't Hooft line operators in $\mathcal{N} = 2$ supersymmetric gauge theories

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Based on
arXiv:1905.11305 and 2012.12275 with H. Hayashi and Y. Yoshida
arXiv:1910.01802 with Y. Yoshida

- The audience of this talk is mixed (condensed matter theorists and high-energy theorists). Early in the talk I will review topological aspects of line operators. I will also try to explain the basics of supersymmetric (BPS) line operators.

- Please do not hesitate to ask questions if there is anything unclear.

Plan of the talk

- Motivations
- Classification(s) of line operators in 4d (review)
- SUSY localization, supersymmetric quantum mechanics, and brane construction
- Wall-crossing and operator ordering
- Summary, conclusions, and future directions

Motivations

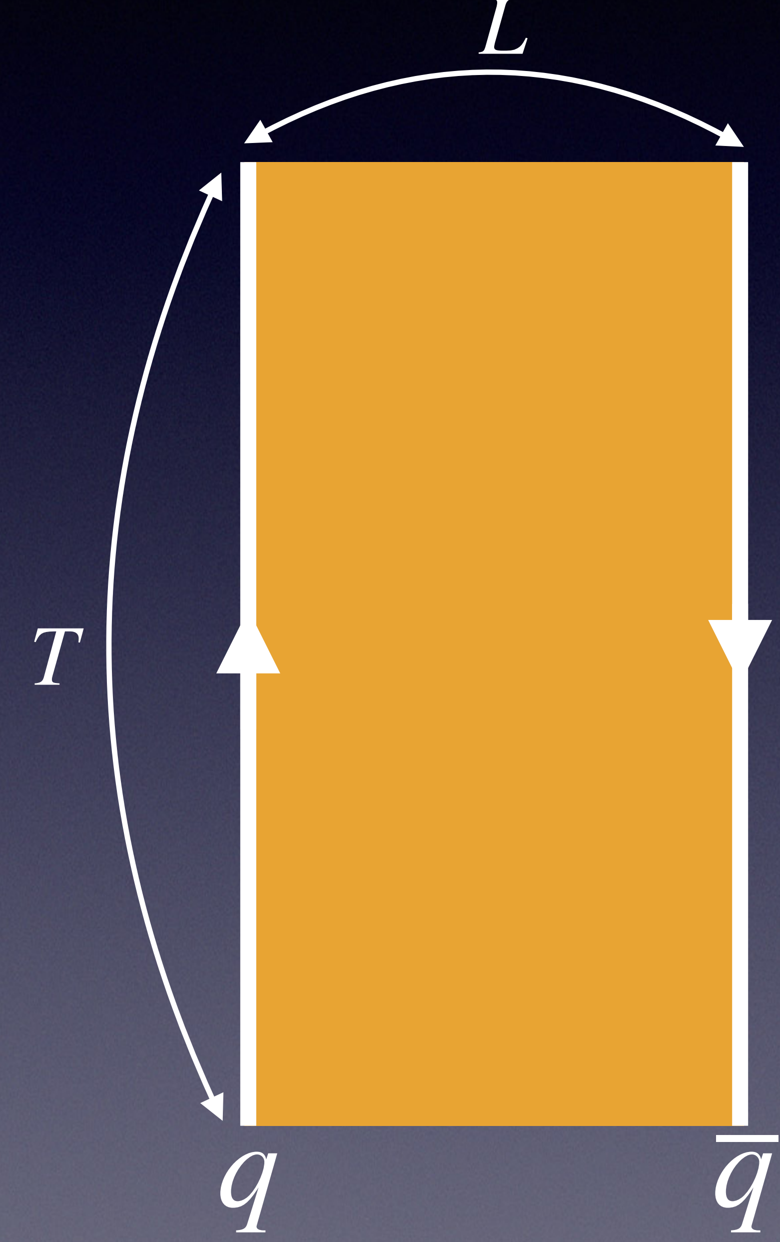
- Wilson and **'t Hooft line** operators are the most basic non-local operators that exist in a general 4d gauge theory. Order parameters for phases.
- Some properties of these line operators are topological, as we will review.
- With $\mathcal{N} = 2$ supersymmetry, very detailed exact calculations are possible for BPS line operators. Such calculations have many non-trivial physical and mathematical implications via dualities.

Wilson line (loop) operator

- $W_R(\gamma) = \mathrm{Tr}_R P \exp(i \oint_\gamma A).$

  World-line of an infinitely heavy electrically charged particle.

- Order parameter for confinement. Area law due to the electric flux tube. $\langle W_R \rangle \sim e^{-V(L)T}, V(L) \propto L.$



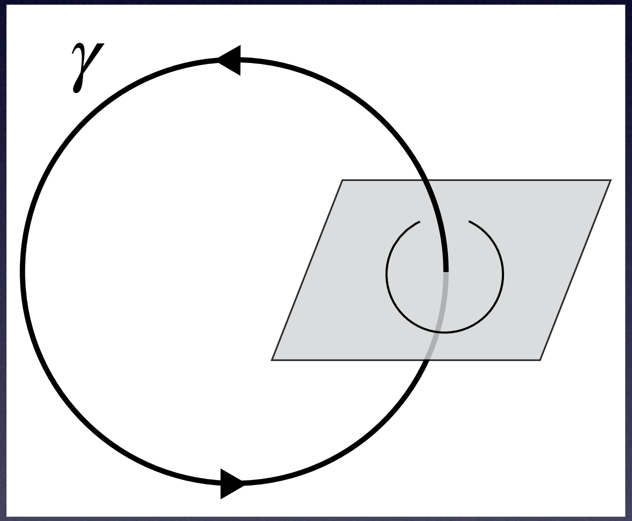
Topological classification of Wilson lines

- Gauge theory with gauge group G with matter
- $\Gamma_E \equiv \{g \in \text{Center}(G)\,|\,g \text{ acts trivially on matter}\}$
- Codimension-2 defects for $g \in \Gamma_E$ are topological. Wilson lines are classified by the charges for the Γ_E one-form symmetry (center symmetry).
- For $G = SU(N)$ with no matter, the charge is the "N-ality" (number of boxes in the Young tableau mod N).

’t Hooft line operator

- World-line of an infinitely heavy magnetically charged particle.

$$\langle T_B(\gamma)\rangle = \int_{F=\frac{B}{2}\mathrm{vol}(S^2)} \mathscr{D}A\ldots e^{-S}$$



- Order parameter for gauge symmetry breaking (Higgs mechanism). Area law due to a magnetic flux tube.

Topological classification of ’t Hooft lines

- The global form of the gauge group G is restricted by matter.
- $\Gamma_M \equiv \pi_1(G)^\vee = \mathrm{Hom}(\pi_1(G), U(1))$.
- ’t Hooft lines are topologically classified by the charges for the Γ_M one-form symmetry.

- Properties as order parameters (area v.s. perimeter laws) depend only on the charges for the one-form symmetries.

- For Wilson lines, this is because gluons and matter fields can screen the electric charges.

- For ’t Hooft lines, smooth (Polyakov-’t Hooft) monopoles can screen the magnetic charges.

Wilson-'t Hooft line operators in 4d $\mathcal{N} = 2$ theories

- Constructions of Wilson and ’t Hooft lines can be generalized (by using scalars) to define BPS line operators in $\mathcal{N} = 2$ supersymmetric gauge theories

  [Maldacena][Rey-Yee][Kapustin].

- They play significant roles in the 4d/2d (AGT) correspondence [Alday,Gaiotto,Tachikawa].

- Wilson-’t Hooft lines in 4d map to Verlinde lines (topological defects) in 2d Liouville CFT

  [Drukker,Gomis,TO,Teschner][Alday,Gaiotto,Gukov,Tachikawa,Verlinde][Drukker,Gaiotto,Gomis].

Wilson-'t Hooft line operators in 4d $\mathcal{N} = 2$ theories

- Expectation values can be computed exactly by **SUSY localization**. Given as a sum over saddle points [Gomis,TO,Pestun][Ito,TO,Taki].
- Via Kronheimer's correspondence between monopoles and instantons, SUSY localization for 't Hooft line operators is intimately connected to instanton counting.

Classification of BPS line operators

- BPS line operators admit more refined classification than the topological (one-form symmetry) classification [Kapustin].
- BPS Wilson lines are classified by representations.
- BPS 't Hooft lines are classified by the magnetic charge B for the Dirac singularity $F \sim (B/2)\mathrm{vol}(S^2)$.

Classification of BPS line operators

- General BPS Wilson-'t Hooft line operators are classified by $(\Lambda_{\text{char}} \times \Lambda_{\text{cochar}})/\text{Weyl}$ [Kapustin].
- Λ_{char}: lattice generated by the weights in all representations R of the gauge group G. Electric charges.
- Λ_{cochar}: dual lattice of Λ_{char}. Magnetic charges B.

- Let Λ_{mat} be the lattice generated by roots and weights in the matter representation. Then $\Lambda_{\text{char}}/\Lambda_{\text{mat}} = \Gamma_E^\vee$. One-form symmetry charges.

- $\Lambda_{\text{cochar}}/\Lambda_{\text{coroot}} = \Gamma_M^\vee$.

- Assumed no discrete theta angles [Aharony et al.].

Coulomb branch of 4d $\mathcal{N} = 2$ theory on $S^1 \times \mathbb{R}^3$

- HyperKähler manifold.
- For a class S theory based on a punctured Riemann surface C, the Coulomb branch is the Hitchin moduli space on C.
- VEVs of BPS Wilson-'t Hooft operators parametrize the Coulomb branch.

Set-up: $S^1 \times_\lambda \mathbb{R}^3$

- $S^1 \times_\lambda \mathbb{R}^3$: As we go around the S^1, rotate along the 3-axis of $\mathbb{R}^3$ by angle $2\pi\lambda$. Insert line operator L around the S^1.

- Analog of omega deformation for 5d instanton counting. $\langle L \rangle = \mathrm{Tr}_{\mathcal{H}(L)}(-1)^F e^{-\beta H} e^{2\pi i \lambda (J_3 + I_3)} \ldots$

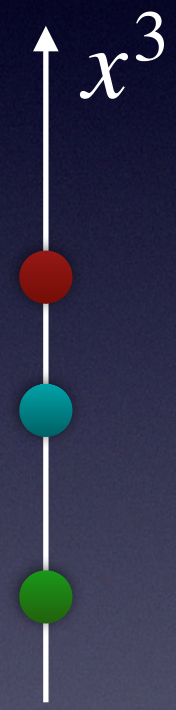
- Set-up of Gaiotto, Moore and Neitzke who studied line operators with spectral networks.

Set-up: $S^1 \times_\lambda R^3$

- Deformation quantization of the Coulomb branch.
- For a class S theoryh, via the AGT correspondence, the VEV $\langle L \rangle$ is mapped to the Wigner transform (Weyl ordering) of the Verlinde operator expressed as a difference operator [Ito,TO,Taki].

Deformation quantization of the Coulomb branch

- Omega deformation quantizes the Coulomb branch.
- 1-dimensional topological sector. Only the **ordering** of operators matters.
- Noncommutative product implemented by the Moyal product [Ito-TO-Taki]. This provides non-trivial checks.



$$\langle \mathcal{O}_1 \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \rangle * \langle \mathcal{O}_2 \rangle$$

$$(f * g)(a, b) \equiv \lim_{a' \to a, b' \to b} e^{i\hbar(\partial_b \partial_{a'} - \partial_a \partial_{b'})} f(a, b) g(a', b') \,, \quad \hbar \propto \lambda$$

Monopole screening/ bubbling

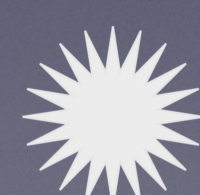
$$D\Phi = *_3 F$$

- Bogomolny equations describe smooth (Pokyakov-'t Hooft) monopoles in non-abelian gauge theories.

- We are interested in a situation where there is a Dirac monopole singularity.

B

- Smooth monopoles may screen and reduce the Dirac singularity.



Localization for 't Hooft operators

- Recall that the instanton partition function receives contributions from torus fixed points on the instanton moduli space.
- Similarly, the 't Hooft operator VEV receives contributions from torus fixed points on the (singular) monopole moduli space. [Gomis, TO & Pestun][Ito, TO & Taki]
- The fixed points occur in the locus where smooth monopoles screen the singular one.

Monopole bubbling contribution

$$\langle T_B \rangle = \sum_v e^{iv\cdot b} Z_{\text{1-loop}}(a,\lambda;v) Z_{\text{mono}}(a,\lambda;B,v)$$

- v = B - coroot : magnetic charge reduced by screening

- $\mathbf{Z_{mono}}$ for the 't Hooft operator is the analog of the 5d instanton partition function for singular monopoles.

- (a,b): Fenchel-Nielsen coordinates of the Hitching moduli space (A_1 class S theories).

Results of localization

Example: U(2) SQCD with 4 flavors
B=diag(1,-1)

$$\langle T_B \rangle = e^{\pi i b_{12}} \left(\frac{\prod_{f=1}^{4} \sin \pi(a_1 - m_f) \sin \pi(a_2 - m_f)}{\sin^2 \pi a_{12} \prod_{\pm} \sin \pi(a_{12} \pm \lambda)} \right)^{1/2}$$

$$+ e^{-\pi i b_{12}} \left(\frac{\prod_{f=1}^{4} \sin \pi(a_1 + m_f) \sin \pi(a_2 + m_f)}{\sin^2 \pi a_{12} \prod_{\pm} \sin \pi(a_{12} \pm \lambda)} \right)^{1/2}$$

$$+ \frac{\prod_{f=1}^{4} \sin \pi \left(a_1 - m_f + \frac{\lambda}{2} \right)}{\sin \pi a_{12} \sin \pi \left(-a_{12} - \lambda \right)} + \frac{\prod_{f=1}^{4} \sin \pi \left(a_2 - m_f + \frac{\lambda}{2} \right)}{\sin \pi a_{21} \sin \pi \left(-a_{21} - \lambda \right)}$$

$Z_{mono}(B, v=0)$

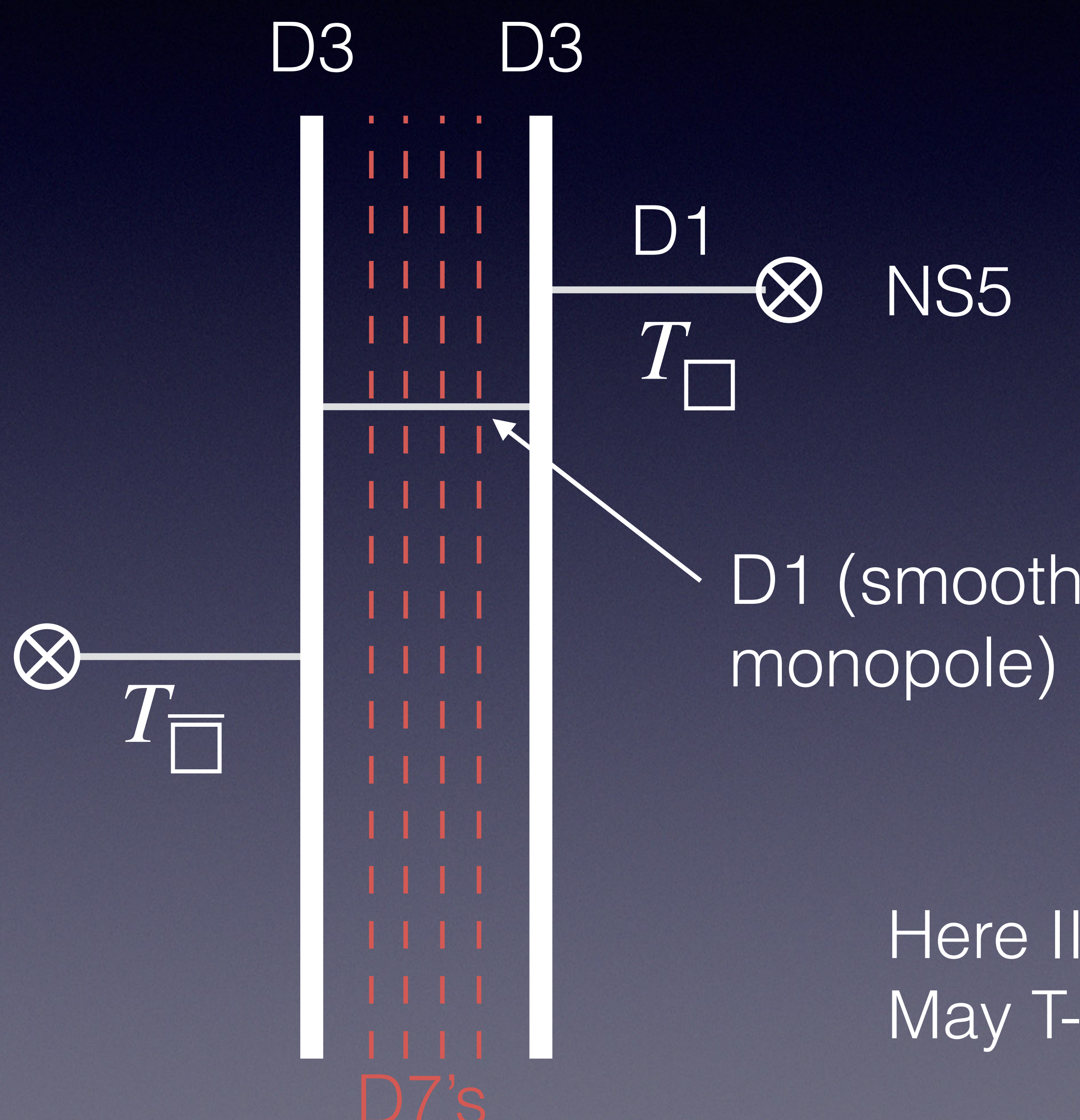
Bubbling contribution from SUSY quantum mechanics

- It is well known that the instanton partition function can be computed using the ADHM matrix model/quantum mechanics.
- Similarly Z_{mono} can be computed as the Witten index of a supersymmetric quantum mechanics (SQM).
- Approach taken by Brennan, Dey, and Moore.

SQM and U(N) SQCD

- The relation between SQM and wall-crossing is simpler for U(N) and SO/USp than for SU(N). [cf. Brennan, Dey, and Moore]
- Here we consider gauge group U(N) rather than SU(N)[Hayashi, TO, Yoshida].
- SQCD: matter hypermultiplet in the fundamental representation.

Brane construction of 't Hooft operators

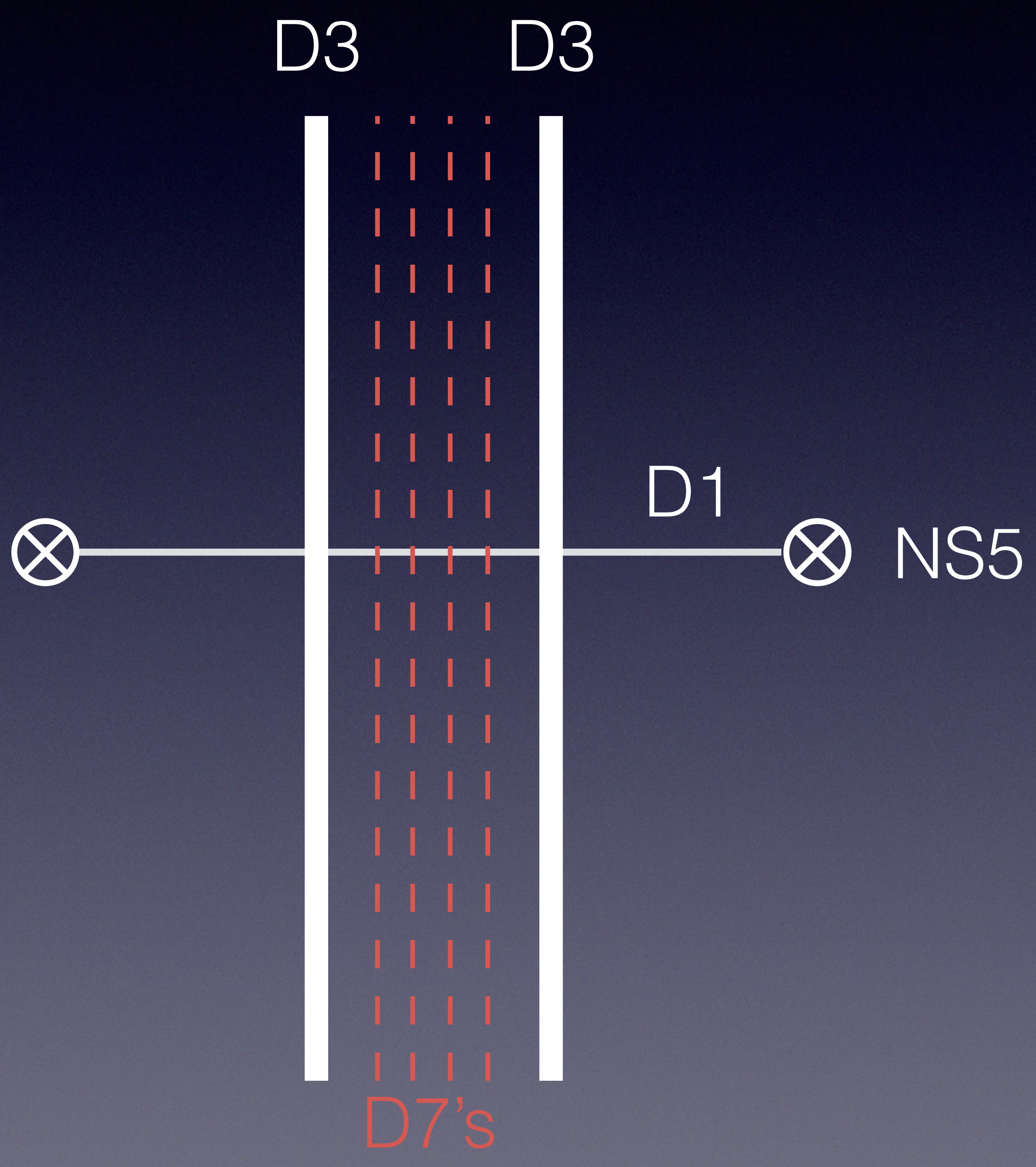


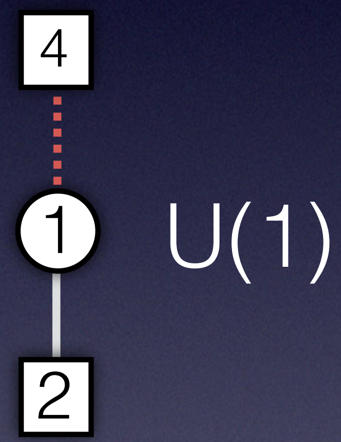
$$T_{B=\mathrm{diag}(-1,1)} \sim T_{\square} \cdot T_{\overline{\square}}$$

Here IIB string with D1/D3/D7.
May T-dualize to IIA with D2/D4/D6.

	0	1	2	3	4	5	6	7	8	9
D3	x	x	x	x						
D7	x	x	x	x			x	x	x	x
D1	x				x					
NS5	x					x	x	x	x	x

Supersymmetric quantum mechanics on D1-branes





1d $\mathcal{N} = (0,4)$ quiver supersymmetric quantum mechanics

SQM partition function in terms of Jeffrey-Kirwan residues

$$Z_{\text{vec}}(\phi,\lambda) = \left[\prod_{1\le i\neq j\le r} 2\sinh\frac{\phi_i-\phi_j}{2}\right]\left[\prod_{1\le i,j\le n} 2\sinh\frac{\phi_i-\phi_j+2\lambda}{2}\right]$$

$$Z_{\text{hyp}}(\phi,\phi',\lambda) = \frac{1}{\prod_{i=1}^{n}\prod_{j=1}^{n'} 2\sinh\frac{\phi_i-\phi'_j+\lambda}{2}2\sinh\frac{-\phi_i+\phi'_j+\lambda}{2}}$$

$$Z_{\text{Fermi}}(\phi,\phi',\lambda) = \prod_{i=1}^{n}\prod_{j=1}^{n'} 2\sinh\frac{\phi_i-\phi'_j+\lambda}{2}$$

[Hwang,Kim,Kim,Park]
[Cordova,Shao][Hori,Kim,Yi]
cf. Benini et al. for 2d

$$Z_{\text{SQM}} = \frac{1}{|W|}\sum_{\phi_*}\text{JK-Res}_{\phi=\phi_*}(\mathbf{Q}_*,\zeta)$$

$$\times\prod Z_{\text{vec}}\prod Z_{\text{hyp}}\prod Z_{\text{Fermi}}\, d\phi_1\wedge\cdots\wedge d\phi_{\text{rank}}$$

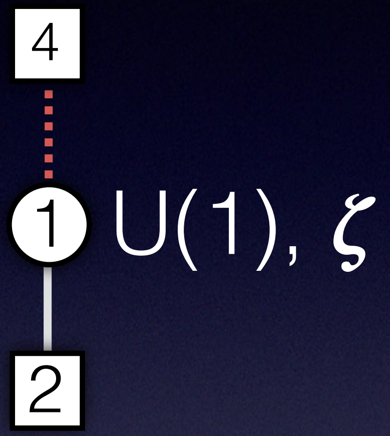
Wall-crossing and operator ordering

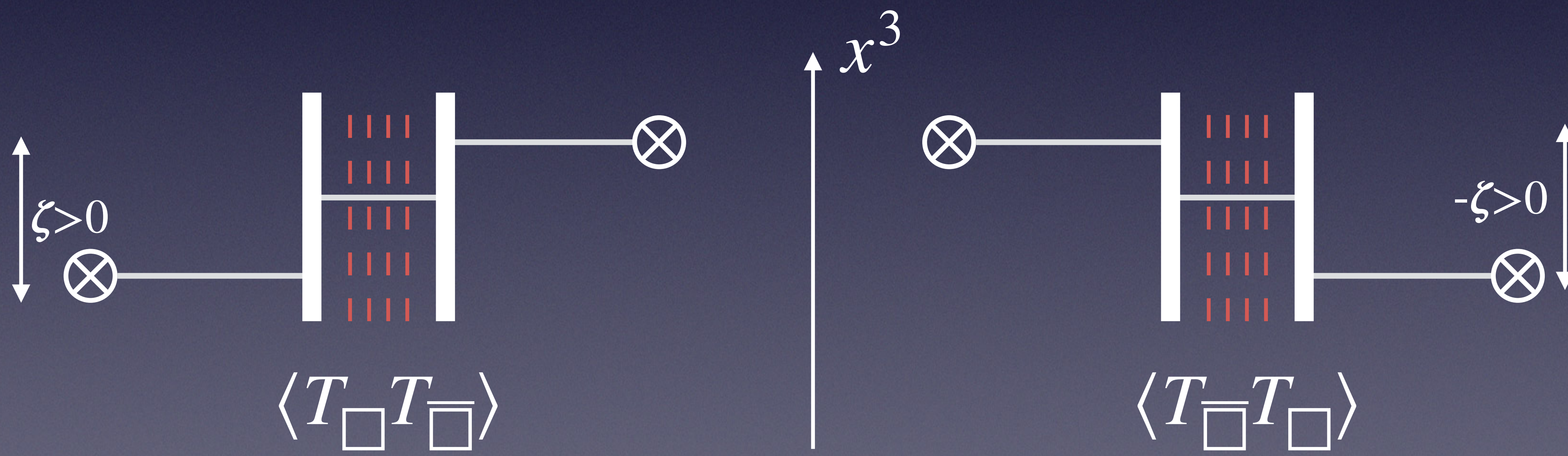
- Wall-crossing (dependence on the choice of a chamber in the space of FI/stability parameters) occurs when the ordering of operators in a correlator changes.

- In the brane construction, this is natural because the FI parameters of the SQM are the locations of the NS5-branes ('t Hooft operators).

Examples

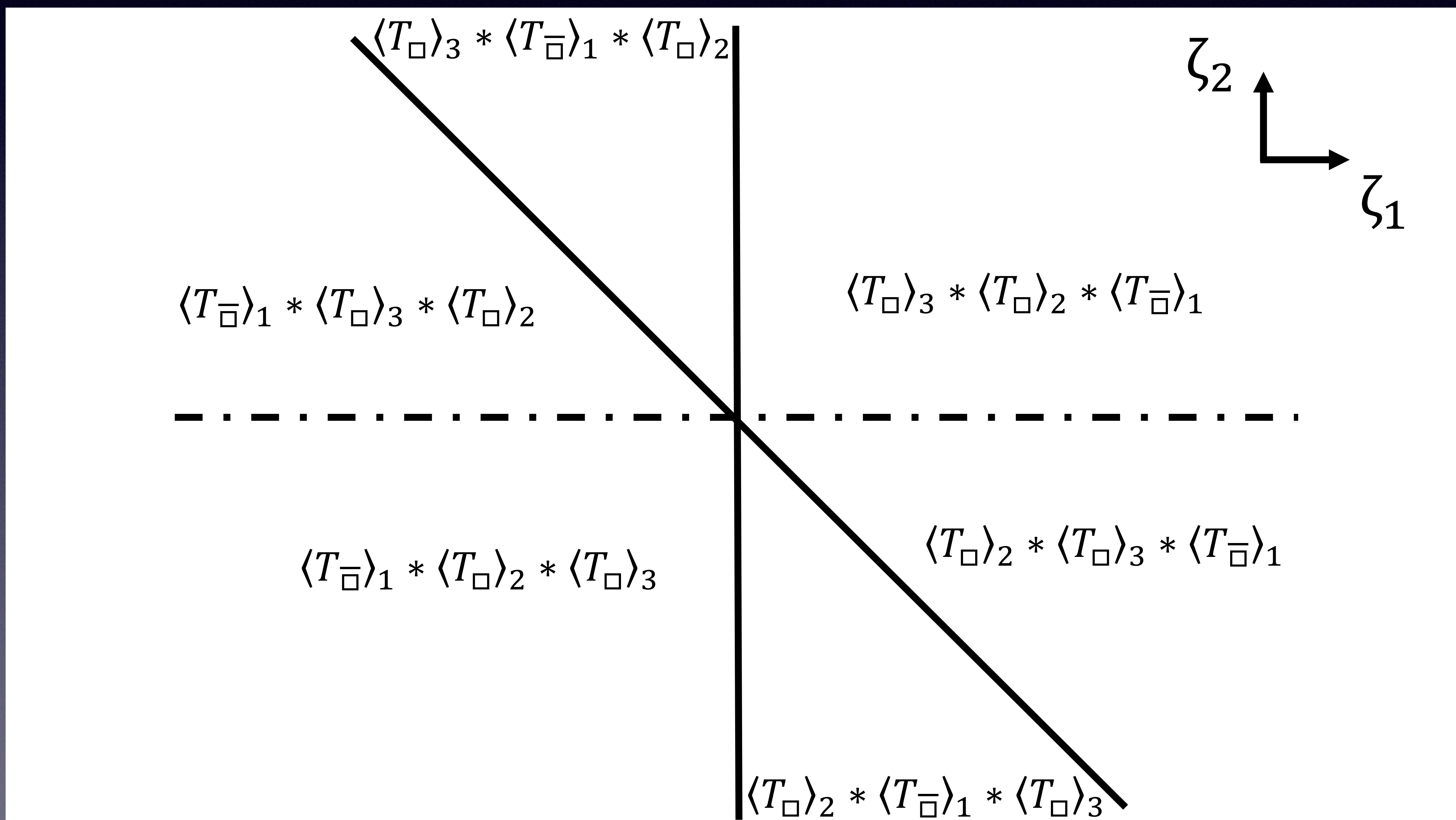
- U(N) SQCD with N_F=2N. N=2 here.
- $T_{\square} = T_{B=\mathrm{diag}(0,+1)}$ $\quad T_{\overline{\square}} = T_{B=\mathrm{diag}(-1,0)}$



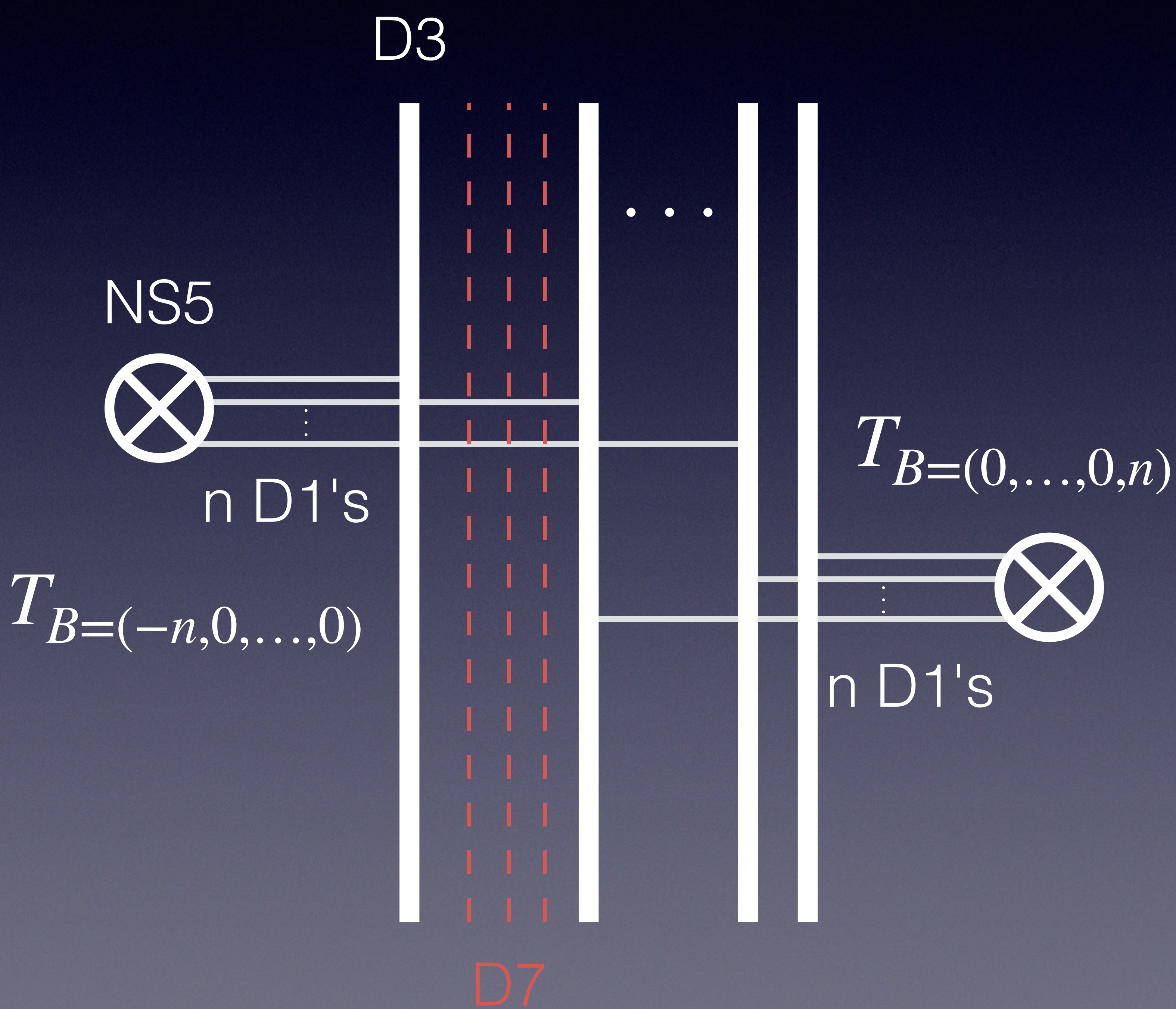


Z_{mono} in $\langle T_{\square} T_{\overline{\square}} \rangle$ = $Z_{\mathrm{SQM}}(\zeta>0)$, $\quad Z_{\mathrm{mono}}$ in $\langle T_{\overline{\square}} T_{\square} \rangle$ = $Z_{\mathrm{SQM}}(\zeta<0)$.

Chamber structure for the product of three operators

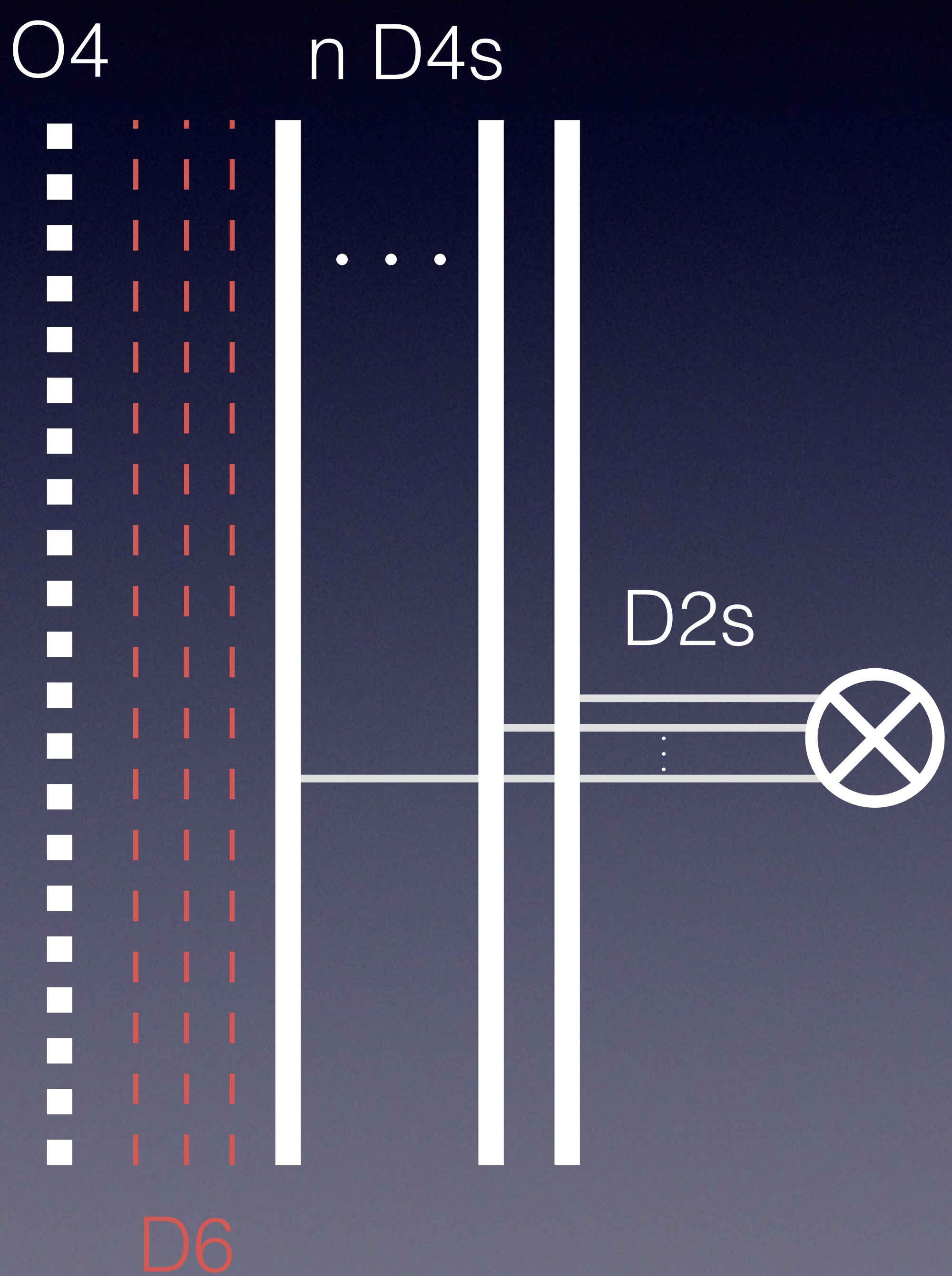


't Hooft operators with higher magnetic charges



- 't Hooft operator corresponding to rank-n anti-symmetric tensor product.
- B=(0,...,0,n) for product of fundamentals.
- B=(-n,0,...,0) for product of anti-fundamentals.

SO/USp gauge theories



- Orientifold 4-plane $O4 = O4^-, \widetilde{O4}^-, O4^+$.
  - SO(2n) for $O4^-$.
  - SO(2n+1) for $\widetilde{O4}^-$.
  - USp(2n) for $O4^+$.
- Computed 't Hooft operator VEVs

[Hayashi,TO,Yoshida].

’t Hooft (monopole) operators in other dimensions

- 3d: monopole operators. Their VEVs can be computed by matrix models. [TO, Yoshida][Assel, Cremonesi, Renwick].
- 5d: ’t Hooft surface operators in $\mathcal{N} = 1^*$ U(N) theory [Yoshida]. VEVs are computed by 2d GLSMs and give rise to type A elliptic Ruijsenaars operators.

Summary and conclusions

- Wilson and 't Hooft line operators admit a topological classification by one-form symmetries. BPS operators admit a more refined classification.
- The BPS line operators on $S^1 \times_\lambda \mathbb{R}^3$ are natural observables. Their VEVs can be computed by SUSY localization. Non-perturbative contributions to 't Hooft operators are computed by supersymmetric quantum mechanics. Computed for $G = U(N), SO(N), USp(N)$.
- Wall-crossing can occur when the ordering of 't Hooft operators changes.

Future directions

- ’t Hooft line operator expectation values for other gauge groups (especially exceptional gauge groups) and matter representations.

- Wall-crossing in instanton counting [Hwang,Kim,Kim,Park] [Ohkawa][Ito,Maruyoshi&TO]. How much of this can we understand using SQMs? [Hwang,Kim,Kim,Park][work in progress: Hayashi,TO&Yoshida]