Lessons from the Large N solution to Matter Chern Simons Theories

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Based on

1110.4386 Giombi, S.M. Prakash, Trivedi, Wadia, Yin CS Fermions. Sovability at large N. Non renormalisation of spectrum. Use of lightcone gauge. Exact soln of gap equation and partition function. Proposed bulk dual. First suggestion of bose fermi duality. 1110.4382 Aharony, Gur Ari, Yacobi CS Bosons. Non renormalization of spectrum. Perturbative β function 1112.1016, 1204.3882 Maldacena Zhiboedov Solution for large N correlators using HS sym. 1207.4593 Aharony, Gur Ari, Yacobi Exact solution of 2 and 3 pt functions. Identification of MZ parameters. First concrete duality conjecture and proposal of duality map. 1207.4485 Chang, S.M., Sharma, Yin Vasiliev dual of susy theories including ABJM 1207.4750 Jain, Trivedi, Wadia, Yokoyama Susy thermal partition functions. 1207.4593 Jain, S.M. , Sharma, Takimi, Wadia, Yokoyama Partition function and phase transitions on S^2. Duality of partition functions from level rank duality. 1211.4843 Aharony, Giombi, Gur Ari, Maldacena Yacobi Correct treatment of holonomy in thermal partition function. Duality of thermal partition functions. 1305.7235 Jain, S.M., Yokoyama Flows from susy to quasi fermionic theories 1404.6373, 1505.6371 Jain, Inbasekar, Mandlik, Mazumdar, S.M. Takimi, Wadia, Umesh, Yokoyama Duality of scattering. Modification of crossing symmetry 1506.05412 Bhedotiya, Prakash Four point functions of lightest scalars, first attempt 1507.04378 Gur Ari, Yacoby Re derivation of flows from susy 1507.04546 Gur Ari, Yacoby Exact flows from quasi bosonic to quasi fermionic theories 1511.01902 Radicevic Identification of monopoles as dual Baryons 1511.04772 Geracy, Goikhman, Son Interpolation between Bose and Fermi Statistics at finite temperature 1512.00161 Aharony Precise conjecture for duality map at finite N and k 1610.08472 Giombi, Gurucharn, Kirilin, Prakash, Skvortsov Anomalous dimensions

Based on

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Introduction

- Chern Simons theories coupled to dynamical matter fields are of interest for several reasons.
- First, in parity non invariant theories, the one derivative Chern Simons Lagrangian generically dominates the two derivative Yang Mills kinetic term and so governs gauge dynamics at low energies.
- Second, the Chern Simons coupling, $\frac{1}{k}$, does not flow under the renormalization group, so fine tuning matter masses to zero often results in conformal dynamics.
- Third, Chern Simons matter theories host anyonic excitations with 'non half integer' spins whose S matrices display unusual crossing properties
- Fourth, some of these theories have conjectured AdS/CFT dual descriptions in large N limits.
- Fifth, some of these theories they enjoy invariance under (conjectured) strong weak coupling Bose Fermi duality even without sypersymmetry.

Sixth, and most importantly for this talk, several exact results are available for two interesting limits of these theories.

- (1) When the mass of the matter fields is taken to infinity, our theories reduce to pure Chern Simons theory which has an intricate, beautiful and very thoroughly understood exact solution.
- (2) When all matter fields are in the fundamental, and N and k are taken to infinity with

$$\lambda = \frac{N}{k + \mathrm{sgn}(k)N} \equiv \frac{N}{\kappa}$$

held fixed, the theory is once again exactly solvable. Several interesting dynamical quantities have been exactly computed in this limit.

Review of Pure Chern Simons Theories: $SU(N)_k$

- $SU(N)$ Chern Simons theories are defined by the action

$$S = \frac{ik}{4\pi} \int d^3x \, \epsilon^{\mu\nu\rho} \, \mathrm{tr}\left(y_\mu \partial_\nu y_\rho - \frac{2i}{3} y_\mu y_\nu y_\rho\right) \tag{1}$$

  where y_μ are $SU(N)$ matrix valued fields, where the UV divergences that arise in this path integral are regulated by adding an infinitesimal Yang Mills term.

- Equivalently these theories are defined by the action

$$S = \frac{i\kappa}{4\pi} \int d^3x \, \epsilon^{\mu\nu\rho} \, \mathrm{tr}\left(y_\mu \partial_\nu y_\rho - \frac{2i}{3} y_\mu y_\nu y_\rho\right) \tag{2}$$

$$\kappa = k + \mathrm{sgn}(k) N. \tag{3}$$

  and UV divergences are regulated in the dimensional regulation scheme. k is called the level of the Chern Simons theory. We refer to κ as the renormalized level of this theory. Despite first appearances, path integral gauge invariant if k (and so κ) are integers.

Review of Pure Chern Simons Theories: $SU(N)_k$

- Note that the metric of the manifold makes no appearance in the action. Follows that the action is topological, atleast classically (and it turns out also quantum mechanically).
- Equation of motion: $F = 0$. Implies that all onshell configurations are locally pure gauge and so are given by maps from the base manifold to group manifold $SU(N)$.
- For instance consider $SU(N)$ Chern Simons theory on $\Sigma_g \times S^1$ with Wilson lines in representation R_1, $R_2 \ldots R_n$ at points on Σ_g but winding the S_1 The representations are constrained to be 'integrable', see below. The phase space is that of Σ_g to the group, with a monodromy (whose equivalence class is determined by the representation R_m) around the m^{th} puncture. Witten demonstrated that the quantization of this phase space yields the Hilbert Space of Σ_g conformal blocks in $SU(N)_k$ WZW theory. All states have zero energy, so the $\Sigma_g \times S^1$ path integral equals the dimensionality of the space of these conformal blocks.

Counting $SU(N)_k$ conformal blocks

- The Verlinde formula that counts the dimensionality of the space of these conformal blocks can, for the case of $SU(N)_k$, be massaged into the following relatively simple form

$$N_{sing} = \frac{1}{N\kappa^{N-1}} \sum_{\{w_i\}} \prod_{i<j} |w_i - w_j|^{2-2g} \prod_{p=1}^{m} \chi_{R_p}(w_i) \qquad (4)$$

- Here w_i are the eigenvalues of the $SU(N)$ holonomy, that are discretized (by the sum over fluxes in the path integral to obey)

$$\prod_{i=1}^{N} w_i = 1, \quad \text{and} \quad |w_i| = 1 \quad \forall i \qquad (5)$$
$$w_i^{\kappa} = w_j^{\kappa}, \qquad w_i \neq w_j, \quad \forall i, j$$

Formula only holds when the maximum row size in the Young Tableaux for each R_m is $\leq k$ "unitary or integrable

Review of $U(N)_{k,k'}$ theories.

- Unlike their $SU(N)$ counterparts, $U(N)$ Chern Simons theories are characterized by two levels k and k'. Roughly these are the levels of the $SU(N)$ part and the $U(1)$ part of the gauge group respectively (in the normalization in which fundamental fields carry unit $U(1)$ charge).

$$\mathcal{S}[y,a] = \frac{ik}{4\pi}\int \mathrm{tr}\left(ydy - \frac{2i}{3}y^3\right) + \frac{ik'}{4\pi N}\int \mathrm{tr}y\, d\left(\mathrm{tr}y\right). \quad (6)$$

  Consistency forces k' to take the form

$$k' = \kappa + qN$$

- In this talk we will be particularly interested in the case $q = 0$ which we call the Type I theory, and $q = -1$, which we call the Type II theory. For Type I theory $k' = \kappa$ while for Type II theory $k' = k$

Counting $U(N)_{k,k'}$ conformal blocks on S^2

- Once again the partition function of the $U(N)_{k,k'}$ theory on $\Sigma_g \times S^1$, in the presence of Wilson lines, is given by the space of $U(N)_{k,k'}$ WZW blocks. The dimensionality of this Hilbert space is, once again, counted by the Verlinde formula, which can be simplified in this particular case to

$$N_{sing} = \frac{1}{(\kappa + qN)\kappa^{N-1}} \sum_{\{w_i\}} \prod_{\mathrm{i<j}} |w_i - w_j|^{2-2g} \prod_{p=1}^{n} \chi_{R_p}(w_i) \tag{7}$$

where

$$\begin{aligned} |w_i| = 1 \quad \forall i, \quad w_i^{\kappa} = w_j^{\kappa} \qquad \forall i,j \\ w_m^{\kappa} \left(\prod_{i=1}^{N} w_i \right)^{q} = (-1)^{N+1} \quad \forall m \end{aligned} \tag{8}$$

Counting Type I $U(N)$ conformal blocks on S^2

- The formula of the previous slide simplifies at $q = 0$, i.e. in the Type I theory, to

$$N_{sing} = \frac{1}{\kappa^N} \sum_{\{w_i\}} \prod_{\mathrm{i<j}} |w_i - w_j|^2 \prod_{p=1}^{n} \chi_{R_p}(w_i) \qquad (9)$$

$$w_m^{\kappa} = (-1)^{N+1} \quad \forall m$$

- For concreteness let us specialize to Type I theory.
- (9) is a very particular discretization of the Weyl integral formula of classical group theory. In the large N limit the spacing between two eigenvalues, $\frac{1}{2\pi\kappa} \to 0$ the discretization spacing $\to 0$ so (9) reduces to the classical Weyl formula except for one constraint;

$$\rho(\theta) \leq \frac{2\pi N}{\kappa} = \frac{2\pi}{\lambda}.$$

$[dU]_{CS}$ from conformal blocks

- In equations, in the t'Hooft large N limit

$$\frac{1}{\kappa^N}\sum_{\{w_i\}}\prod_{\mathrm{i<j}}|w_i-w_j|^2\prod_{p=1}^{n}\chi_{R_p}(w_i)\to\int[dU]_{CS}\prod_{p=1}^{n}\chi_{R_p}(w_i)$$

- Here $[dU]_{CS}$ is the usual Haar measure subject to the constraint

$$\rho(\alpha)\leq\frac{1}{2\pi|\lambda|}\,,\qquad\rho(\alpha)=\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\delta(\alpha-\alpha_j)\,.\qquad(10)$$

($e^{i\alpha_j}$ are the eigenvalues of U and $\rho(\alpha)$ is the eigenvalue distribution function).

Level Rank Duality of Chern Simons Theories

- Pure Chern Simons theories enjoy invariance under a remarkable strong weak coupling duality called level rank duality. The ones we will use are summarized in this table

Type I-Type I	$U(N)_{k,\,\kappa} \longleftrightarrow U(\vert k\vert)_{\epsilon N,\,\epsilon\kappa}$
Type II-$SU(N)$	$U(N)_{k,k} \longleftrightarrow SU(\vert k\vert)_{\epsilon N}$

Table: Here $\epsilon = -\mathrm{sgn}(\mathrm{k})$ and $\kappa = k + \mathrm{sgn}(k)N$.

- Part of the level rank duality map is a map between representations. Roughly the rule is that Young Tableaux are transposed (rows $\leftrightarrow$ columns). In particular, fundamentals map to fundamentals, while n index symmetrical tensor map to n index antisymmetrical tensors under level rank duality. Not difficult to convince oneself, for instance, that (9) with only fundamental and and antifundamental insertions is level rank selfdual

Matter Chern Simons Theories

- So far we have studied only pure Chern Simons theories. These are beautiful and completely solvable theories; nonetheless they lack the richness of genuine quantum field theories precisely because they are topological and so have no local degrees of freedom
- In this talk we study Chern Simons theories minimally coupled to matter. The theories we study have all the richness of standard quantum field theories. The price we pay for this richness, however is that these theories are generically not solvable.
- For this reason, in much of this talk we will focus on theories with fundamental matter that turn out to be effectively solvable in the t'Hooft large N limit. Before describing the simplification of the large N limit, however, we first present the simplest theories we study and understand some simple properties of these theories even at finite N and k.

Regular Fermion and Critical Boson Theories

- The simplest and best studied large N matter CS theories. So called Quasi Fermionic theories. This talk: single matter flavour. (Generalization to finite number of flavour N_f also solvable and studied. Ignore in this talk. Generalization to N_f of order N not solvable using our large N techniques).
- Atleast order by order in $1/N$, theories appear sensibly defined by path integral using actions

$$U(N_F)_{(k_F,k_F)} + \int \bar{\psi} D_\mu \gamma^\mu \psi + m_F^{\text{reg}} \bar{\psi}\psi$$

$$SU(N_B)_{k_B} + \int [D_\mu \bar{\phi} D^\mu \phi + \sigma_B \left(\bar{\phi}\phi + \frac{N_B}{4\pi} m_B^{\text{cri}} \right)$$

in the dimensional regulation scheme.

- Refer to these as the 'regular fermion' and critical boson theories respectively. Can also interchange fermionic SU and bosonic U theories- or study U U theories with shifted $U(1)$ levels. All very similar at large N.

Regular Boson and Critical Fermion Theories

- More general regular boson and critical fermion theories

$$SU(N_B)_{k_B} + \int D_\mu\bar{\phi}D^\mu\phi + m_B^2\bar{\phi}\phi + \frac{4\pi b_4}{\kappa_B}(\bar{\phi}\phi)^2 + \frac{(2\pi)^2}{\kappa_B^2}\left(x_6^B + 1\right)(\bar{\phi}\phi)^3, \tag{11}$$

$$S_{RF}(\psi) + \int -J_0^F\zeta + +y_2^2\zeta - \frac{4\pi y_4}{\kappa_F}\zeta^2 + \frac{(2\pi)^2}{\kappa_F^2}x_6^F\zeta^3, \quad J_0^F = \frac{4\pi\bar{\psi}\psi}{\kappa_F} \tag{12}$$

- Theories well defined? Two potential issues. 1) Stability of vacuum (leading N). 2) Existence of RG fixed point for x_6^B and x_6^F (β functions first subleading order in $1/N$). Both issues studied in detail at large N. Upshot: theories well defined in window of λ atleast at large N. Will come back to this point.

Duality and notation

- As most of you know these two classes of theories are conjecturally dual.
- Notation

$$\kappa_B = \mathrm{sgn}(\mathrm{k_B})(N_B + |k_B|), \quad \lambda_B = \frac{N_B}{\kappa_B}, \quad (\text{and} \quad B \leftrightarrow F)$$

- Conjectured duality map (e.g. for CB and RF theories)

$$k_B = -\mathrm{sgn}(\mathrm{k_F})\mathrm{N_F}, \quad \mathrm{N_B} = |\mathrm{k_F}|, \quad \mathrm{m_B^{cri}} = \left(\frac{|\mathrm{k_F}| + \mathrm{N_F}}{\mathrm{k_F}}\right) \mathrm{m_F^{reg}}$$

- Equivalently

$$\kappa_B = -\kappa_F, \quad \lambda_F = \lambda_B - \mathrm{sgn}(\lambda_B), \quad -\lambda_B m_B^{\mathrm{cri}} = m_F^{\mathrm{cri}}$$

- Level and rank duality map believed to be exact. Mass map known only at leading order in large N. Note large N results nontrivial function of effective t'Hooft coupling λ.

CB and RF theories: Phases 1

- The CB and RF theories are conformal at $m_B^{\rm cri} = m_F = 0$. Stable particle like excitations do not exist in conformal theories.
- Motivates the study of massive phases where gauge charged particle like excitations appear meaningful.
- Mass term only relevant operator. Phase diagram with two distinct massive phases (sign of deformation) separated by a second order phase transition.
- Bosonic side. Positive mass deformation $m_B^{\rm cri} > 0$. $SU(N_B)$ spins in the paramagnetic phase, gauged. Elementary excitations the $SU(N_B)_{k_B}$ spins created by the boson ϕ^a.
- Negative mass deformation $m_B^{\rm cri} < 0$. CS gauged $SU(N_B)_{k_B}$ spins in the ferromagnetic phase. Higgs phenomenon. Use unitary gauge to put ϕ in N_B^{th} direction. ϕ degrees of freedom eaten up. $SU(N_B - 1)_{k_B}$ CS gauge fields coupled to a massive fundamental W_μ boson plus massve neutral Z_μ boson. 'Vector Excitations'.

RB and CF theory: Phases 2

- $m_B^{\rm cri} > 0$ maps to $m_F k_F > 0$. Two phases, massive fermions with masses of opposite signs. Low energy theories topological.

$$U(N_F)_{(k_F,k_F)} \leftrightarrow SU(N_B)_{k_B}$$

$$U(N_F)_{(\tilde{k}_F,\tilde{k}_F)} \leftrightarrow SU(N_B-1)_{k_B}$$

$$\tilde{k}_F = \mathrm{sgn}(k_F)(|k_F|-1), \quad \leftrightarrow \quad \text{means level rank dual to}$$

- Excitations on both signs are the elementary fermions.
- How do the elementary charged excitations map across duality?
- Claim. First phase the fermions map to the elementrary bosonic spins. Second phase the fermions map to the W_μ bosons.

Matching spins

- Puzzle. ϕ W_μ and ψ excitations appear to have spin 0, $\mathrm{sgn}(k_B)$ and $\frac{m_F}{2}$ respectively. Does this falsify matching?
- Answer. Intrinsic (or classical) spins are additively renormalized by a statistical Chern Simons (analogy $E \times B$) contribution. $s_{stat} = \frac{c_2(R)}{2\kappa}$. Physical requirement

$$s^B_{intrinsic} + s^B_{stat} = s^F_{intrinsic} + s^F_{stat} \tag{13}$$

- It turns out (group theory)

$$s^F_{stat} - s^B_{stat} = \frac{\mathrm{sgn}(k_B)}{2} = -\frac{\mathrm{sgn}(k_F)}{2}$$

- Follows that (13) works provided

$$s^B_{intrinsic} = \frac{1}{2}\left(\mathrm{sgn}(m_F) - \mathrm{sgn}(k_F)\right).$$

But easy to see its true in both phases.

Excitations: number of components

- There is an obvious issue with the discussion of the last 5 slides. How can (for instance) thhe ϕ particles map to ψ particles in the unHiggsed phase when there are N_B ϕ particles but N_F ψ particles?
- One - perhaps cop out - answer to this question is that CS theories on R^2 need to be carefully defined (what are the boundary conditions on gauge fields at infinity? Are there WZW type edge modes on $\mathcal{I}^+$ and $\mathcal{I}^-$). A safe (if bit boring) way to define the theory is by regarding R^2 as S^2 in the limit of an infinitely large radius.
- With this definition single particle states dont exist. Gauss law. Contradiction disappears at level of states with few particles. One might, however, suspect that the contradiction will return when the number of particles becomes much larger than N_B or N_F. Will return when we quantitatively study thermodynamics on S^2.

Large N

- We would like to study these matter Chern Simons theories more quantitatively. But they are complicated quantum field theories. How do we do this?
- Well known that 'vector' like large N limits are easy to solve, but 'matrix' like large N limits are typically intractable. Large N $SU(N)$ gauge theories always have gauge bosons which are matrix like fields. Typically hard to solve.
- Exceptions. Pure Chern Simons theory in $d = 3$ or pure YM in $d = 2$. As we have reviewed above, solvable at finite N. So also at large N. Price you pay for this: the theory has no local degrees of feedom.
- Now consider CS theories coupled to matter in the fundamental rep. Now genuine QFT. Realized in 2011 theory still solvable at large N. Rest of the talk: will review some of the lessons learnt from this study.

Large N: List of results

The computations that have successfully been performed in the large N limit are

- Existence and StabilityBeta function and Quantum Effective action of RB and CF theories.
- Correlators and duality2, 3 and 4 point functions of gauge invariant opearators at conformal points in these theories.
- Hilbert Space and dualityThe computation of the S^2 thermal partition function of these theories
- Statistics and Crossing S matrices
- Duality from SusyOne Boson and one Fermion
- Quantum Hall?The study of these theories in a uniform magnetic field as a function of chemical potential

Beta function for x_6

The RG flows of x_6 can be plotted as follows

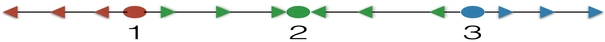


Figure: The points 2 and 1 coincide at $\lambda_B = 0$. They split up at small λ_B. At $\lambda_F = 0$, the point 2 is exactly centred between 1 and 3.

Stability of Vacuum for RB and CF theories

- Since ϕ has a floppy potential, there is a chance that the gauge invariant operator $\bar{\phi}\phi$ develops a quantum effective potential with run away behaviour.
- To test for this we simply compute the exact quantum effective potential for $\bar{\phi}\phi$. Have done this computation at leading order in N_B but for all λ_B.

Quantum Effective Potential: Result

- Classically

$$U_(\bar{\phi}\phi) = m_B^2\bar{\phi}\phi + \frac{4\pi b_4}{\kappa_B}(\bar{\phi}\phi)^2 + \frac{(2\pi)^2}{\kappa_B^2}\left(x_6^B + 1\right)(\bar{\phi}\phi)^3$$

- Quantum mechanically we find

$$U_{\text{eff}}(\bar{\phi}\phi) = \begin{cases} m_B^2\bar{\phi}\phi + + \frac{4\pi b_4}{\kappa_B}(\bar{\phi}\phi)^2 + \frac{(2\pi)^2}{\kappa_B^2}(x_6 - \theta_2)(\bar{\phi}\phi)^3 & \text{for } \sigma_B < 0 \,, \\ m_B^2\bar{\phi}\phi + + \frac{4\pi b_4}{\kappa_B}(\bar{\phi}\phi)^2 + \frac{(2\pi)^2}{\kappa_B^2}(x_6 - \theta_1)(\bar{\phi}\phi)^3 & \text{for } \sigma_B > 0 \,, \end{cases} \tag{14}$$

- The constants ϕ_1 and ϕ_2 are given by

$$\phi_1 = \frac{4}{3}\left(\frac{1}{(2-|\lambda_B|)^2} - 1\right) \,, \quad \phi_2 = \frac{4}{3}\left(\frac{1}{\lambda_B^2} - 1\right) \,. \tag{15}$$

Quantum Effective potential: boundedness

- Classically $\bar{\phi}\phi$ takes values from 0 to ∞. Quanum mechanically the operator is defined with a subtraction and the renormalized operator varies from $-\infty$ to ∞.
- Potential bounded from below only when

$$\phi_1 < x_6 < \phi_2 \quad (16)$$

Range non empty because $\phi_1 < \phi_2$.

- Effective potential for is unbounded from below at large negative values of $\bar{\phi}\phi$ when x_6 is too large and from unbounded from below at large negative positive values of $\bar{\phi}\phi$ when x_6 is too small. However there is an interval x^6 whithin which there is no run away direction.

Phase Diagram

Turns out FP 1 is in the stable range at small λ_B but not at large λ_B. FP 3 stable neither at small nor large λ_B. FP 2 is stable at both small and large λ_B. Phase diagram for FP 2 takes following form (quantiatively known as function of x_6 which itself is unknown). Space of phases topologically a circle. Single second order and sinlge first order phase transition between un Higgsed and Higgsed phases massive phases. 2nd order phase transition governed by quasi fermionic theory.

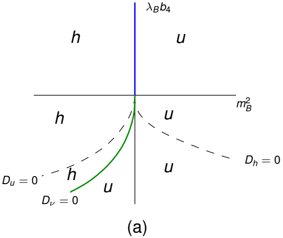


Figure: Blue curve= second order phase transition. Green curve first order phase transition. Two curves at different values of x_6.

Conformal limit 1

- At $m_F^{\text{reg}} = 0$ and $m_B^{\text{cri}} = 0$ the CB and RF theories are conformal. Similarly for the RB and CF theories with all massive parameters set to zero.
- Incredibly simple spectrum of single trace operators. Made up entirely of the traceless symmetric 'currents'

$$J^s_{\mu_1 \dots \mu_s}, \quad s = 0, 1 \dots \infty$$

Schematically

$$J^s = \sum \partial \dots \partial \bar{\phi} \partial \dots \partial \phi, \qquad J^s = \sum \partial \dots \partial \bar{\psi} \partial \dots \partial \psi$$

Turns out

$$\Delta(J^0) = 2 + \frac{\delta^0(\lambda)}{N} \quad \Delta(J^s) = s + 1 + \frac{\delta^s(\lambda)}{N} \quad (s \geq 1)$$

- $\delta^s(\lambda)$ known for $s \geq 1$. Conjectured for $s = 0$.
- Note: gauge invariant operator spectrum also includes $SU(N_B)$ baryons dual to $U(N_F)$ monopoles. Both op dimensions $\sim N$. Much less studied at large N.

‘Quasi Fermionic’: Conformal limit 2

- Obey the nonlinear current algebra (which captures weak breaking of higher spins symmetry)

$$\partial . J^s = \frac{1}{N} \sum c_{s,s_1,s_2}(\lambda) J^{s_1} J^{s_2} + \frac{1}{N^2} \sum c_{s,s_1,s_2,s_3}(\lambda) J^{s_1} J^{s_2} J^{s_3}$$

- All coefficients known. Partial large N eoms (fewer equations than variables). Full (nonlocal) large N eoms from conjectured holographic AdS_4 Vasiliev dual.
- The three point functions $\langle J^{s_1} J^{s_2} J^{s_3} \rangle$ are explicitly known as functions of λ at leading order in the large N limit. Some (very few) four point functions also explicitly known. Intriguing interplay with inversion formula.
- All these quantities are known in explicit detail independently for the regular fermion and critical boson CFTs. Match perfectly under conjectured duality map. Powerful calculational evidence for duality at N.

Properties of J_0

- As one example I present the two point functions of the scalar dimension 2 opearators, $\tilde{J}_0 = \sigma$ maps to $J_0^F = 4\pi \frac{\bar{\psi}\psi}{\kappa_F}$, in momentum space
- At leading order in large N

$$\langle \tilde{J}_0(q)\tilde{J}_0(-q')\rangle = (2\pi)^3\delta^3(q-q') \left(\frac{-4\pi|q|}{\kappa_B}\right)\frac{1}{\tan(\frac{\pi\lambda_B}{2})}.$$
$$\langle J_0^F(q)J_0^F(-q')\rangle = (2\pi)^3\delta^3(q-q')\left(\frac{-4\pi|q|}{\kappa_F}\right)\tan\left(\frac{\pi\lambda_F}{2}\right), \tag{17}$$

- Two point functions match under duality. Similar simple results are available for all 2 and 3 point functions. All results agree perfectly under duality.

Thermal partition function

- I will now present, and then interpret in great detail, the results of thermal partition functions of our theories at large N.
- The aim of this exercise is to find a simple effective description of the Hilbert Space of large N Matter Chern Simons theories, and to answer, in part, the confusion of 'number of components' we encountered above.
- Most of the analysis of the next few slides is taken from work, yet to be published, in collaboration with A. Mishra, N. Prabhakar and T. Sharma

$S^2 \times S^1$ Partition Function: Structure

- In the limit $N \to \infty$, $k \to \infty$, $\mathcal{V}_2 \to \infty$, with all ratios of these quantities, as well as the temperature T, chemical potential μ, and all masses and couplings held fixed, the S^2 times S^1 partition function $\mathcal{Z}_{S^2\times S^1}$ is given by an integral over the unitary matrix U, the zero mode of the holonomy around the time circle
- $$\mathcal{Z}_{S^2\times S^1} = \int [dU]_{\rm CS}\; e^{-\mathcal{V}_2 v[\rho]}\ , \qquad (18)$$
- where $[dU]_{CS}$ is the usual Haar measure subject to the constraint

$$\rho(\alpha) \leq \frac{1}{2\pi|\lambda|}\ , \qquad \rho(\alpha) = \lim_{N\to\infty} \frac{1}{N}\sum_{j=1}^{N} \delta(\alpha - \alpha_j)\ . \qquad (19)$$

(Recall we encountered $[DU]_{CS}$ early in this talk).

- $v[\rho]$ in (18) above depends on details of the matter Chern Simons theory under study. In this talk we focus on theories whose matter content includes a single fundamental fermion or boson.
- For each theory, computation reveals that the quantity $v[\rho]$ is given by the extremization, over two auxilliary variables, of a so called off shell free energy. Schematically

$$v[\rho] = \mathrm{min}_{\zeta_i} F(\zeta_i)$$

- Next few slides: present the explicit expressions for the offshell free energies for each of the four theories described above.

$S^2 \times S^1$ Partition Function: RF offshell free energy,

- In the case of the Regular Fermion theory, $F[\zeta_i]$ is given by

$$F_{\rm RF}(\hat{c}_F, \tilde{\mathcal{C}}) = \frac{N_F T^2}{6\pi}\Bigg[-8\lambda_F^2\tilde{\mathcal{C}}^3 - 3\tilde{\mathcal{C}}\left(\hat{c}_F^2 - \left(2\lambda_F\tilde{\mathcal{C}} + \hat{m}_F\right)^2\right)$$
$$-6\lambda_F\hat{m}_F\tilde{\mathcal{C}}^2 + \hat{c}_F^3$$
$$-3\int_{\hat{c}_F}^{\infty} d\hat{\epsilon}\,\hat{\epsilon}\int_{-\pi}^{\pi} d\alpha\,\rho_F(\alpha)\left(\log\left(1+e^{-\hat{\epsilon}-\hat{\mu}-\mathrm{i}\alpha}\right) + \log\left(1+e^{-\hat{\epsilon}+\hat{\mu}+\mathrm{i}\alpha}\right)\right)\Bigg] . \tag{20}$$

Here N_F is the gauge rank, λ_F is the t'Hooft coupling, $\tilde{\mathcal{C}}$ and $\hat{c}_F$ are auxilliary variables that have to be extremized over ($\hat{c}_F$ has the interpretation of the thermal mass in units of temperature) and

$$\hat{\mu} = \frac{\mu}{T}, \quad \hat{m}_F = \frac{m_F}{T}$$

- Note that only the third line of (22) depends on $\rho(\theta)$ or μ

$S^2 \times S^1$ Partition Function: CB offshell free energy

- In the case of the Critical Boson theory, $F[\zeta_i]$ is given by

$$
\begin{aligned}
& F_{\rm CB}(c_B, \tilde{\mathcal{S}}) \\
& = \frac{N_B T^2}{6\pi}\Bigg[\frac{3}{2}\hat{c}_B^2 \hat{m}_B^{\rm cri} - 4\lambda_B^2\left(\tilde{\mathcal{S}} - \tfrac{1}{2}\hat{m}_B^{\rm cri}\right)^3 + 6|\lambda_B|\hat{c}_B\left(\tilde{\mathcal{S}} - \tfrac{1}{2}\hat{m}_B^{\rm cri}\right)^2 - \hat{c}_B^3 \\
& + 3\int_{\hat{c}_B}^{\infty} d\hat{\mathrm{e}}\ \hat{\mathrm{e}} \int_{-\pi}^{\pi} d\alpha\ \rho_B(\alpha)\left(\log(1 - e^{-\hat{\mathrm{e}}+\hat{\mu}+i\alpha}) + \log(1 - e^{-\hat{\mathrm{e}}-\hat{\mu}-i\alpha})\right) \\
& \qquad - \Theta(|\mu| - c_B)\frac{(|\hat{\mu}| - \hat{c}_B)^2(|\hat{\mu}| + 2\hat{c}_B)}{2|\lambda_B|}\Bigg]. \qquad (21)
\end{aligned}
$$

Here N_B and λ_B are the gauge rank and t' Hooft coupling, $\tilde{\mathcal{S}}$ and $\hat{c}_B$ are auxilliary variables that have to be extremized over ($\hat{c}_B$ has the interpretation of the thermal mass in units of temperature) and

$$\hat{\mu} = \frac{\mu}{T}, \qquad \hat{m}_B^{\rm cri} = \frac{m_B^{\rm cri}}{T},$$

- Note that only last two lines of (22) depends on $\rho(\theta)$ or μ

$S^2 \times S^1$ Partition Function: CF offshell free energy,

- In the case of the Regular Fermion theory, $F[\zeta_i]$ is given by

$$F_{\rm CF}(\hat{c}_F, \tilde{\mathcal{C}}, \zeta_F) = \frac{N_F T^2}{6\pi}\Bigg[-8\lambda_F^2\tilde{\mathcal{C}}^3 - 3\tilde{\mathcal{C}}\left(\hat{c}_F^2 - \left(2\lambda_F\tilde{\mathcal{C}} - \frac{4\pi\hat{\zeta}_F}{\kappa_F}\right)^2\right) + 6\lambda_F\tilde{\mathcal{C}}^2\left(\frac{4\pi\hat{\zeta}_F}{\kappa_F}\right) + 3\left(\frac{\hat{y}_2^2}{2\lambda_F}\frac{4\pi\hat{\zeta}_F}{\kappa_F} - \frac{\hat{y}_4}{2\lambda_F}\left(\frac{4\pi\hat{\zeta}_F}{\kappa_F}\right)^2 + \frac{x_6^F}{8\lambda_F}\left(\frac{4\pi\hat{\zeta}_F}{\kappa_F}\right)^3\right) + \hat{c}_F^3 - 3\int_{\hat{c}_F}^{\infty} d\hat{\epsilon}\,\hat{\epsilon}\int_{-\pi}^{\pi} d\alpha\,\rho_F(\alpha)\left(\log\left(1+e^{-\hat{\epsilon}-\hat{\mu}-i\alpha}\right) + \log\left(1+e^{-\hat{\epsilon}+\hat{\mu}+i\alpha}\right)\right)\Bigg]. \tag{22}$$

Here

$$\hat{\zeta}_F = \frac{\zeta_F}{T}, \quad \hat{y}_2 = \frac{y_2}{T}, \quad \hat{y}_4 = \frac{y_4}{T}$$

- Note that only the third line of (22) depends on $\rho(\theta)$ or μ

$S^2 \times S^1$ Partition Function: RB offshell free energy

- In the case of the Critical Boson theory, $F[\zeta_i]$ is given by

$$
\begin{aligned}
&F_{\rm RB}(c_B, \tilde{\mathcal{S}}, \sigma) \\
&= \frac{N_B T^2}{6\pi}\Bigg[-3\hat{c}_B^2\hat{\sigma}_B + \lambda_B^2\hat{\sigma}_B^3 - 4\lambda_B^2(\tilde{\mathcal{S}} + \hat{\sigma}_B)^3 + 6|\lambda_B|\hat{c}_B(\tilde{\mathcal{S}} + \hat{\sigma}_B)^2 \\
&\qquad + 3(\hat{m}_B^2\hat{\sigma}_B + 2\lambda_B\hat{b}_4\hat{\sigma}_B^2 + (x_6+1)\lambda_B^2\hat{\sigma}_B^3) - \hat{c}_B^3 \\
&+ 3\int_{\hat{c}_B}^{\infty} d\hat{\rm e}\, \hat{\rm e} \int_{-\pi}^{\pi} d\alpha\, \rho_B(\alpha)\, \left(\log(1 - e^{-\hat{\rm e}+\hat{\mu}+{\rm i}\alpha}) + \log(1 - e^{-\hat{\rm e}-\hat{\mu}-{\rm i}\alpha})\right) \\
&\qquad - \Theta(|\mu| - c_B)\frac{(|\hat{\mu}| - \hat{c}_B)^2(|\hat{\mu}| + 2\hat{c}_B)}{2|\lambda_B|}\Bigg]\,. \qquad (23)
\end{aligned}
$$

Here

$$\hat{b}_4 = \frac{b_4}{T}, \quad \hat{m}_B = \frac{m_B}{T}, \quad \hat{\sigma}_B = \frac{\sigma_B}{T}$$

- Note that only last two lines of (22) depends on $\rho(\theta)$ or μ

$S^2 \times S^1$ Partition Function: Interchanging Orders

- In summary, for all four theories

$$\mathcal{Z}_{S^2\times S^1} = \int [dU]_{\rm CS} \, \min_{\zeta_i} \left[e^{-\mathcal{V}_2 F(\zeta_i,\rho)} \right] \, , \qquad (24)$$

- At leading order in the large N limit, the integral over U reduces to a saddle point extremization over $\rho(\theta)$. The extremization over ζ_i and $\rho(\theta)$ can be performed in any order, so (24) can be rewritten as

$$\mathcal{Z}_{S^2\times S^1} = \min_{\zeta_i} \left[\int [dU]_{\rm CS} e^{-\mathcal{V}_2 F(\zeta_i,\rho)} \right] \, , \qquad (25)$$

- To evaluate $\mathcal{Z}_{S^2\times S^1}$ we must thus
  Step (1): Evaluate $I(\zeta_i) = \int [dU]_{\rm CS} e^{-\mathcal{V}_2 F(\zeta_i,\rho)}$ at fixed ζ_i.
  Step (2): Extremize $I(\zeta_i)$ over ζ_i.
- Step I is universal (indep of details of contact interactions). Step 2 is non universal and accounts for these interactions.

The computation for Step 1

- In Step 1 for fermions we are required to evaluate

$$I_F = \int [dU]_{\rm CS} \; Z_{NS}^F(U) \; ,$$

$$Z_{NS}^F(U) = e^{\frac{N_F T^2 \mathcal{V}_2}{2\pi} \left[\int_{\hat{c}_F}^{\infty} d\hat{\epsilon} \; \hat{\epsilon} \int_{-\pi}^{\pi} d\alpha \, \rho_F(\alpha) \left(\log\left(1+e^{-\hat{\epsilon}-\hat{\mu}-i\alpha}\right) + \log\left(1+e^{-\hat{\epsilon}+\hat{\mu}+i\alpha}\right) \right) \right]} . \tag{26}$$

The exponent of $Z_{NS}^F(U)$ includes all terms in $F^F(\zeta_i)$ that depend on either ρ or μ.

- Similarly, in Step 1 for bosons we are required to evaluate

$$I_B = \int [dU]_{\rm CS} \; Z_{NS}^B(U)$$

$$Z_{NS}^B(U) = e^{-\frac{N_B T^2 \mathcal{V}_2}{2\pi} \int_{\hat{c}_B}^{\infty} d\hat{\epsilon} \; \hat{\epsilon} \int_{-\pi}^{\pi} d\alpha \, \rho_F(\alpha) \left(\log\left(1-e^{-\hat{\epsilon}-\hat{\mu}-i\alpha}\right) + \log\left(1-e^{-\hat{\epsilon}+\hat{\mu}+i\alpha}\right) \right)} \times$$
$$e^{-\frac{N_B T^2 \mathcal{V}_2}{2\pi} \Theta(|\mu| - c_B) \frac{(|\hat{\mu}|-\hat{c}_B)^2(|\hat{\mu}|+2\hat{c}_B)}{6|\lambda_B|}} . \tag{27}$$

Once again the exponent of $Z_B(U)$ includes all terms in $F^B(\zeta_i)$ that depend on either ρ or μ. Note both I_F and I_B depend on no contact data and no auxilliary parameters other than c_F.

Partition function in terms of I_B and I_F

- Let us suppose that we have mananged to evaluate $I_B(c_B)$ and $I_F(c_F)$. The final free energy is then given by a relatively simple extremization. For instance, in the case of the RB theory
- $$Z_{RB} = \left[e^{-N_B \mathcal{V}_2 \beta \left(\frac{1}{2\pi}(m_B^2 - c_B^2)\sigma + \frac{b_4 \lambda_B}{\pi}\sigma^2 + \frac{\lambda_B^2}{2\pi}(x_6^B + \frac{4}{3})\sigma^3 - 4\lambda_B^2(\mathcal{S}+\sigma_B)^3 + 6|\lambda_B| c_B (\mathcal{S}+\sigma_B)^2 \right)} I_B(c_B) \right]_{c_B, \sigma, \mathcal{S}} \tag{28}$$
- Where, in our notation, $[A(\alpha)]_\alpha$ denotes the extremization of $A(\alpha)$ over the variables α.
- Extremizing (29) over $\mathcal{S}$ gives the equation

$$(\mathcal{S} + \sigma_B)\left(|\lambda_B|(\mathcal{S} + \sigma_B) - c_B\right) = 0.$$

This equation has two solutions for $\mathcal{S}$; the one that describes the so called unHiggsed phase is $\mathcal{S} = \sigma_B$.

Gauged versus ungauged: I

- In the unHiggsed phase (29) simplifies to

$$Z_{RB} = \left[e^{-N_B \mathcal{V}_2 \beta \left(\frac{1}{2\pi}(m_B^2 - c_B^2)\sigma + \frac{b_4 \lambda_B}{\pi}\sigma^2 + \frac{\lambda_B^2}{2\pi}(x_6^B + \frac{4}{3})\sigma^3 \right)} I_B(c_B) \right]_{c_B, \sigma} \quad (29)$$

- Now notice that the ungauged scalar large N matter theory

$$S_{\rm RB}^{eff}[\phi] = \int d^3x \left((\partial_\mu \bar{\phi})(\partial^\mu \phi) + m_B^2 \bar{\phi}\phi \right) + \frac{4\pi b_4}{\kappa_B}(\bar{\phi}\phi)^2 + \frac{(2\pi)^2}{\kappa_B^2}(x_6^B + \frac{4}{3})(\bar{\phi}\phi)^3 \Big) \quad (30)$$

can be recast, with the aid Lagrange Multipliers c_B and σ, as

$$S = \int d^3x \left[(\partial_\mu \bar{\phi} \partial^\mu \phi) + c_B^2(\bar{\phi}\phi) + \frac{N_B}{2\pi}(m_B^2 - c_B^2)\sigma + \frac{N_B \tilde{b}_4 \lambda_B}{\pi}\sigma^2 + \frac{N_B \lambda_B^2}{2\pi}(x_6^B + \frac{4}{3})\sigma^3 \right]. \quad (31)$$

It follows that the thermal partition function of this theory is

$$Z^{eff} = \left[e^{-N_B \mathcal{V}_2 \beta \left(\frac{1}{2\pi}(m_B^2 - c_B^2)\sigma + \frac{\tilde{b}_4 \lambda_B}{\pi}\sigma^2 + \frac{\lambda_B^2}{2\pi}(x_6^B + \frac{4}{3})\sigma^3 \right)} \mathrm{tr}_{\mathcal{H}_{c_B}^{Fock}} \left(e^{-\beta H_{c_B}} \right) \right]_{\{c_B, \sigma\}} \quad (32)$$

where $\mathcal{H}_{c_B}^{Fock}$ is the free Fock space of a scalar of mass c_B

Gauged versus ungauged: 2

- Comparing (29) and (32) we see that the partition function of the CB theory has exactly the same form as the partition function of the ungauged theory (30) but with the replacement

$$\mathrm{tr}_{\mathcal{H}^{Fock}_{c_B}} \left(e^{-\beta H_{c_B}} \right) \to I_B(c_B)$$

- (32) expresses the fact that the Hilbert Space of a vector like large N matter theory is a Fock Space with all states receiving forward scattering or mean field type renormalizations.
- We have thus discovered that the Hilbert Space of Matter Chern Simons theories is that of an 'I_B or I_F system corrected by forward scattering interactions. Over the next 25 or so slides we focus our attention on thoroughly understanding I_B and I_F, returning to the full partition function only at the very end of the talk.

Simplification of I_F

- Recall that

$$I_F = \int [dU]_{\rm CS}\; Z^F_{NS}(U)\ ,$$

$$Z^F_{NS}(U) = e^{\frac{N_F T^2 \mathcal{V}_2}{2\pi}\left[\int_{\hat{c}_F}^{\infty} d\hat{\rm e}\ \hat{\rm e} \int_{-\pi}^{\pi} d\alpha\, \rho_F(\alpha)\left(\log\left(1+e^{-\hat{\rm e}-\hat{\mu}-{\rm i}\alpha}\right)+\log\left(1+e^{-\hat{\rm e}+\hat{\mu}+{\rm i}\alpha}\right)\right)\right]}. \tag{33}$$

- It is not difficult to verify

$$Z^F_{NS}(U) = {\rm Tr}_{H_{NS}}\left(U e^{-\beta(H-\mu Q)}\right)\ , \tag{34}$$

Where the trace is taken over H_{NS} the free Fock Space of free fermions of mass $c_F = \hat{c}_F T$ propagating on a (very large) S^2.

- Consequently

$$I_F = \int [dU]_{\rm CS}\ {\rm Tr}_{H_{NS}}\left(U e^{-\beta(H-\mu Q)}\right) \tag{35}$$

Simplification of I_B

- Recall that

$$I_B = \int [dU]_{\rm CS}\, Z^B_{NS}(U)$$

$$Z^B_{NS}(U) = e^{-\frac{N_B T^2 \mathcal{V}_2}{2\pi} \int_{\hat{c}_B}^{\infty} d\hat{\epsilon}\ \hat{\epsilon} \int_{-\pi}^{\pi} d\alpha\, \rho_F(\alpha) \left(\log\left(1-e^{-\hat{\epsilon}-\hat{\mu}-i\alpha}\right)+\log\left(1-e^{-\hat{\epsilon}+\hat{\mu}+i\alpha}\right)\right)} \times$$
$$e^{-\frac{N_B T^2 \mathcal{V}_2}{2\pi} \Theta(|\mu|-c_B)\ \frac{(|\hat{\mu}|-\hat{c}_B)^2(|\hat{\mu}|+2\hat{c}_B)}{6|\lambda_B|}}\ . \tag{36}$$

- This can be rewritten as

$$I_B = \int [dU]_{\rm CS}\, {\rm Tr}_{H_{NS}} \left[\left(U e^{-\beta(H-\mu Q)}\right) \left(e^{-\frac{N_B T^2 \mathcal{V}_2}{2\pi} \Theta(|\mu|-c_B)\ \frac{(|\hat{\mu}|-\hat{c}_B)^2(|\hat{\mu}|+2\hat{c}_B)}{6|\lambda_B|}}\right)\right] \tag{37}$$

The second term in the trace is a new element missing in the case of fermions.

Toy Model for I_F

- Let us momentarily consider a quantity similar to I_F.

$$\tilde{I}_F = \int [dU]\, \mathrm{Tr}_{H_{NS}} \left(U e^{-\beta(H-\mu Q)} \right) \tag{38}$$

where dU is the usual unmodified Haar measure.

- $\tilde{I}_F$ has a simple and familiar Hilbert Space interpretation. The free fermion Fock Space can be decomposed into a sum over irreducible representations of $U(N_F)$. The integral over U in (39) simply projects this Fock Space onto the $U(N_F)$ singlets.
- In other words

$$\tilde{I}_F = \mathrm{Tr}_{H_{Sing}} \left(e^{-\beta(H-\mu Q)} \right) \tag{39}$$

where H_{Sing} is the projection H_{NS} to $U(N_F)$ singlets.

- Question: Does I_F have an interpretation similar to $\tilde{I}_F$? And whats the story with I_B?

$[dU]_{CS}$ from conformal blocks

- We have already seen at the beginning of this talk that (36) and (37) have similar interpretations. Recall that, e.g. for the case of Type I theory in the large N limit

$$\frac{1}{\kappa^N}\sum_{\{w_i\}}\prod_{\mathrm{i<j}}|w_i - w_j|^2\prod_{p=1}^{n}\chi_{R_p}(w_i) \to \int [dU]_{CS}\prod_{p=1}^{n}\chi_{R_p}(w_i)$$

- It follows that the quantity I_F above does indeed admit the interpretation we proposed: it is the partition function of the Fock Space restricted to the WZW singlet sector.
- While thats great, it raises the obvious question: what about the Bosons? What is the origin of the extra factor for bosons in step 1? We now turn to this question

$\theta(\mu - c_B)$... from conformal blocks

- Consider a Type I $U(N)$ Chern Simons theory coupled to fundamental bosons on S^2. Let a parameterize the positive energy solutions of the Klein Gordon equation, with mass c_B, on S^2. Clearly the U twisted partition function over the free bosonic Fock Space is given by a product of partition functions, one for every free particle state
- Explicitly

$$\mathrm{Tr}\left(Ue^{-\beta(H-\mu Q)}\right) = \prod_a \left[\left(\prod_{i_a=1}^{N_B} \frac{1}{1-e^{-\beta(E_a-\mu)}w_{i_a}}\right)\left(\prod_{i_a=1}^{N_B} \frac{1}{1-e^{-\beta(E_a+\mu)}w_{i_a}^*}\right)\right] \tag{40}$$

- Note that

$$\frac{1}{1-e^{-\beta(E_a-\mu)}w_{i_a}} = \sum_{n=0}^{\infty} e^{-n\beta(E_a-\mu)}\chi_n^S(U) \tag{41}$$

Truncation for Bosons

- Now the partition function of the bosonic Fock space restricted to WZW singlets is *not* given simply by

$$\frac{1}{\kappa^N}\sum_{\{w_i\}}\prod_{\mathrm{i<j}}|w_i - w_j|^2 \mathrm{Tr}\left(Ue^{-\beta(H-\mu Q)}\right) \tag{42}$$

- Terms with $n > k_B$ in (41) are non integrable insertions. Conformal blocks involving such insertions should vanish. (42) does not correctly account for this fact (see comment in red before (4)), which must thus be inserted by hand.
- The correct truncation of the free boson Fock space to WZW singlets is given by

$$I_B^l = \frac{1}{\kappa^{N_B}}\sum_{\{z_i\}}\prod_{\text{pairs}}|w_i - w_j|^2 \prod_a \left[\left(\prod_{i_a=1}^{N_B}\frac{1}{1-e^{-\beta(E_a-\mu)}w_{i_a}}\right)\Bigg|_{k_B} \left(\prod_{i_a=1}^{N_B}\frac{1}{1-e^{-\beta(E_a+\mu)}w^*_{i_a}}\right)\Bigg|_{k_B}\right]$$

Implication of the Truncation

- It is not difficult to prove that

$$Q(y) = \prod_{i=1}^{N_B} \frac{1}{1 - w_i y}\Bigg|_{k_B} = \left(1 + (-1)^N y^\kappa\right) \prod_{i=1}^{N_B} \frac{1}{1 - z_i y} \qquad (44)$$
$$= \exp\left(-\operatorname{tr}\ln\left(1 - yU\right) + \ln\left(1 - y^\kappa\right)\right)$$

- In the large N limit it follows that

$$\ln Q(y) = -\operatorname{tr}\ln\left(1 - yU\right) + \kappa\Theta(y - 1)\ln w\,. \qquad (45)$$

- In the physical problem of interest $y = e^{-\beta(E-\mu)}$. The second term in (45) is thus nonzero only for states with $E < \mu$. Such states exist only if $c_B < \mu$. Adding up the contribution of all such states (accounting for the density of states) reproduces the extra term in (37) with all factors.

The Bosonic Exclusion Principle

- We have learnt something important here. In large N matter Chern Simons theories, no single particle bosonic state can be occupied more than k_B times. We call this the 'Bosonic Exclusion Principle'. It is the direct level rank dual of a more obvious result for fermionic theories, namely that no single particle fermionic state can be occupied more than N_F.
- Recall that ordinary free boson theories are ill defined at values of the chemical potential greater than the mass, as all states with energies between the mass and the chemical potential are infinitely occupied in such theories. The bosonic exclusion principle cures this singularity in matter Chern Simons theories, rendering Bosonic theories with chemical potential larger than the mass well defined.

I_F and I_B at large volume

- Let us recap. The partition function of large N matter Chern Simons theories is given by an expression of the same form as their ungauged counterparts, with the replacement of the Fock Space partition function to its WZW singlet projected counterpart.
- It is clear that the WZW singlet constraint has a huge effect on the partition function of the theory at small values of the sphere volume. But one might naively expect its impact to disappear in the large volume limit. This is indeed what happens for the Gauss Law constraint. Interestingly enough this expectation is incorrect.

‘Saddle Point’ at large Volume

- At finite N and k I_B is given by the formula (43) (I_F is given by a similar formula). The key simplification of the large volume limit is that the summation over choices of eigenvalues in that formula is dominated by a single eigenvalue configuration (this is a sort of saddle point approximation for the summation in that formula).
- The eigenvalue configuration that dominates the sum is

$$w_i = e^{i(\alpha_i)},$$

$$\{\alpha_i\} = \frac{2\pi}{\kappa}\left\{ -\frac{N-1}{2}, -\frac{N-3}{2}, -\frac{N-5}{2}, \ldots \frac{N-5}{2}, \frac{N-3}{2}, \frac{N-1}{2} \right\} \tag{46}$$

This configuration is the correct ‘saddle point’ for all cases; the $SU(N)$, Type II and Type I theories and for fermions and bosons. There is a one to one map between integrable representations and the collection of discretized eigenvalues that are summed over in the Verlinde formula. (46) is the eigenvalue configuration that maps to the Identity representation

Factorization at large volume

- At general values of the volume, the formula (43) expresses the partition function as a sum over products. As we have explained, at large volume the sum local to a single term, leaving us with a simple product, one term for each single particle state.

$$I_B \propto \prod_a Z_B^a \bar{Z}_B^a , \quad Z_B^a = \left(\prod_{i=1}^{N_B} \frac{1}{1 - e^{-\beta(E_a - \mu)} w_i} \right) \Bigg|_{k_B} , \qquad \bar{Z}_B^a = \left(\prod_{i=1}^{N_B} \frac{1}{1 - e^{-\beta(E_a + \mu)} w_i} \right) \Bigg|_{k_B} , \tag{47}$$

(the eigenvalues that apear in (47) are those listed on the previous slide)

- $$Z_B^a = \sum_{r=0}^{k_B} \chi_n^S(U)\, e^{-r\beta(E_a - \mu)} = \sum_{r=0}^{k_B} d_n^S\, e^{-r\beta(E_a - \mu)} \tag{48}$$

q numbers and quantum dimensions

- d_n^S is the so called quantum dimension of the n box symmetric representation. As mentioned above it equals the character of this representation evaluated on our special 'saddle point' unitary matrix This connection works for every representation, not justthe completely symmetric representation. Explicitly

$$
\begin{aligned}
d_n^S &= \binom{n}{m}_q = \frac{[n]_q!}{[m]_q![n-m]_q!} \\
[m]_q! &= [1]_q[2]_q \cdots [m]_q \\
[r]_q &= \frac{q^{r/2} - q^{-r/2}}{q^{1/2} - q^{-1/2}}
\end{aligned}
\tag{49}
$$

with $q = e^{\frac{2\pi i}{\kappa}}$

- Similar expressions hold for fermions. Using identities involving q factorials, the bosonic and fermionic expressions can be shown to be level rank dual.

Product but not Free

- Ignoring details, for the moment, an immediately striking aspect of the large volume limit is that the partition factorizes (its a product of partition functions, one for each single particle state).
- This feature may appear to suggest that our system is free in the infinite volume limit (whats going on in one single particle state does not affect the partition function of another single particle state).
- While this suggestion sounds initially reasonable, it is not correct. We can see this by noting that the coefficients of $e^{-\beta(E_n-\mu)}$ in the expansion of Z_n above are not integers. It follows that the different Z_n are not partition functions over independently defined Hilbert Spaces.

Explanation of the Product Structure

- We believe that this product structure is a manifestation of an interesting universality in the fusion rule algebra in the large insertion limit.
- Consider any generic collection of integrable representations of the WZW algebra $R_1 \ldots R_n$.
- Let us now sequentially fuse our representations with each other. Once this process is completed let us suppose we are left with n_{R_i} representations of type R_i for each integrable representation R_i.
- In the limit that n is the largest number in the problem, we believe that

$$\frac{n_{R_i}}{n_{R_j}} = \frac{d_{R_i}}{d_{R_j}}$$

  Independent of the details of the participating representations R_i.

Explanation of factorization

- This universality explains the factorization of our partition functions as follows
- The coefficient of $e^{-n\beta(E_n-\mu)}$ in Z_n is actually proportional to the number of sea particles in the representation conjugate to the n box symmetric representation.
- The conjecture of the previous slide explains why this number is independent of the precise state of the 'sea', explaining why the product structure of single particle
- The universality described in our conjecture is tightly connected to the fact that the unitary matrix U localizes on the same universal matrix in the $\mathcal{V}_2 \to \infty$ limit, independent of the temperature, chemical potential and masses together with the fact that $d_R = \chi_R(U)$.

Implications of Factorization

- Recall that

$$Z_B^a = \sum_{m=0}^{k_B} \binom{N_B}{m}_q e^{-r\beta(E_a-\mu)} \tag{50}$$

- In the limit $\lambda_B \to 0$ $q \to 1$ and

$$\binom{N_B}{m}_q \to \binom{N_B}{m}$$

  Also in this limit k_B so the upper limit on the summation in (50) $\to \infty$. We thus reproduce usual Bose statistics.

- (50) can be thought of as a one parameter deformation of usual Bose thermal 'statistics'; one that changes the details of occupation probabilities at low occupation numbers, and imposes the Bose Exclusion Principle at high occupation numbers.

New Single particle Thermal Statistics

- As a consequence we obtain a one parameter deformation of many of the familiar rules of free statistical physics that we learn about as undergraduates.
- For instance, in the t'Hooft Large N limit we find the following formula for the average occupation number of any given single particle state at temperature T and chemical potential μ

$$\bar{n}_B(\epsilon,\mu) = \frac{1-|\lambda_B|}{2|\lambda_B|} - \frac{1}{\pi|\lambda_B|}\tan^{-1}\left(\frac{e^{\beta(\epsilon-q\mu)}-1}{e^{\beta(\epsilon-q\mu)}+1}\cot\frac{\pi|\lambda_B|}{2}\right),$$

- Generalizing the familiar free boson result

$$\bar{n}_B(\epsilon,\mu) = \frac{1}{e^{\beta(\epsilon-q\mu)}-1}$$

- Similar results apply for fermions and respect duality

Putting it all together: 1

- Armed with our thorough physical understanding of I_B and I_F, we now return to the comparison of the formulae

$$Z_{RB} = \left[e^{-N_B \mathcal{V}_2 \beta \left(\frac{1}{2\pi}(m_B^2 - c_B^2)\sigma + \frac{b_4 \lambda_B}{\pi}\sigma^2 + \frac{\lambda_B^2}{2\pi}(x_6^B + \frac{4}{3})\sigma^3 \right)} \; I_B(c_B) \right]_{c_B, \sigma, \mathcal{S}}$$

for the RB theory and

$$Z^{eff} = \left[e^{-N_B \mathcal{V}_2 \beta \left(\frac{1}{2\pi}(m_B^2 - c_B^2)\sigma + \frac{\tilde{b}_4 \lambda_B}{\pi}\sigma^2 + \frac{\lambda_B^2}{2\pi}(x_6^B + \frac{4}{3})\sigma^3 \right)} \; \mathrm{tr}_{\mathcal{H}^{Fock}_{c_B}} \left(e^{-\beta H_{c_B}} \right) \right]$$

for the related effective ungauged scalar theory

Putting it all together: 2

- We see that the full partition function of the RB theory is precisely the partition function of Fock Space projected down to WZW singlets, corrected by the mean field or forward scattering interactions of the ungauged effective scalar theory.
- Starting with the RB theory it is possible to integrate out the gauge fields (this is how we originally solved these theories at large N). However this process yields a highly nonlocal (and at first sight extremely ugly) scalar effective action .
- The fact that this highly nonlocal theory was solvable at large N always suggested that the effective scalar theory was simpler than it appeared. We now see in what sense that is true. Atleast as far as the partition function is concerned, the entire effect of the ugly looking nonlocal interactions is to impose (nonlocal but beautiful) WZW constraint on free fock space. The apparent ugliness probably had to do with our choice of gauge.

‘Quasi Fermionic’: Scattering of Excitations

- The (non gauge invariant) two point functions $\langle \phi \bar{\phi} \rangle$,, $\langle \bar{W}_\mu W_\nu \rangle$ and $\langle \psi \bar{\psi} \rangle$ have all been computed as functions of λ at leading order in large N. Answers contain (gauge invariant) poles. Pole masses of proposed dual excitations match across duality.
- Worry: how about the apparent mismatch of statistics between Bosons and Fermions? To address this we computed the exact (large N) S matrix of fermionic and bosonic excitations - defined by the sum over diagrams and implementing the LSZ procedure.
- Scattering has the following 4 inequivalent channels.

$$FF \to FF\,(sym), \quad FF \to FF\ as, \quad FA \to FA\,(adj), \quad FA \to FA\,(sing)$$

‘Quasi Fermionic’: Statistics

- Direct computation.

$$S^{Boson} = a\mathcal{P}_{sym} + b\mathcal{P}_{as}, \quad S^{Fermion} = b\mathcal{P}_{sym} + a\mathcal{P}_{as}.$$

- Lesson: the difference between Bose and Fermi statistics is compensated for by the fact that the duality map is not straightforward in gauge indices. A state symmetric in gauge indices maps under duality to a state antisymmetric in gauge indices. Allows for the two states to have identical statistics in ‘non hidden’ indices.
- This is (in my opinion) an important qualitative insight into how this duality works at the partonic level. Gives a physical ‘explanation’ of the well known map between representations of Wilson loops under level rank duality.

Anyons and Statistics

- Note that the resolution of the Bose Fermi dichotomy is not that the scattering particles are 'neither bosons nor fermion but anyons. To see clearly consider the non relativistic limit.
- Problem equivalent to scattterig of a non relativistic particle of a flux tube of magnitude

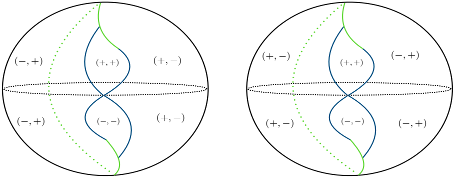
$$\nu = \frac{c_2(R_1) + c_2(R_2) - c_2(R)}{\kappa}.$$

- $2\pi\nu$ is the effective anyonic phase seen by the S matix. Turns out $\nu = \mathcal{O}(1/N)$ in both $FF \to FF$ channels. Scattering of two identical fundamentals effectively non anyonic.
- Infact at large N the only effectively anyonic scattering channel is $AF \to AF$ in the singlet sector. There is an interesting related issue - the usual rules of S matrix crossing symmetry are violated in this sector. No time to discuss. Prob related to subtleties at infinity (see below).

One Boson one Fermion and Susy

- Theories describing the interaction of one fundamental boson with one fundamental fermion.
- Lagrangian long and complicated. 3 highly relevant parameters. 4 aproximately marignal parameters. Marginal at large N. Flow at finite N. Full space of flows and fixed points not worked out. Definitely includes $N = 2$ fixed point.
- The $N = 2$ susy theory has only 3 relevant operators. Elaborate 2 dimensional phase diagram. Recently fully worked out. Generic low energy behaviour massive (4 distinct massive phases). 2 parameter fine tuning allows for massless low energy dynamics, including quasi bosonic CFT dynamics as well as fixed points governing the interaction of one quasi fermionic and one quasi bosonic theory.

Schematic Phase diagram



$0 < \lambda < \frac{1}{2}$ $\frac{1}{2} < \lambda < 1$

Figure: Schematic of the phase structure of the $N = 2$ theory. This phase diagram is known in full quantitative detail at large N.

- The main point of interest here is that the $\mathcal{N} = 2$ susy theory is known to enjoy invariance under a version of Giveon Kutasov type duality (Benini et al duality). Fantastic evidence that this duality persists for finite N.
- The phase diagram above demonstrates that Bose Fermi duality (both quasi bosonic and quasi fermionic) can be understood as consequences of this susy duality. If you take two dual CFTs, and follow flows seeded by dual operators, the IR theories will also be dual.
- These arguments strongly suggest that duality is not somehow a large N artifact but persists also at finite (but large) N. As you know there is also independent evidence that the duality actually persists all the way down to N and k equals unity.

- Consider, e.g. regular fermions in background magnetic field (of the $U(1)$ global symmetry of this theory).
- We work at zero temperature but sometimes at nonzero value of the chemical potential. What we actually compute is the fermion two point function of the theory, dressed with a background open Wilson line to make our quantity background gauge invariant.
- Our correlator is not $SU(N_F)$ gauge invariant, but we expect the locations of its singularities (e.g. poles) to be physical as in the discussion above.

Results

- Using standard large N techniques we are able to demonstrate that our two point function obeys a nonlinear integral equation.
- Quite remarkably it turned out to be possible to find a completely exact and reasonably explicit solution to this equation. The Moyal star product - and noncommutative solitons - play a leading role in this solution. I now describe its key features.
- At leading order in large N it turns out that the only singularities of our propagators are poles (no cuts).
- Working first at zero chemical potential, the poles are located at $E = \chi_\nu^+$ and $-\chi_\nu^-$.

Solution at zero chemical potential

- $$(\chi_\nu^+)^2 = c_F^2 + 2b\left(\nu + \frac{1}{2} - \frac{\mathrm{sgn}(gs)}{2}\right)$$
  $$(\chi_\nu^-)^2 = c_F^2 + 2b\left(\nu + \frac{1}{2} + \frac{\mathrm{sgn}(gs)}{2}\right) \quad (51)$$
  $$g = 2, \quad s = \frac{\mathrm{sgn}(m_F) - \lambda_F}{2}$$

  Where c_F is a solution to the equation:

$$m_F = sgn(m_F)\sqrt{c_F^2 + \lambda_F b\, sgn(m_F)} + \sum_{n=0}^{\infty}\left[c_F^2 + 2b(n+1) + \lambda_F b\right]^{1/2} - b\int dx \frac{1}{2((c^0)_F^2 + 2bx)^{1/2}} - \lambda_F |c_F^0| \quad (52)$$

s is full fermion spin. *B.S* Zeeman effect. Landau Levels of 'free relativistic particle of spin s and mass c_F' in a magnetic field.

Turning on a chemical potential

- The procedure for obtaining our solution is modified only slightly at nonzero chemical potentia. At some point in the computation we had to do an integral over a frequency variable ω. Turns out that the only effect of turning on a chemical potential is to change the contour of the ω integral.
- If we assume that μ lies in the window between $\zeta_+(M)$ and $\zeta_+(M+1)$ we find a similar solution to the one described above, but with a modified equation for c_F (the new equation depends on M). The new solution works only when the assumptions under which the equation is derived is obeyed, i.e. in a range $\mu_{up}^M < \mu < \mu_{down}^M$.

Turning on a chemical potential

- μ_{up}^{M} is the lowest potential upto which it is consistent to assume that the M^{th} Landau Level is totally filled. On the other hand μ_{down}^{M-1} is the highest potential upto which it is consistent to assume that the M^{th} Landau Level is completely empty.
- It follows that
$$\delta\mu^{M} = \mu_{up}^{M} - \mu_{down}^{M-1}$$
is the 'thickness' in energy space of the M^{th} Landau level. When $\delta\mu^{M}$ is positive it follows that the M^{th} Landau Level - which is perfectly degenerate in the free theory - broadens out as you start filling the level.
- When $\delta\mu^{M}$ is negative, on the other hand we have a 'first order phase transition' type situation. In this case there is a range of μ over which both solutions (completely filled and completely empty) are consistent. The system will pick out the state with the lower free energy.

Gaps in Landau Levels

- Actually determining $\delta\mu^M$ requries explicit solutions for the quantities c_F^M. In general we have only been able to solve these equations numerically. Our paper contains lots of plots.
- Easy to find $\delta\mu^M$ perturbatively in λ_F. Our result

$$\delta\mu^M = -2\lambda_F b \left(1 - \frac{m_F}{\sqrt{m_F^2 + 2b(M+1)}}\right) + \mathcal{O}(\lambda_F^2) \quad (53)$$

- Note the sign of this quantity is opposite to that of λb. Note also that the bracket behaves very differently at small b depending on the sign of m_F.
- Many generalizations of this computation: e.g. finite temperature free energy - may well be possible to compute.

Summary of Magnetic Field

- Exact solution of theory at leading order in large N in presence of magnetic field. Spectrum as a function of μ has gaps and bands. Inside a gap the system is 'incompressible'. Changing μ inside a gap does nothing. Know the locations of all bands and gaps.
- When $\delta\mu > 0$ have no solution in band. Integral over ω hits a pole. Deform contour above - completely fill Landau Level. Deform contour below, LL completely unfilled. Natural prescription for partially filled level: f times the contour above + $(1 - f)$ times the contour below. Adopting find system is 'compressible' within the band (single particle energies vary continuously from the lower to the upper end of each band as f varies from 0 to unity).
- Have presented only the leading large N solution. Possible that there are IR singularities at subleading order in $1/N$ that somehow modify physics within the band leading to creation of mass gaps. What is this good for?

Discussion

- Large N Chern Simons theories with fundamental matter are remarkable theories. They are simple enough that they can be exactly solved. But they also exhibit a fair degree of richness in their dynamics.
- The study of these theories has taught us many lessons that go beyond these theories. E.g. Dualities appear to be ubiquitous in non supersymmetric $D = 3$ theories.
- However the results obtained from our study have thrown up several new questions, for which we do not have clear answers. For instance, what is the precise definition of S matrices in Anyonic theories? What are the correct crossing symmetry rules at finite N and k?
- I think it would be very interesting to persue the 'exact' large N solutions of our theories in magnetic field backgrounds, so search for, for instance, fractional quantum hall type mass gaps in the middle of our bands.

Discussion

- Duality was first speculated upon in these theories by the study of their Vasiliev bulk duals. Perhaps it is time to return to this study.
- It would be interesting if the new 'free thermal statistics' of this talk showed up in a real two dimensional material in an experimental context, though I have no clue of how or where this might happen.
- All Large N computations in this theory have been perfomred in a particular light cone gauge. It would be useful to reproduce and generalize some of these results in another gauge, e.g. the $A_0 = 0$ gauge. This could allow for new computations like the thermal partition function on a genus g surface

Discussion

- The WZW singlet condition is intimately connected to the mathematical structure of quantum groups. It would be interesting if this connection had dynamical implications, perhaps in governing the structure of interaction matrix elements on the almost free structures encountered in ths talk.
- It would be very interesting to see how far this understanding persists away from the large N limit and also from the large volume limit. It would be fantastic if we could rewrite the Index of superconformal Chern Simons matter theories in a language similar to this talk (i.e. free theory subject to an effective constraint, which may be a deformation of the WZW singlet condition).

Discussion

- The Bose condensate encountered in our analysis is an extremely simple stabilization of the run away instability of free theory. The sharp cut off at k_F plus the Bose exclusion principle gives this phase all the properties of a Fermi Sea. I would be very intersted to know of any other situation in which such a stabilized Bose condensate has been encountered.
- It would be interesting to investigate the dynamical implications of the Bose condensation principle. Cut off lasers? Connection with quantum groups.
- It would be interesting to better understand how the path integral ‘knows’ that mixed ‘correlators’ of non integrable and integrable Wilson lines must vanish, even in the case of pure Chern Simons theory (see remark in red before (4)). It would also be satisfying to reproduce the phase $(-1)^{N+1}$ in (8) directly from the Blau and Thompson path integral.

Discussion

- Finally, perhaps we are buiding up to a point where we get to understand these theories well enough so that we have a realisitic chance of shooting for a proof of the Bose Fermi duality. Recall in this context that no nontrivial strong weak coupling duality in higher than two dimensions has ever yet been proven.