspaces of TQFTs (cont...)

• Can say more using Tannaka-Krein reconstruction. In particular, one can prove that acting with $q \in \mathrm{Gal}(\mathcal{T}_G)$ yields

$$q(\mathcal{T}_G) = \mathcal{T}'_G \ , \tag{24}$$

i.e., another discrete gauge theory with the *same* gauge group, G (possibly with a different DW twist).

• In general, there can be other Lagrangian subcategories $\mathrm{Rep}(H_i)$ for $H_i \neq G$. Then the theory can also be thought of as an H_i discrete gauge theory. These are dualities. Since the number of Lagrangian subcategories can't change under Galois action, the duality structure of theories before and after conjugation is isomorphic.

Galois invariant TQFTs

• Regarding our question **(2)**, we saw that the twisted $\mathbb{Z}_2$ discrete gauge theory is invariant under the Galois action. A similar story holds in some other simple and famous examples like Toric code TQFT (a.k.a. untwisted $\mathbb{Z}_2$ discrete gauge theory). What can one say more generally?

• Well, the twisted $\mathbb{Z}_2$ discrete gauge $= SU(2)_1 \times (E_7)_1$. Can get Galois-invariant theories from products over Galois orbits.

• But this yields an immense and uncontrolled space if we consider non-unitary examples like

$$\mathcal{T} = (G_2)_1 \times \cdots \ , \tag{25}$$

where the ellipses contain non-unitary theories with $d_\tau = \frac{1-\sqrt{5}}{2}$. We will naively need to involve every prime theory.

Galois invariant TQFTs (cont...)

• Let us therefore restrict to unitary theories.

• Here things are already under better control: the quantum dimensions must be integers.

• To see this, we should first spell out what we mean by a unitary theory. At the level of the MTC, we require that it is possible to write F and R as unitary matrices and have all $d_a > 0$.

• Recall that the d_a satisfy

$$d_a \times d_b = \sum_c (N_a)_b{}^c d_c \ . \qquad (26)$$

Galois invariant TQFTs (cont...)

• Since $(N_a)_b{}^c$ is a non-negative integer matrix, the Perron-Frobenius theorem tells us that there is a unique eigenvector with positive entries and that it corresponds a maximal norm positive eigenvalue (the FP dimension). This means that d_a is the FP dimension in a unitary theory.

• The quantum dimensions are algebraic integers (they are maximal eigenvalues of a polynomial over the integers) and it follows that any Galois action on the quantum dimensions satisfies

$$|q(d_a)| \leq d_a \ , \quad \forall q \in \mathsf{Gal}(\mathcal{T}) \ , \quad \forall a \in \mathcal{T} \ . \tag{27}$$

• If the theory is Galois-invariant then $q(d_a) = d_b$ for some b.

Galois invariant TQFTs (cont...)

• Since q must be invertible, we require that

$$q^{-1}(d_b) = d_a \ . \tag{28}$$

But imposing (27) on this inverse action means that $d_a = d_b$. Therefore, $d_b \in \mathbb{Q}$. In fact, the rational root theorem implies that $d_b \in \mathbb{Z}$.

• Can we constrain such theories further?

• Theories with integer quantum dimensions are believed to be quite constrained. All known examples are in a class of theories called "weakly group theoretical."

Galois invariant TQFTs (cont...)

• A result of [Natale (2018)] building on work of [Drinfeld et. al. (2010)] shows that any "weakly group theoretical" TQFT can be constructed from gauging 0-form symmetries of theories built out of 8 different classes of abelian theories (some of these classes are infinite).

• We can say more by using a result of ours I won't explain:

Theorem (M.B., Radhakrishnan): If we start from a unitary Galois-invaraint TQFT and gauge a bosonic 1-form symmetry (or, more generally, condense some Rep(G)) the resulting theory is also Galois invariant.

Galois invariant TQFTs (cont...)

• Natale's result, combined with the above theorem, implies that all Galois-invariant weakly group theoretical TQFTs can be gotten by gauging zero-form symmetries of theories involving products of arbitrary numbers of abelian (untwisted) discrete gauge theories, theories along with Spin$(8)_1$, and certain abelian theories with $\mathbb{Z}_p \times \mathbb{Z}_p$ fusion rules (with p an odd prime).

• If indeed all integeral theories are weakly group theoretical, then we have shown that all unitary Galois invariant theories are of the above type. This is considerably simpler than the non-unitary case!

Conclusions

• Galois conjugation is a phenomenon that is simultaneously "wild" and "tame." It allows us to explore the space of TQFTs in interesting yet well-organized ways.

• Can we use Galois invariance to define interesting equivalence classes of Galois-non-invariant theories? Will this lead to progress on classification of TQFTs?

• Saw that Galois invariant unitary TQFTs were better controlled than non-unitary siblings. Can this be fully proven and extended?

• How does Galois theory work in higher-dimensional TQFT. For example, in 4D, we know that discrete gauge theories play a much more prominent role... In 3D these have very nice properties.