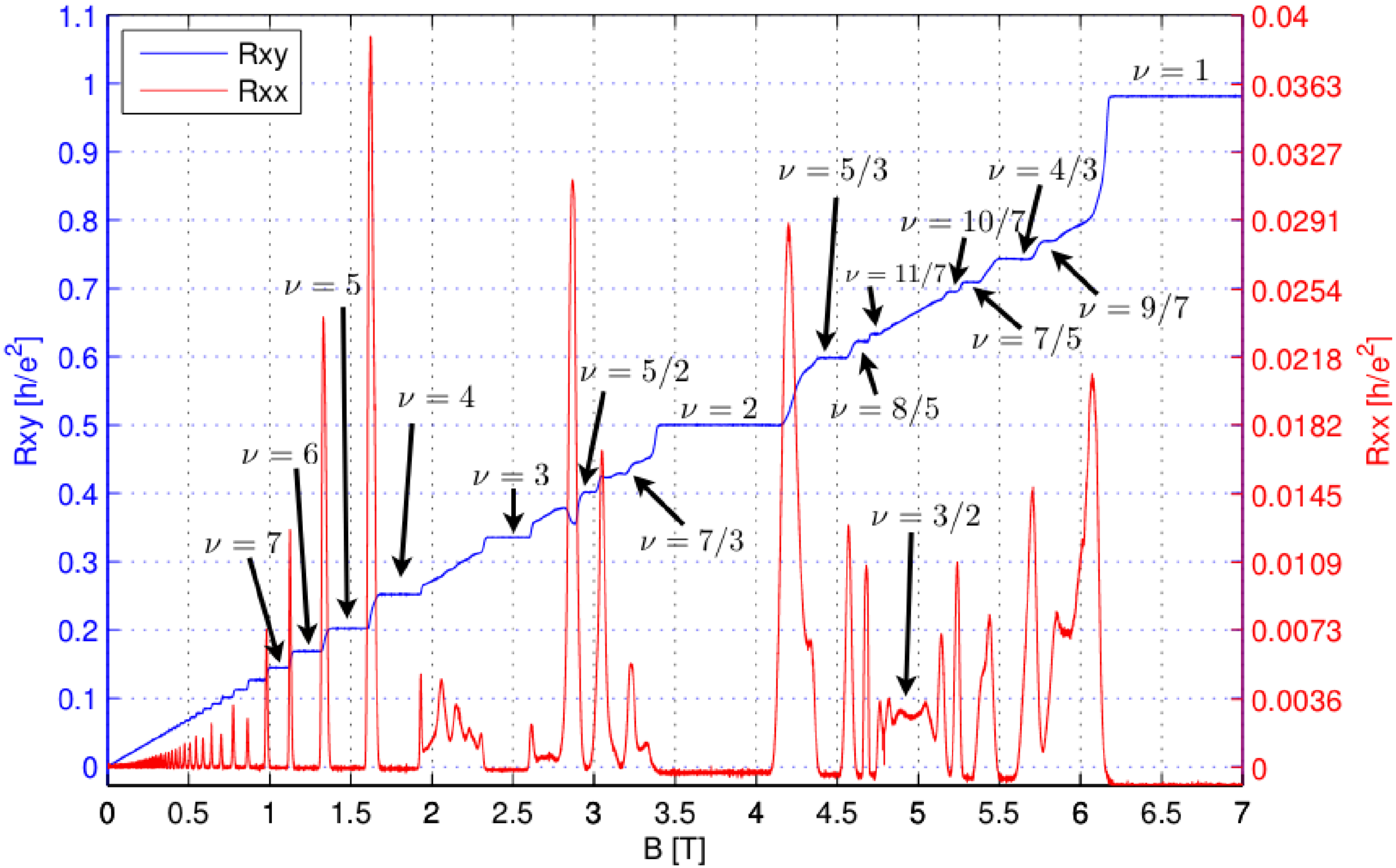
Disentangling Topological States with the Entanglement Negativity

Michael Mulligan

Based on work with Pak Kau Lim, Hamed Asasi, and Jeffrey Teo in arXiv:2106.07668.

What are the types of entanglement in QH states?



(GaAs quantum well)

Source: Marcus et al., ’13.

topological degeneracy

e.g., $\nu = 1/m$ Laughlin state, i.e., $U(1)_m$ Chern-Simons theory, has ground state deneracy m^g, where g is the spatial genus

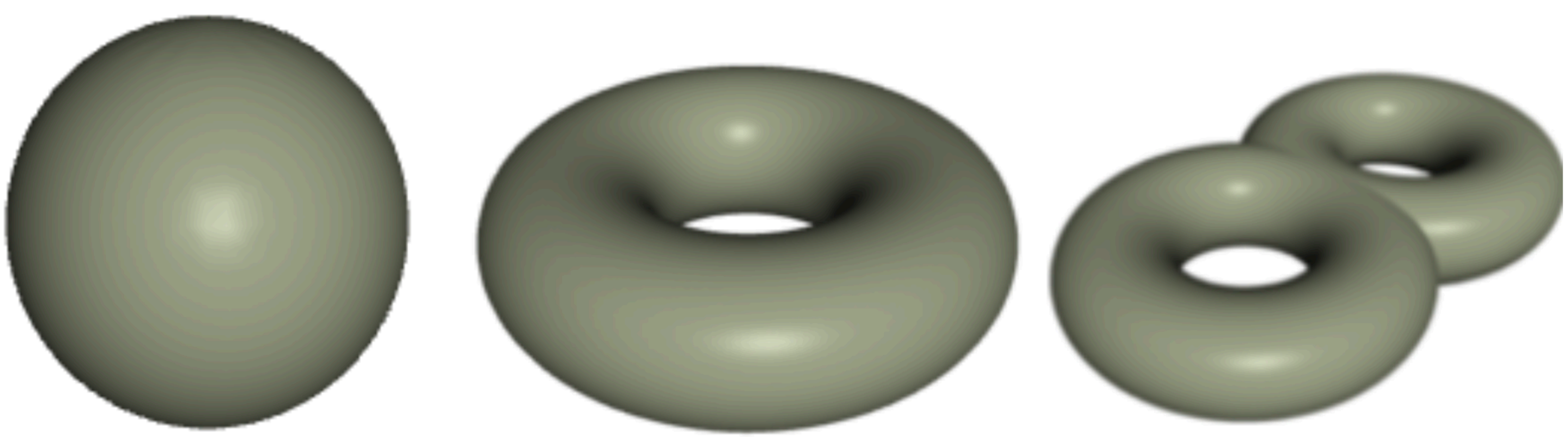
Witten (1989);
Wen & Niu (1990)

such phases are said to be long-range entangled

e.g., Wen's review (2013)

phases without such degeneracy (e.g., $m = 1$) are said to be short-range entangled

Lu & Vishwanath (2012);
Chen, Gu, Liu, & Wen (2013)



entanglement as a phase diagnostic

$$\rho \in \mathscr{H}_A \otimes \mathscr{H}_B$$

entanglement entropy : $S_A = -\,\mathrm{tr}_A \rho_A \ln \rho_A, \quad \rho_A = \mathrm{tr}_B \rho$

in top phase : $S_A = \alpha L - \gamma + \mathcal{O}(1/L)$

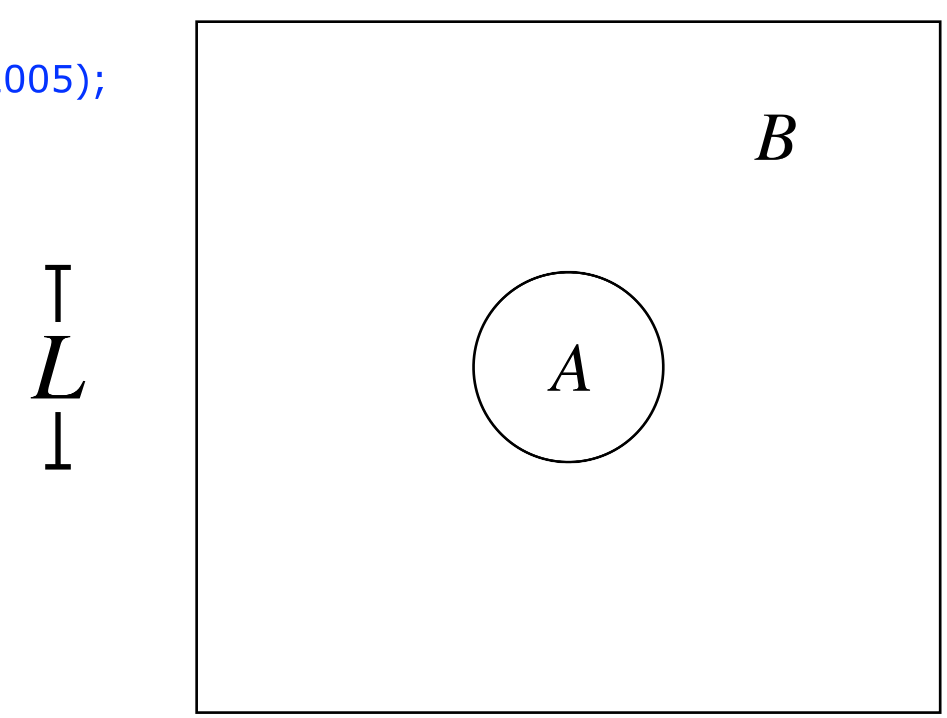
related to the "a-theorem"

Casini, Huerta, & Myers (2011);
Casini & Huerta (2012)

topological entanglement entropy

$$\gamma = \frac{1}{2} \ln \sum_a d_a^2$$

Hamma, Ionicioiu, & Zanardi (2005);
Levin & Wen (2006);
Kitaev & Preskill (2006)



quantum dimension of quasiparticle a

e.g., Laughlin: $d_a = 1$ for all a;

Moore-Read: $d_a \in \{1, \sqrt{2}\}$

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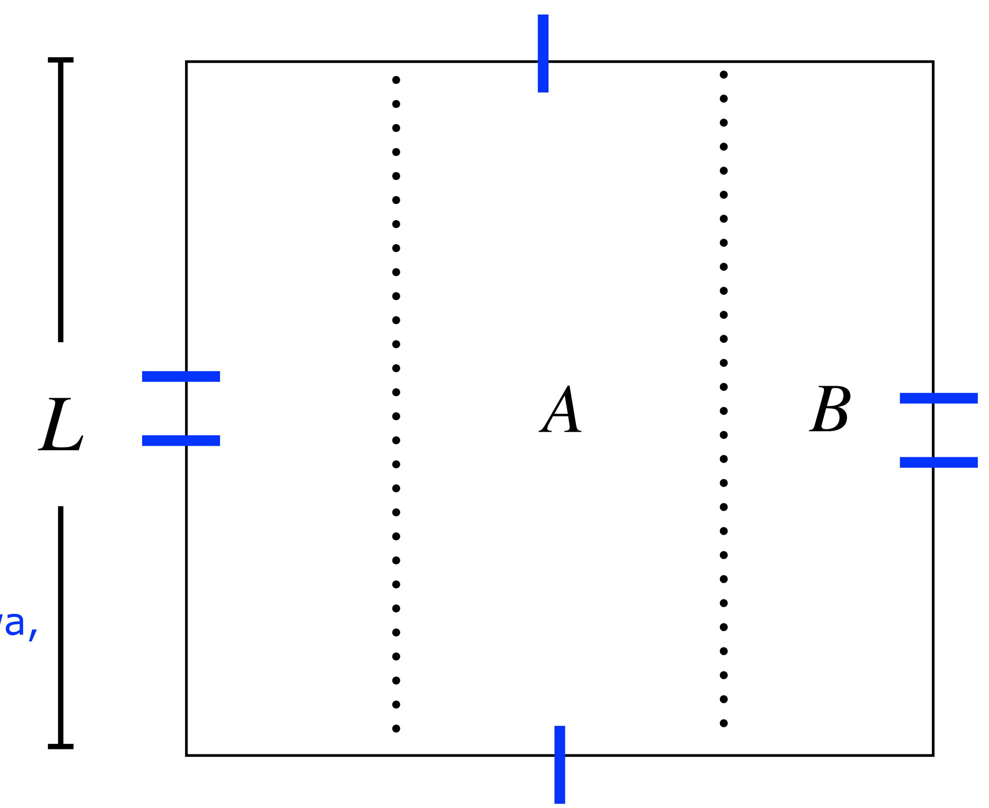
$$\gamma = \ln \sum_a d_a^2 - \sum_a |\psi_a|^2 \ln \frac{|\psi_a|^2}{d_a^2}$$

Dong, Fradkin, Leigh, & Nowling (2008); Zhang, Grover, Turner, Oshikawa, & Vishwanath (2012)

amplitude to be in degenerate torus state a



torus

overview of our results

1. topological sector contribution is distilled by the negativity
2. it constrains the real-space structure of the wave function

$$\gamma = \ln \sum_a d_a^2 - \sum_a |\psi_a|^2 \ln \frac{|\psi_a|^2}{d_a^2}$$

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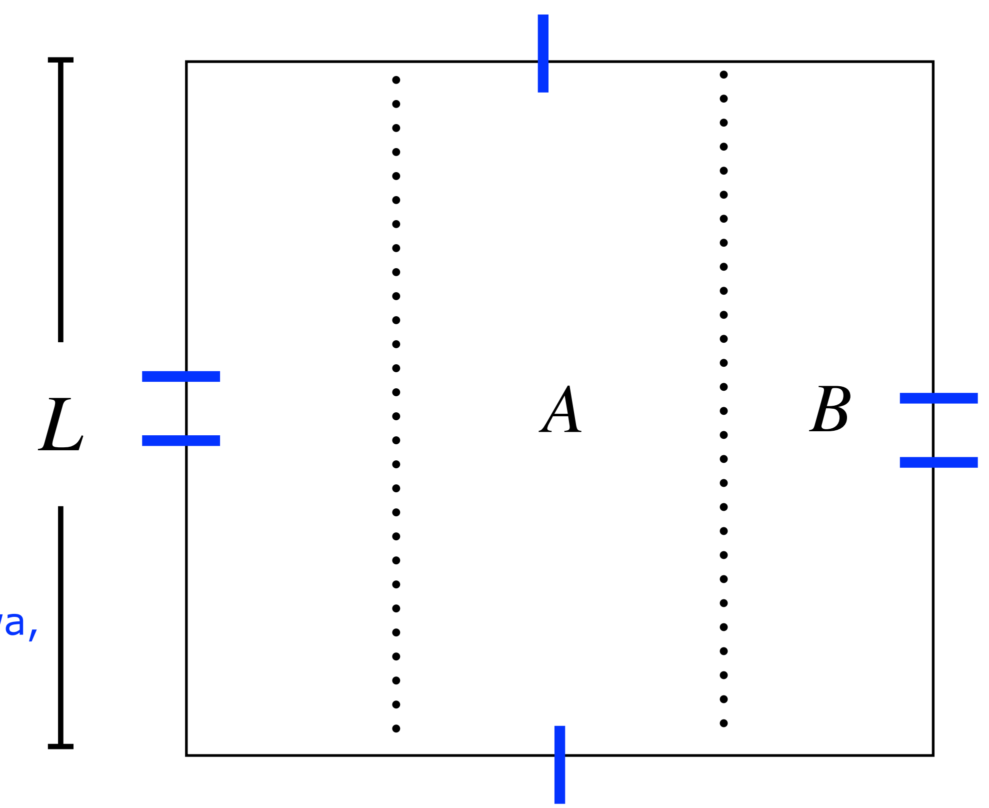


torus

entanglement negativity

Vidal & Werner (2002);
Peres (1996);
Plenio (2007)

1. a measure of quantum correlations in mixed states
2. a measure of multipartite entanglement in pure states

$$\rho \in \mathscr{H}_A \otimes \mathscr{H}_B$$

partial transpose w.r.t. A : $\langle i_A j_B | \rho^{T_A} | k_A l_B \rangle = \langle k_A j_B | \rho | i_A l_B \rangle$

negativity : $\mathscr{N}_{A:B}(\rho) = (||\rho^{T_A}||_1 - 1)/2, \quad ||\rho||_1 \equiv \mathrm{tr}\sqrt{\rho^\dagger \rho}$

entanglement negativity : $\mathscr{E}_{A:B}(\rho) = \ln ||\rho^{T_A}||_1$

entanglement negativity: example

representatives of distinct 3-qubit entanglement:

Dur, Vidal & Cirac (2000)

$$|W\rangle = \frac{1}{\sqrt{3}}\big(\,|001\rangle + |010\rangle + |100\rangle\big)$$

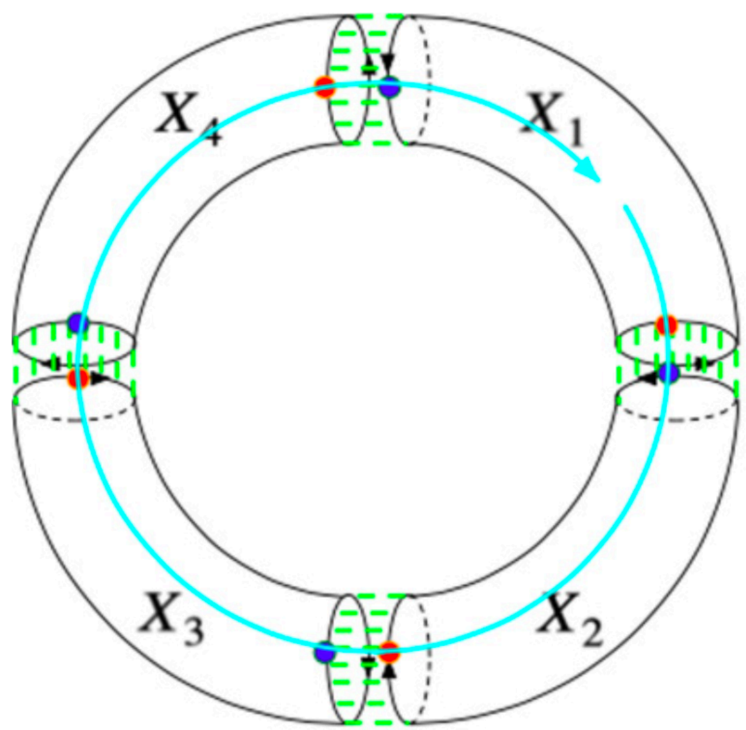
$$|GHZ\rangle = \frac{1}{\sqrt{2}}\big(\,|000\rangle + |111\rangle\big)$$

- not related by LOCC
- $S_A(\rho_W) = S_A(\rho_{GHZ})$
- distinguished by entanglement negativities:

$$\mathcal{E}_{A:BC}(\rho_{\mathrm{GHZ}}) > \mathcal{E}_{A:BC}(\rho_{\mathrm{W}}) > \mathcal{E}_{A:B}(\rho_{\mathrm{W}}) > \mathcal{E}_{A:B}(\rho_{\mathrm{GHZ}}) = 0$$

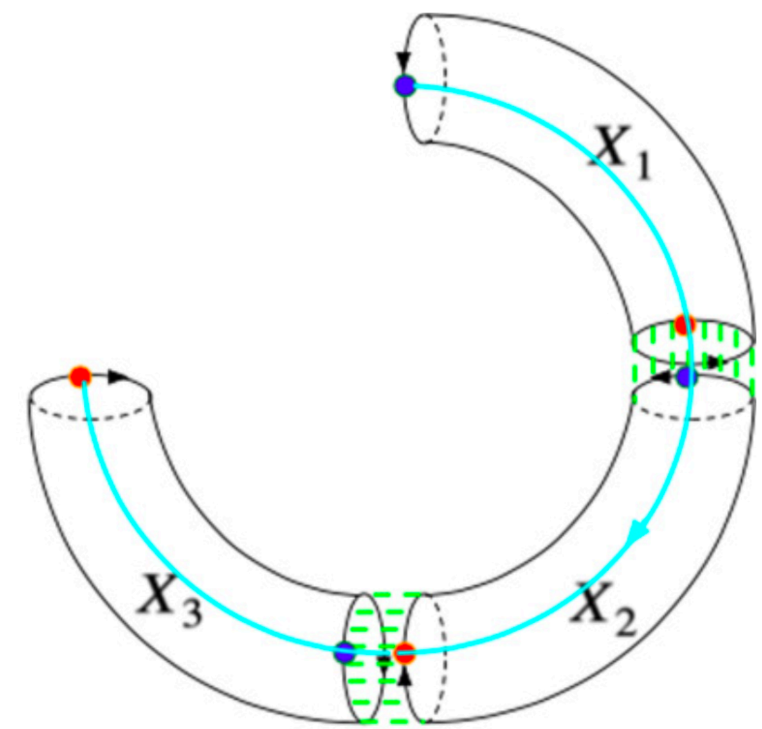
1. Abelian Laughlin state at $\nu = 1/m$
2. Non-Abelian Moore-Read state at $\nu = 1/m$

Torus Geometry $(M = 2)$



$$\mathcal{E}_{X_{\text{odd}}:X_{\text{even}}} = 2\ln \sum_a |\psi_a| \zeta_a^{2M}$$

Cylinder Geometry $(R = 2)$

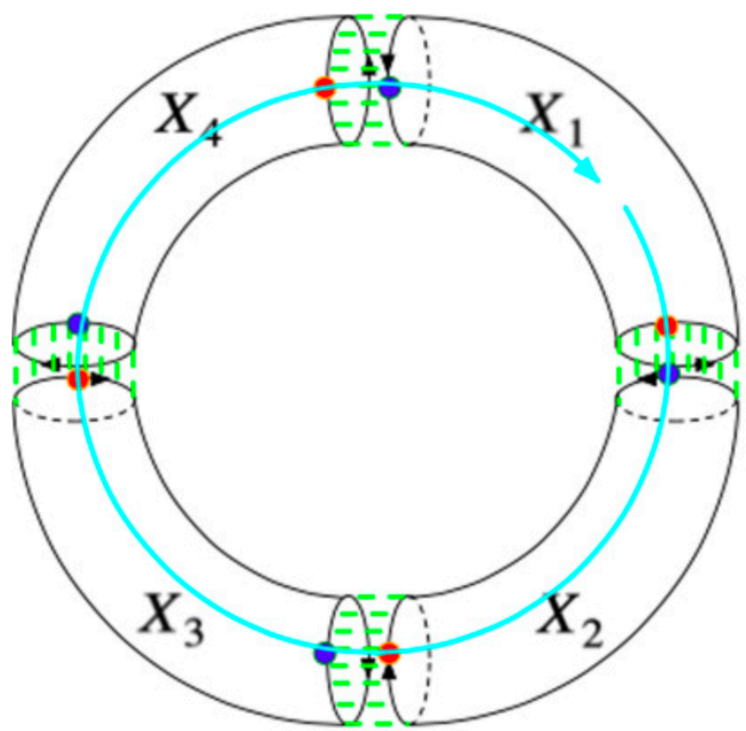


$$\mathcal{E}_{Y_{\text{odd}}:Y_{\text{even}}} = \ln \sum_a \left(|\psi_a| \zeta_a^R \right)^2$$

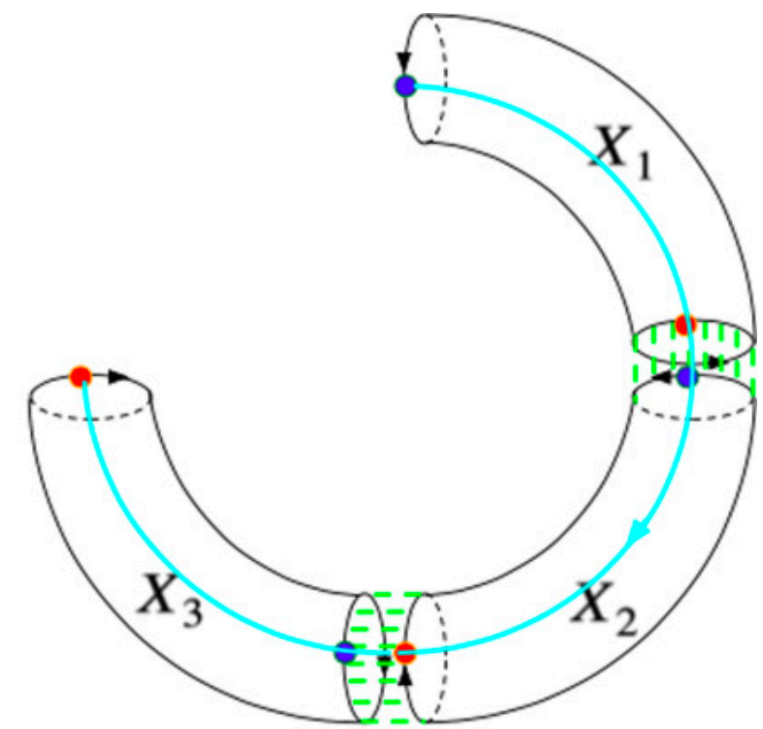
$$\zeta_a = \frac{\mathrm{tr} e^{-H_a/2}}{\sqrt{\mathrm{tr} e^{-H_a}}}$$

ratio of edge-state partition functions at $T = 1$ and $T = 2$

Torus Geometry



Cylinder Geometry



$$\text{torus}: \ \mathscr{E}_{X_{\text{odd}}:X_{\text{even}}} = M\alpha L - M \ln \mathscr{D}^2 + 2 \ln \sum_a |\psi_a| d_a^M$$

$$\text{cylinder}: \ \mathscr{E}_{Y_{\text{odd}}:Y_{\text{even}}} = \frac{R}{2}\alpha L - \frac{R}{2} \ln \mathscr{D}^2 + \ln \sum_a |\psi_a|^2 d_a^R$$

$$\mathscr{D}^2 = \sum_a d_a^2$$

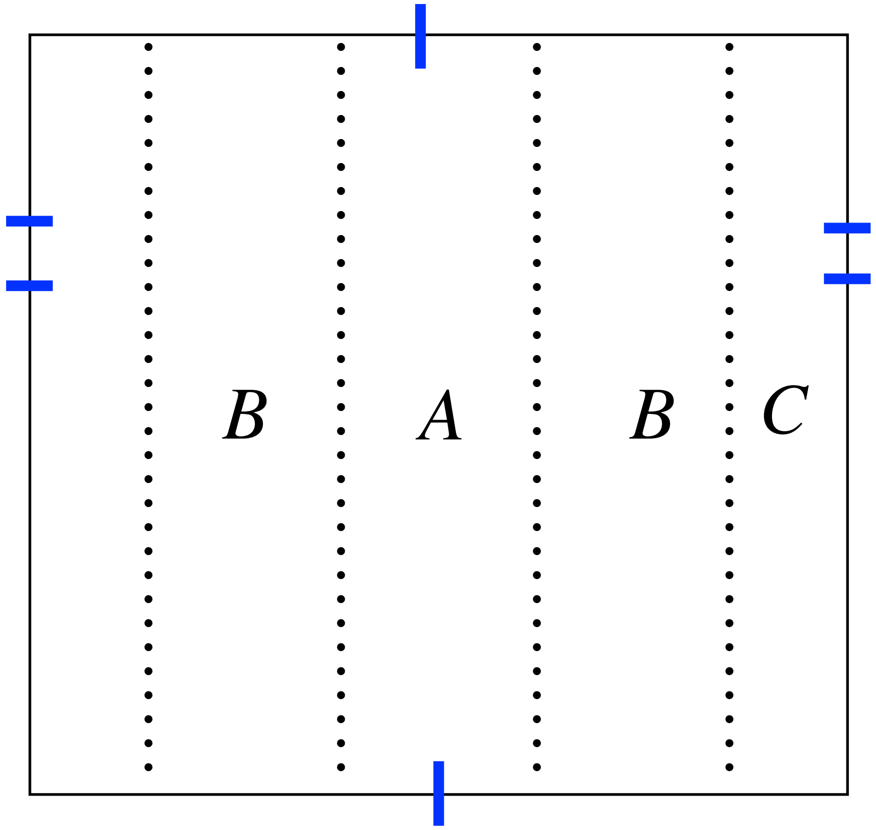
consistent with:
Lee & Vidal (2013);
Castelnovo (2013);
Wen, Matsuura, & Ryu (2016)

disentangling and monogamy

$$\rho \in \mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_C$$

Coffman, Kundu, & Wootters (2000);
Osborne, & Verstraete (2006)

a monogamy relation : $\mathcal{N}^2_{A:BC}(\rho) \geq \mathcal{N}^2_{A:B}(\rho_{AB}) + \mathcal{N}^2_{A:C}(\rho_{AC})$



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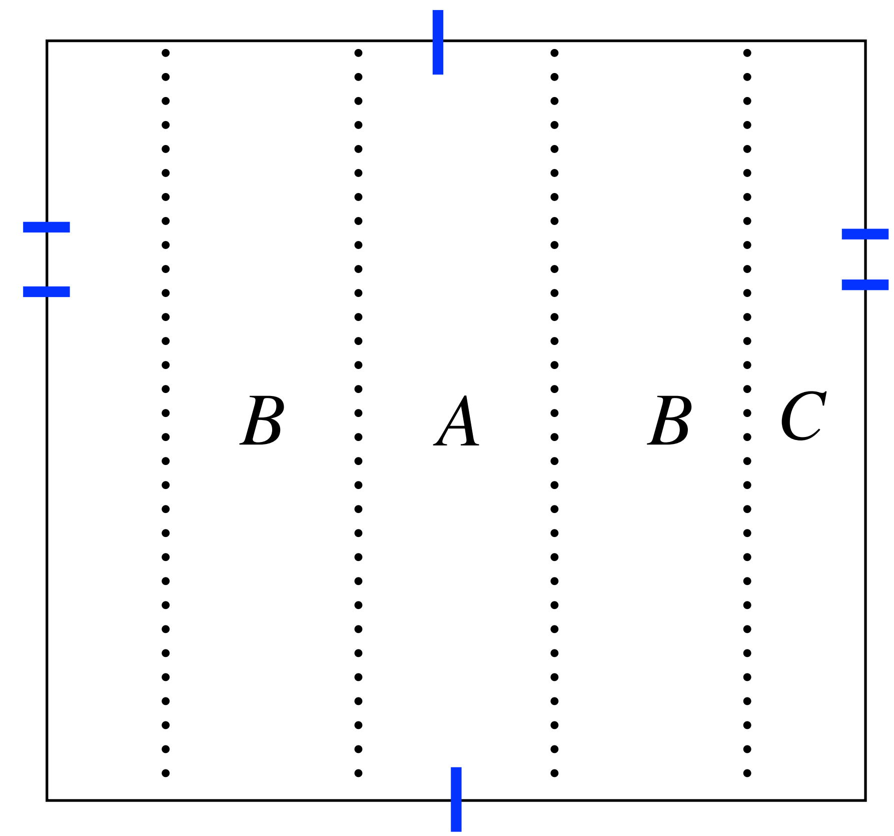
$\updownarrow$

He & Vidal (2015)

disentangling condition : $\mathscr{E}_{A:BC}(\rho) = \mathscr{E}_{A:B}(\rho_{AB})$



torus

real-space disentanglement

disentangling condition, $\mathscr{E}_{A:BC}(\rho) = \mathscr{E}_{A:B}(\rho_{AB})$, implies

for pure states : $|\Psi_{ABC}\rangle \in \mathscr{H}_A \otimes \left(\mathscr{H}_{B_L} \otimes \mathscr{H}_{B_R}\right) \otimes \mathscr{H}_C$

condition implies : $|\Psi_{ABC}\rangle = |\Psi_{AB_L}\rangle \otimes |\Psi_{B_RC}\rangle$

He & Vidal (2015);
see also Ou & Fan (2007)

product state

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product state

for mixed states : $\rho \in \mathscr{H}_A \otimes \left(\sum_j \mathscr{H}_{B_L^j} \otimes \mathscr{H}_{B_R^j}\right) \otimes \mathscr{H}_C$

condition implies : $\rho = \sum_j p_j \rho_{AB_L^j} \otimes \rho_{B_R^j C}$.

Hayden, Jozsa, Petz, & Winter (2004);
Gour & Guo (2018)

separable state

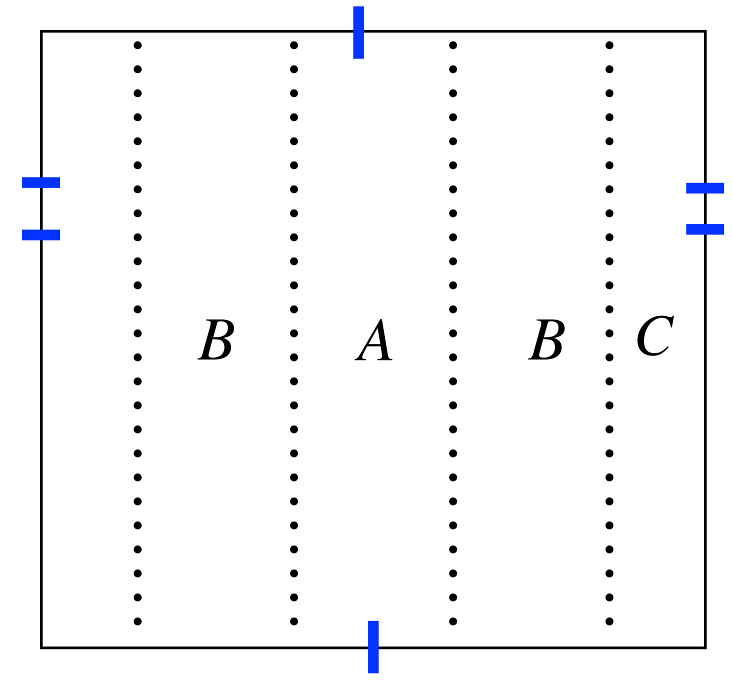
$(\sum_i p_i = 1)$

disentangling topological states

$$\rho \in \mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_C$$

disentangling condition : $\mathscr{E}_{A:BC}(\rho) = \mathscr{E}_{A:B}(\rho_{AB})$

$$\mathscr{E}_{A:BC}(\rho) - \mathscr{E}_{A:B}(\rho_{AB}) = \log \frac{\left(\sum_a |\psi_a| d_a\right)^2}{\sum_a |\psi_a|^2 d_a^2}$$



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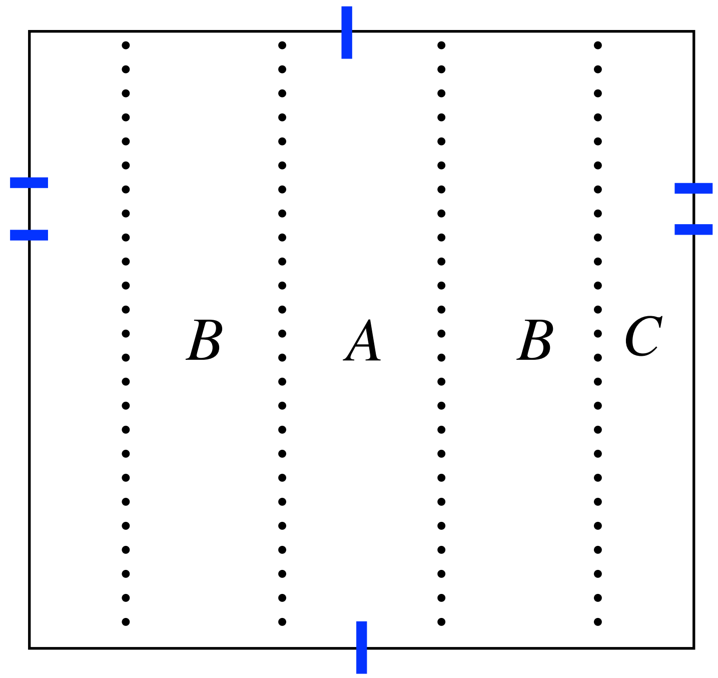
condition $\Longrightarrow$ $\psi_a = 1$ for some a

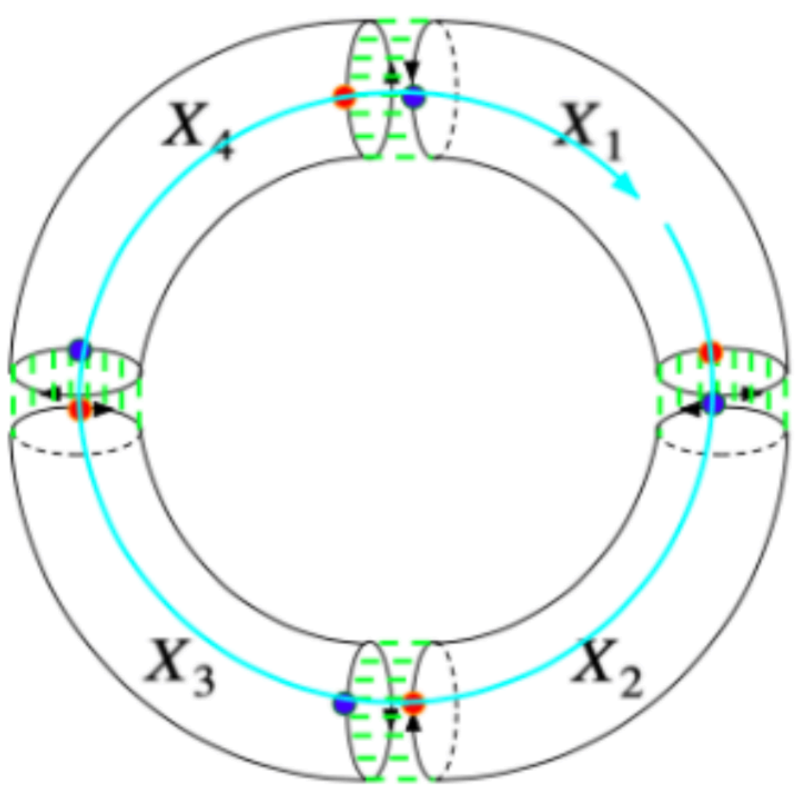
1. sufficient to disentangle Laughlin
2. only necessary for Moore-Read



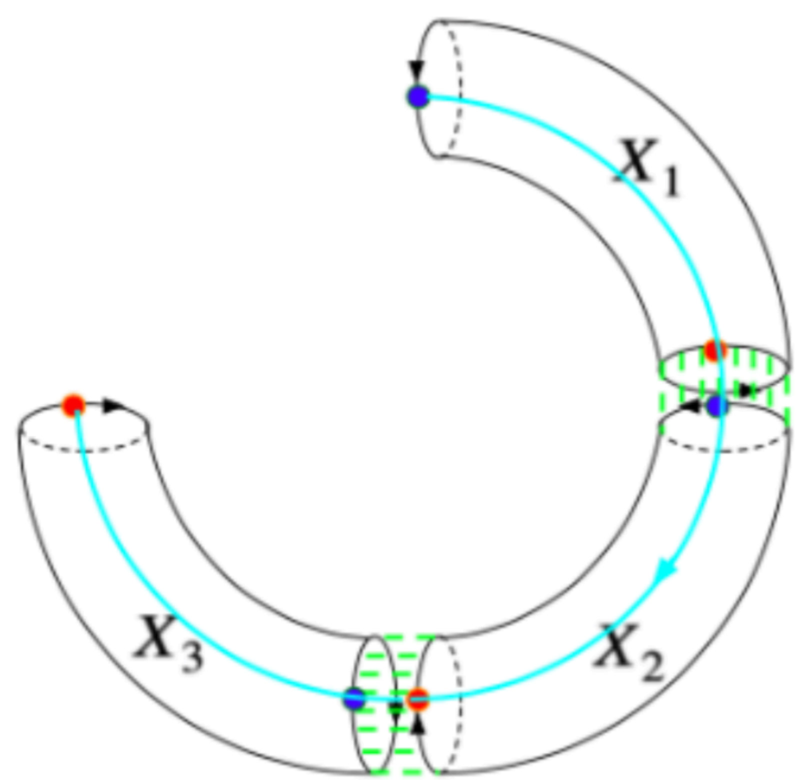
torus

outline for the remainder

- cut and glue construction of topological states
- entanglement negativity calculations
- disentanglement of ground states



Torus Geometry



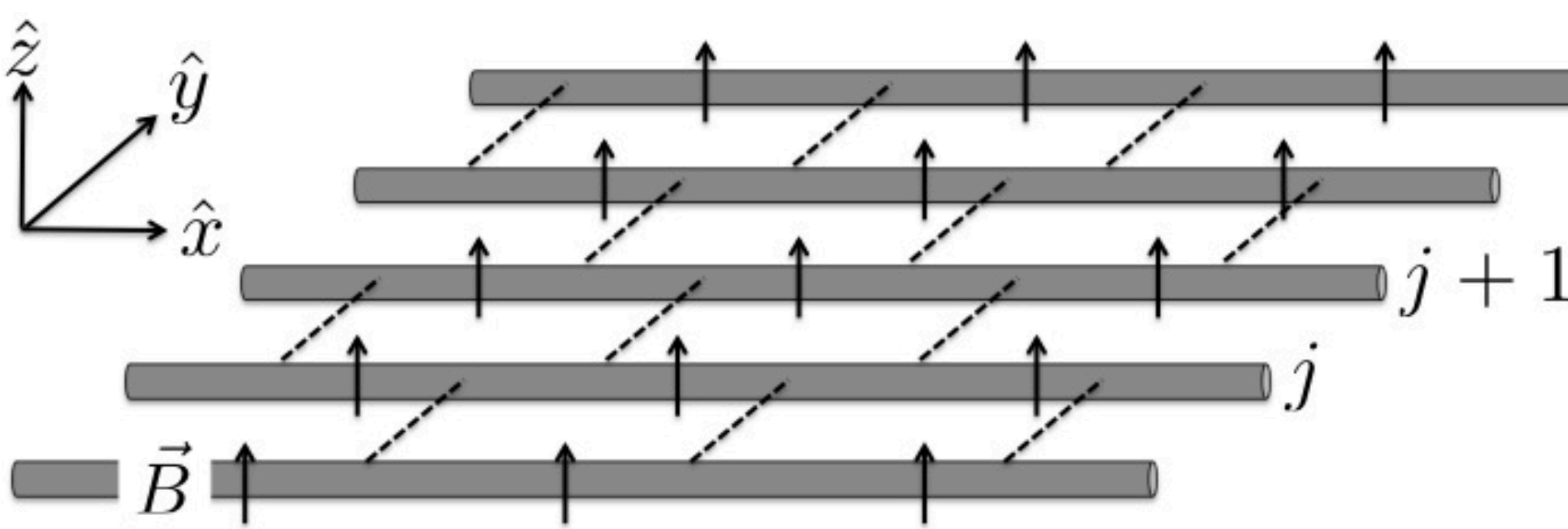
Cylinder Geometry

cut and glue contruction of topological states

Elitzur, Moore, Schwimmer, & Seiberg (1989);
Qi, Katsura, & Ludwig (2012);
Lundgren, Fuji, Furukawa, & Oshikawa (2013);
Teo & Kane (2014)

two equivalent methods:

1. couple together wires hosting 1d nonchiral electrons
2. glue together cylinder states in the target phase by appropriate edge-state interactions



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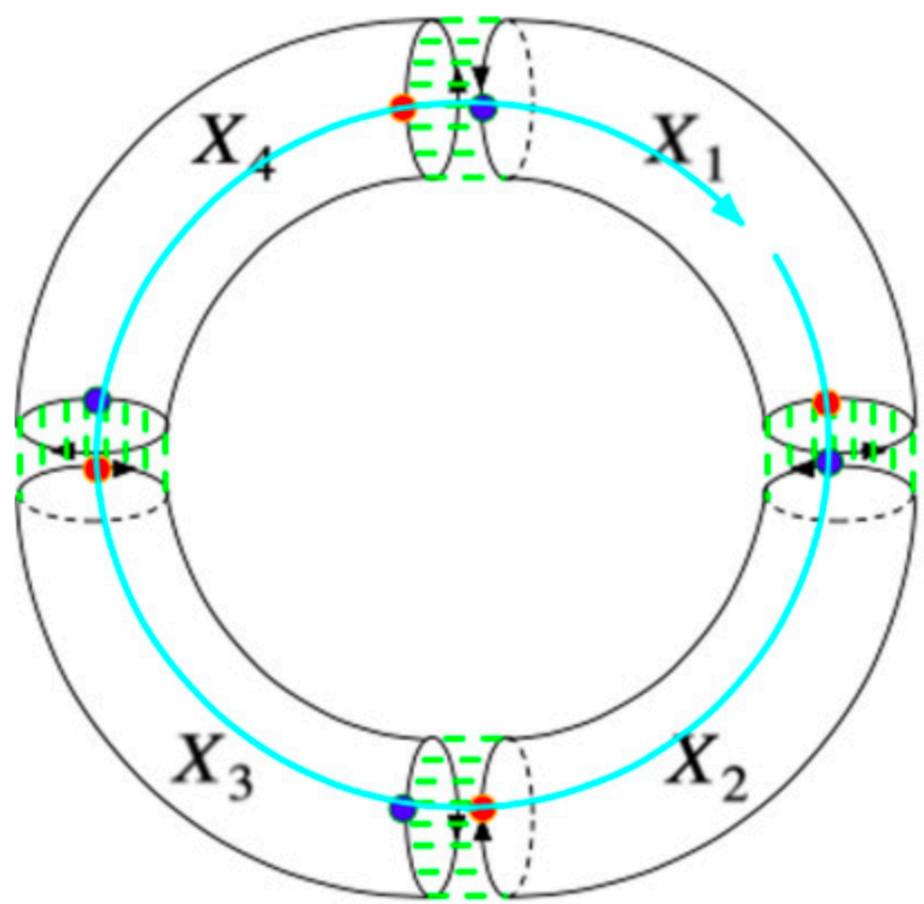


Laughlin

low-energy excitations live along cylinder boundaries

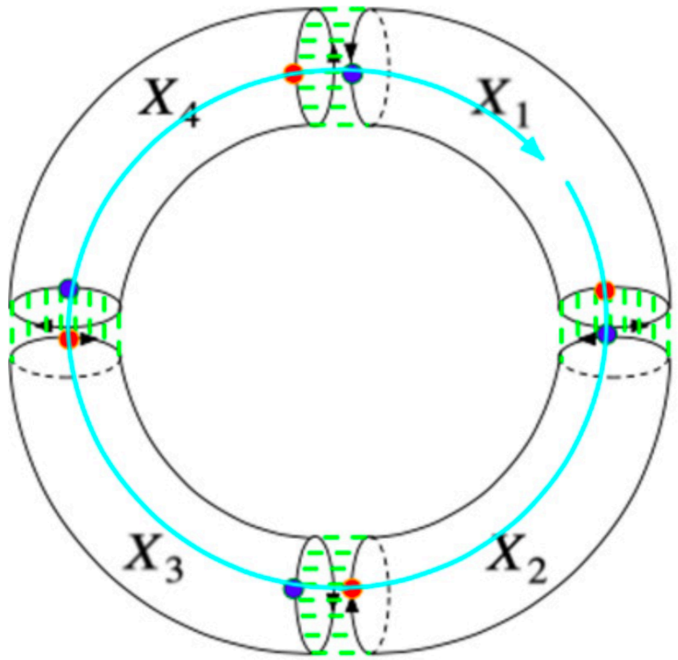
$$H = \sum_{i \in \text{interfaces}} \left(H_i^{(0)} + H_i^{(1)} \right)$$

$$H_i^{(0)} = \frac{m v_c}{4\pi} \int_0^L dx \left[(\partial_x \phi_{i-1}^R)^2 + (\partial_x \phi_i^L)^2 \right], \quad \phi_i^{R/L} \sim \phi_i^{R/L} + 2\pi,$$

$$[\phi_i^{R/L}(x), \partial_{x'} \phi_i^{R/L}(x')] = \pm \frac{2\pi i}{m} \delta(x - x')$$

e.g., Wen's review (1992)

$$H_i^{(1)} = -\frac{2g}{\pi} \int_0^L dx \cos\left[m \left(\phi_{i-1}^R + \phi_i^L \right) \right]$$



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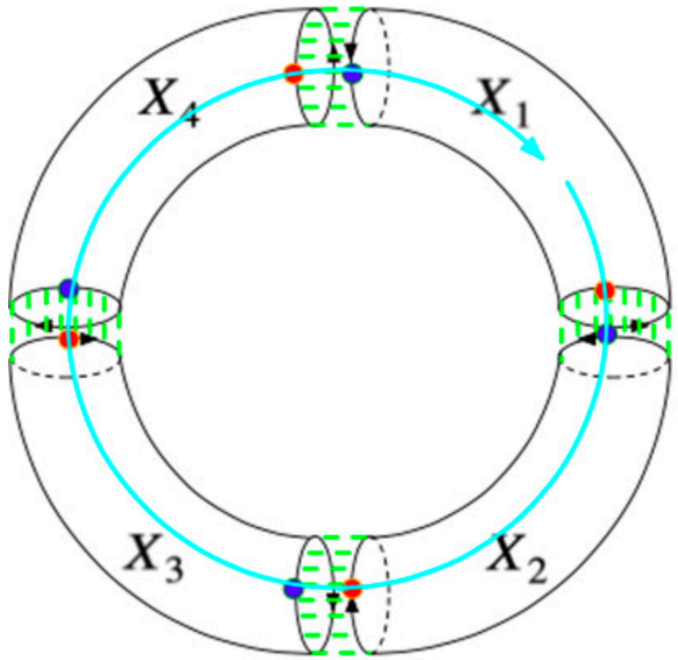
$$H_i^{(1)} = -\frac{2g}{\pi} \int_0^L dx \cos \left[m \left(\phi_{i-1}^R + \phi_i^L \right) \right]$$

$$\approx \frac{g m^2}{\pi} \int_0^L dx \left(\phi_{i-1}^R + \phi_i^L \right)^2 + \text{const} \,.$$

quadratic approx.



Lundgren, Fuji, Furukawa, & Oshikawa (2013)

$$\phi_i^{R/L} = \phi_{i,0}^{R/L} + 2\pi N^{R/LX_i}\frac{x}{L} + \sum_{\pm k>0}\sqrt{\frac{2\pi}{mL|k|}}\left(a_{i,k}e^{ikx} + (a_{i,k})^\dagger e^{-ikx}\right)$$

$$N^{R/LX_i} \in \mathbb{Z} \pm \frac{a}{m}$$

decoupling of zero and oscillator modes

harmonic oscillator

$$H_{i,b}^{\text{zero}} = \frac{\pi m v_c}{2L}\left(N^{RX_{i-1}} - N^{LX_i}\right)^2 + \frac{\pi\lambda v_c L}{2}\left(\phi_{i-1,0}^R + \phi_{i,0}^L\right)^2$$

$$H_{i,b}^{\text{osc}} = v_c\sum_{k>0}\left(a_{i-1,k}^\dagger a_{i,-k}\right)\begin{pmatrix} |k| + \frac{2\lambda\pi^2}{m|k|} & \frac{2\lambda\pi^2}{m|k|} \\ \frac{2\lambda\pi^2}{m|k|} & |k| + \frac{2\lambda\pi^2}{m|k|}\end{pmatrix}\begin{pmatrix} a_{i-1,k} \\ a_{i,-k}^\dagger\end{pmatrix}$$

$$\lambda = 2gm^2/\pi^2 v_c > 0$$

Chen & Fradkin (2013)
Lundgren, Fuji, Furukawa, & Oshikawa (2013)

diagonalize by Bogoliubov

interface ground state $|b^{\text{zero}}_{a,i}\rangle \otimes |b^{\text{osc}}_{i}\rangle$:

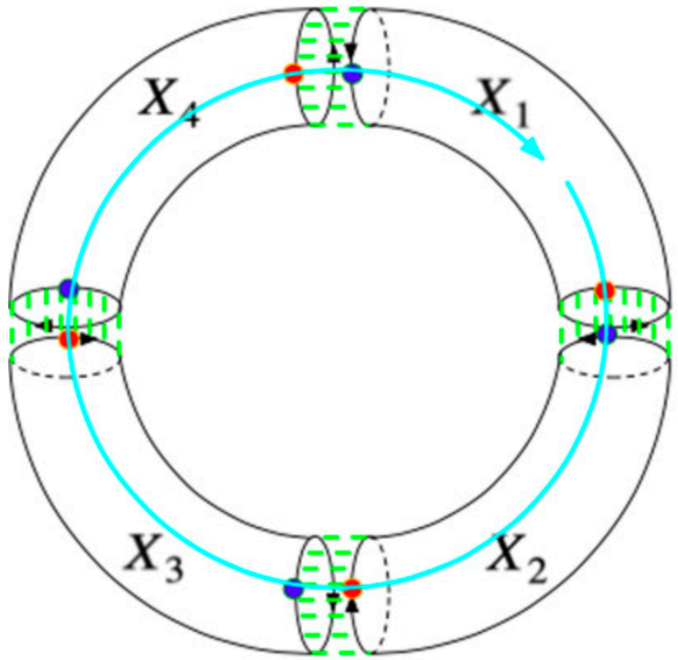
$$|b^{\text{zero}}_{a,i}\rangle = \sum_{N_{a,i}\in\mathbb{Z}-\frac{a}{m}} e^{-\frac{v_e\pi m}{2L}N^2_{a,i}} |N^{RX_{i-1}} = N_{a,i}\rangle_{RX_{i-1}} \otimes |N^{LX_i} = -N_{a,i}\rangle_{LX_i}$$

$$|b^{\text{osc}}_{i}\rangle = \sum_{\{n_{i,k}\in\mathbb{Z}^+\}} e^{-\sum_{k>0}\frac{v_e k}{2}(n_{i,k}+1/2)} |\{n^{RX_{i-1}}_{b,k} = n_{i,k}\}_{k>0}\rangle_{RX_{i-1}} \otimes |\{n^{LX_i}_{b,-k} = n_{i,k}\}_{k>0}\rangle_{LX_i}$$

torus ground state

$$|\Psi_a\rangle = \bigotimes_{i\in\text{interfaces}} |b^{\text{zero}}_{a,i}\rangle \otimes |b^{\text{osc}}_{i}\rangle$$

labels topological sector: $a = 0,\ldots,m-1$



Moore-Read

Moore & Read (1991)
review: Nayak, Simon, Stern, Freedman, & Das Varma (2008)

$$H = \sum_{i \in \text{interfaces}} \left(H_i^{(0)} + H_i^{(1)}\right)$$

$$H_i^{(0)} = \int_0^L dx \left[\frac{mv_c}{4\pi}\left((\partial_x \phi_{i-1}^R)^2 + (\partial_x \phi_i^L)^2\right) \pm \frac{i}{2}\chi_i^{R/L} v_m \partial_x \chi_i^{R/L}\right]$$

$$\{\chi_i^{R/L}(x), \chi_i^{R/L}(x')\} = \delta(x - x'),$$

untwisted sectors: antiperiodic fermions

twisted sectors: periodic fermions

electron operator $\chi e^{im\phi}$ invariant under $\mathbb{Z}/2$:

$$\chi_i^{R/L} \to -\chi_i^{R/L}, \quad \phi_i^{R/L} \to \phi_i^{R/L} \pm \frac{\pi}{m}$$

$$H = \sum_{i \in \text{interfaces}} \left(H_i^{(0)} + H_i^{(1)} \right)$$

Sohal, Han, Santos, & Teo (2020)

$$H_i^{(1)} = -\frac{2g}{\pi} \int_0^L dx \left[i\chi_i^L \chi_{i-1}^R \cos\left[m\left(\phi_{i-1}^R + \phi_i^L \right) \right] \right]$$

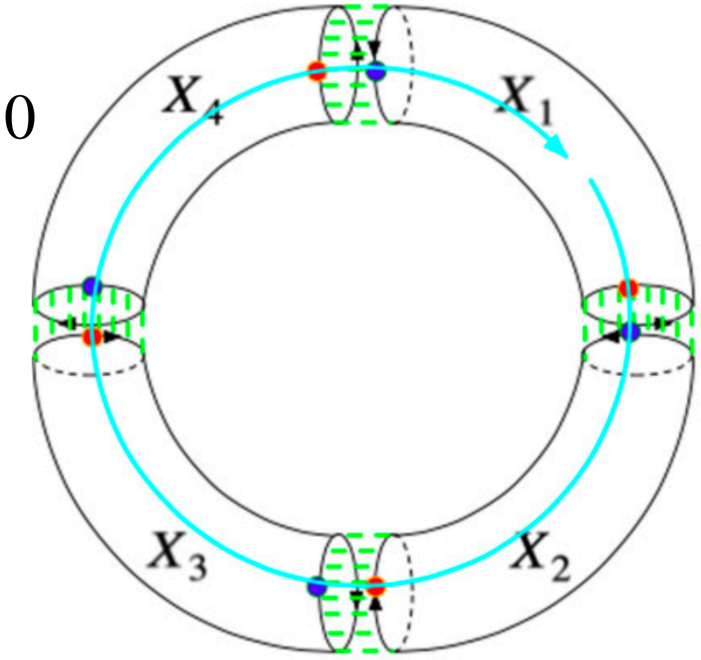
$$\approx \int_0^L dx \left[\frac{v_c \lambda \pi}{2} \left(\phi_{i-1}^R + \phi_i^L \right)^2 + v_m \tilde{g} i \chi_i^L \chi_{i-1}^R + \text{const.} + \dots \right]$$

violates $\mathbb{Z}/2$ symmetry;

restored by projection $\mathscr{P}$ of decoupled ground state

$\tilde{g} = -2g/v_m\pi < 0$

$$\mathscr{P}_a = \otimes_i \mathscr{P}_{a,X_i} = \otimes_i \frac{1}{2}\left(1 + (-1)^{F^{LX_i}+F^{RX_i}}(-1)^{N^{LX_i}+N^{RX_i}}\right)$$



untwisted sector: antiperiodic fermions

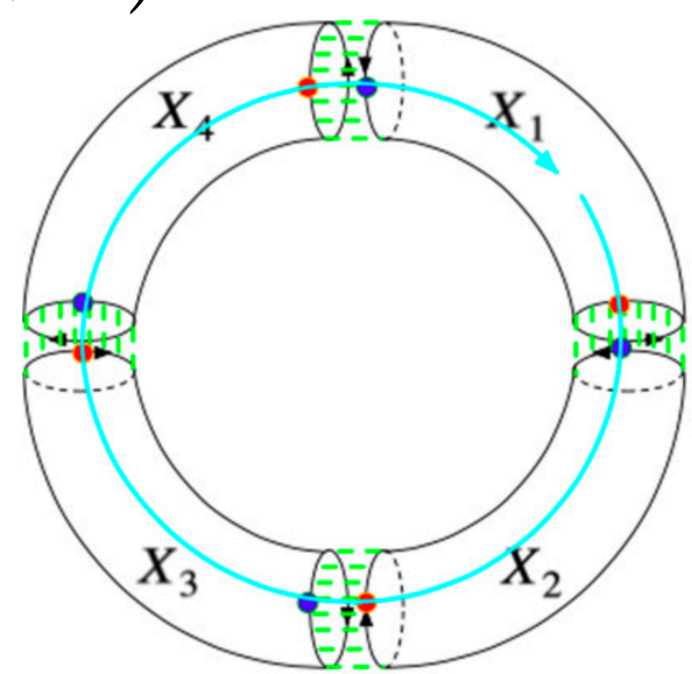
$$|\Psi_a\rangle = \mathscr{P}_a \bigotimes_{i\in\text{interfaces}} |b_{a,i}^{\text{zero}}\rangle \otimes |b_i^{\text{osc}}\rangle \otimes |f_i^{\text{osc}}\rangle$$

$$|f_i^{\text{osc}}\rangle = \sum_{\{\tilde{n}_{i,k}\in\mathbb{Z}_2\}} i^{\sum_{k>0}\tilde{n}_{i,k}} e^{-\sum_{k>0}\frac{\tilde{v}_e k}{2}(\tilde{n}_{i,k}+1/2)} |\{n_{f,k}^{RX_{i-1}} = \tilde{n}_{i,k}\}_{k>0}\rangle_{RX_{i-1}} \otimes |\{n_{f,-k}^{LX_i} = \tilde{n}_{i,k}\}_{k>0}\rangle_{LX_i}$$

$$\mathscr{P}_a = \otimes_{i\in\text{interfaces}} P_{a,i} = \frac{1}{2}\left(1 + (-1)^{N_{a,i} - a/m + \sum_{k>0}\tilde{n}_{i,k}}\right)$$

factorizes for each interface

an interface fermion number operator

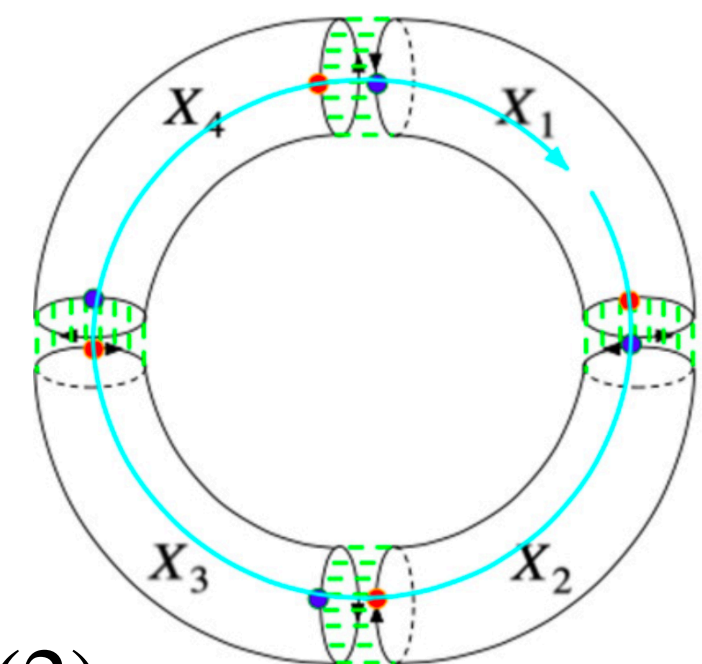


twisted sector: periodic fermions

$$|\Psi_a\rangle = \mathscr{P}_a \bigotimes_{i\in\text{interfaces}} \left(|b^{\text{zero}}_{a,i}\rangle \otimes |b^{\text{osc}}_i\rangle \otimes |f^{\text{osc}}_i\rangle\right) \otimes |f^{\text{zero}}\rangle$$

"interface Dirac fermion" : $f_i = \frac{1}{\sqrt{2}}\left(c^R_{i-1,0} + ic^L_{i,0}\right)$

$$H^{\text{zero}}_f = -i\tilde{g}\sum_i c^R_{i-1,0}c^L_{i,0} = -\tilde{g}\sum_i f^\dagger_i f_i$$



"interface fermion" occupation numbers

$$|f^{\text{zero}}\rangle = |0'0'0'0'\rangle$$
$$= \frac{1}{\sqrt{8}}\sum_{\vec{\gamma}} |\gamma_1\gamma_2\gamma_3\gamma_4\rangle, \quad \sum_i \gamma_i = 1 \bmod(2)$$

"cylinder Dirac fermion" occupation numbers allow for straightforward $\mathbb{Z}/2$ projection

torus state = product of cylinder states

$$|\Psi_a\rangle = \bigotimes_{i=1}^{2M} \sum_{\mathcal{N}_{a,i}} \lambda(\mathcal{N}_{a,i})\,|\mathcal{N}_{a,i}\rangle_{RX_{i-1}} \otimes |-\mathcal{N}_{a,i}\rangle_{LX_i}, \quad \mathcal{N}_{a,i} \equiv \left(N_{a,i}, \{n_{i,k}\}_{k>0}\right)$$

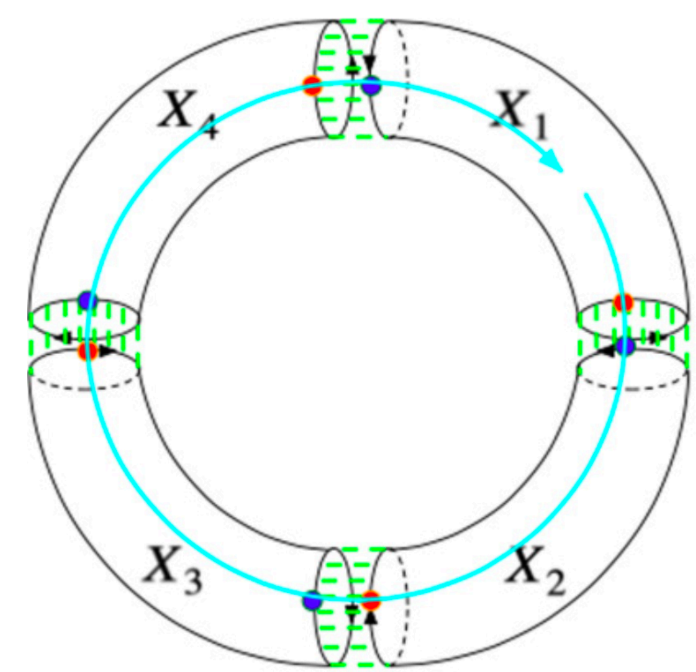
zero and oscillator mode quantum numbers

$$\lambda(\mathcal{N}_{a,i}) = \exp\left[-\frac{v_e \pi m}{2L} N_{a,i}^2 - \sum_{k>0} \frac{v_e k}{2}\left(n_{i,k} + \frac{1}{2}\right)\right], \quad v_e = \frac{2}{\pi}\sqrt{\frac{m}{\lambda}}$$

$$\langle \Psi_a | \Psi_a \rangle \equiv (Z_a)^{2M} = \left(\sum_{\mathcal{N}_a} \lambda^2(\mathcal{N}_a)\right)^{2M}$$

introduces the $U(1)_m$ partition function

$$Z_a(\beta) = \mathrm{tr}\, e^{-\beta H_a}, \text{ with spectrum}\left(H_a\right) = -2\ln\lambda(\mathcal{N}_a)$$



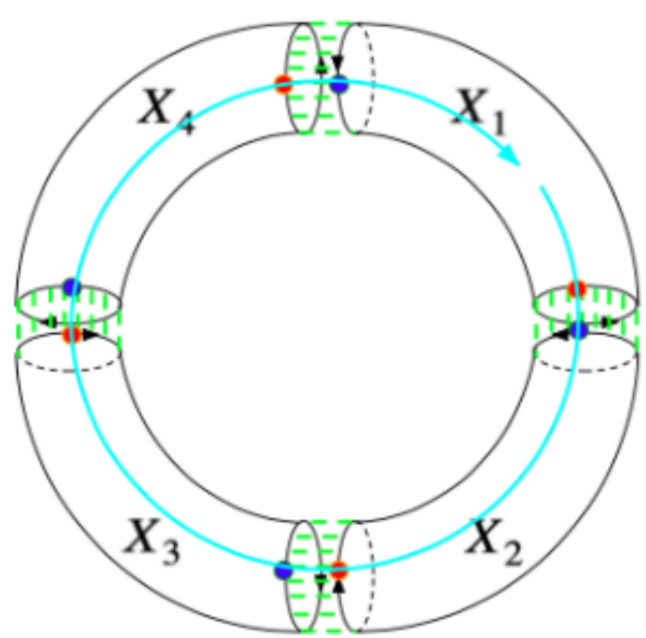
similar to: Li & Haldane (2008); Chandran, Hermanns, Regnault, & Bernevig (2011)

Lemma 1: Torus Geometry

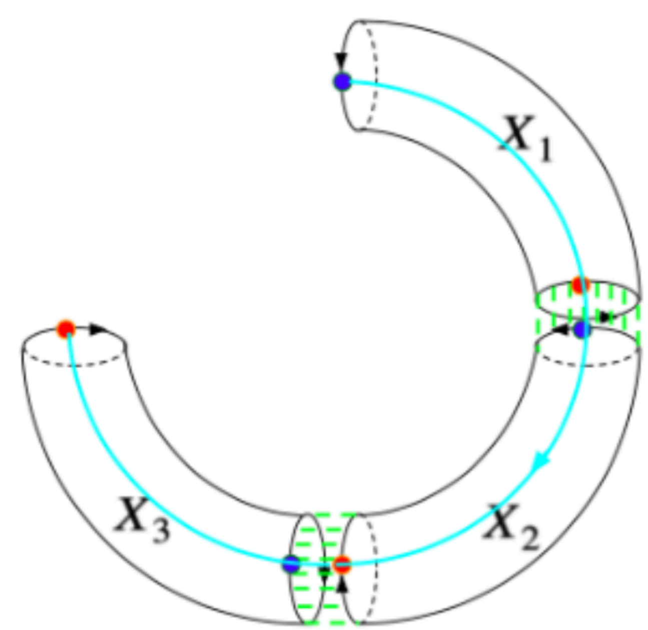
$$\mathcal{E}_{X_{\mathrm{odd}}:X_{\mathrm{even}}} = 2\log \sum_a |\psi_a| \left(\frac{Z_a(1/2)}{\sqrt{Z_a(1)}} \right)^{2M}$$

Lemma 2: Cylinder Geometry

$$\mathcal{E}_{Y_{\mathrm{odd}}:Y_{\mathrm{even}}} = \log \sum_a \left(|\psi_a| \left(\frac{Z_a(1/2)}{\sqrt{Z_a(1)}} \right)^{R} \right)^2$$



Torus Geometry



Cylinder Geometry

Lemma 1 Proof for 2M = 2 cylinders (Laughlin)

compute

$$||\rho^{T_{\text{odd}}}||_1 = \operatorname{tr}\sqrt{(\rho^{T_{\text{odd}}})^\dagger \rho^{T_{\text{odd}}}}$$

torus state density matrix

$$\rho = \sum_{a,a'} \sum_{\overrightarrow{\mathcal{N}}_a, \overrightarrow{\mathcal{N}}'_{a'}} \frac{\psi_a^* \psi_{a'}}{Z_a Z_{a'}} \prod_i \lambda(\mathcal{N}_{a,i})\lambda(\mathcal{N}'_{a',i}) \times$$

$$|\mathcal{N}'_{a',1}\mathcal{N}'_{a',2}\rangle\langle\mathcal{N}_{a,1}\mathcal{N}_{a,2}|_{X_1} \otimes |\mathcal{N}'_{a',2}\mathcal{N}'_{a',1}\rangle\langle\mathcal{N}_{a,2}\mathcal{N}_{a,1}|_{X_2}$$

partial transpose

$$\rho^{T_{\text{odd}}} = \sum_{a,a'} \sum_{\overrightarrow{\mathcal{N}}_a, \overrightarrow{\mathcal{N}}'_{a'}} \frac{\psi_a^* \psi_{a'}}{Z_a Z_{a'}} \prod_i \lambda(\mathcal{N}_{a,i})\lambda(\mathcal{N}'_{a',i}) \times$$

$$|\mathcal{N}_{a,1}\mathcal{N}_{a,2}\rangle\langle\mathcal{N}'_{a',1}\mathcal{N}'_{a',2}|_{X_1} \otimes |\mathcal{N}'_{a',2}\mathcal{N}'_{a',1}\rangle\langle\mathcal{N}_{a,2}\mathcal{N}_{a,1}|_{X_2}$$

$(\rho^{T_{\rm odd}})^\dagger \rho^{T_{\rm odd}}$ is diagonal :

$$\sqrt{(\rho^{T_{\rm odd}})^\dagger \rho^{T_{\rm odd}}} = \sum_{a,a'} \sum_{\vec{\mathcal{N}}_a, \vec{\mathcal{N}}'_{a'}} \left| \frac{\psi_a^* \psi_{a'}}{Z_a Z_{a'}} \prod_i \lambda(\mathcal{N}_{a,i}) \lambda(\mathcal{N}'_{a',i}) \right| \times$$

$$|\mathcal{N}'_{a',1}\mathcal{N}'_{a',2}\rangle\langle\mathcal{N}'_{a',1}\mathcal{N}'_{a',2}|_{X_1} \otimes |\mathcal{N}_{a,2}\mathcal{N}_{a,1}\rangle\langle\mathcal{N}_{a,2}\mathcal{N}_{a,1}|_{X_2}$$

thus (with $M = 1$) :

$$||\rho^{T_{\rm odd}}||_1 = \left(\sum_a |\psi_a| \sum_{\vec{\mathcal{N}}_a} \prod_{i=1}^{2M} \frac{\lambda(\mathcal{N}_{a,i})}{\sqrt{Z_a}} \right)^2$$

$Z_a(1/2)$

$$= \left(\sum_a |\psi_a| \left(\frac{\sum_{\mathcal{N}_a} \lambda(\mathcal{N}_a)}{\sqrt{\sum_{\mathcal{N}_a} \lambda^2(\mathcal{N}_a)}} \right)^{2M} \right)^2$$

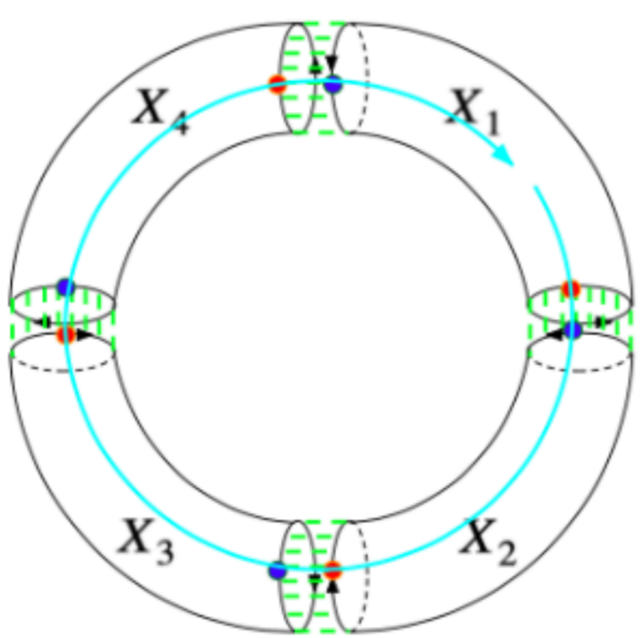
$\sqrt{Z_a(1)}$

Lemma 1: Torus Geometry

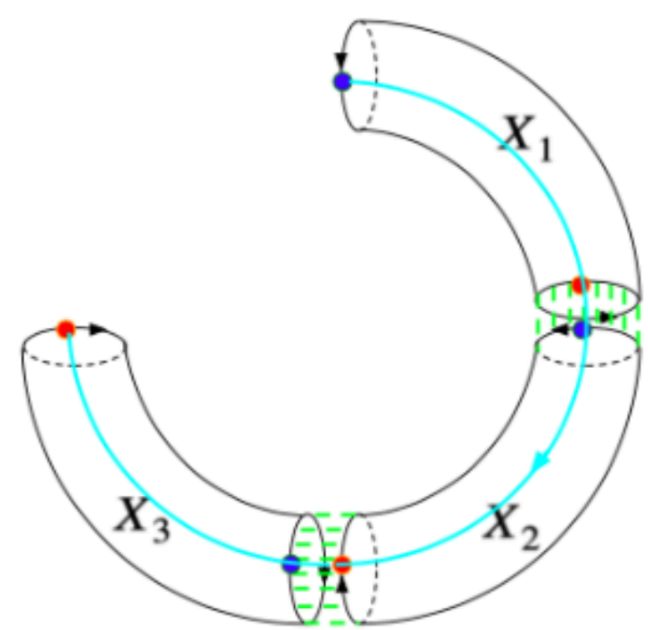
$$\mathscr{E}_{X_{\mathrm{odd}}:X_{\mathrm{even}}} = 2\log\sum_a |\psi_a| \left(\frac{Z_a(1/2)}{\sqrt{Z_a(1)}}\right)^{2M}$$

Lemma 2: Cylinder Geometry

$$\mathscr{E}_{Y_{\mathrm{odd}}:Y_{\mathrm{even}}} = \log\sum_a \left(|\psi_a| \left(\frac{Z_a(1/2)}{\sqrt{Z_a(1)}}\right)^{R}\right)^2$$



Torus Geometry



Cylinder Geometry

Lemma 2 Proof for $R = 1$ shared interfaces (Laughlin)

compute

$$||\rho_Y^{T_{\rm odd}}||_1 = \mathrm{tr}\sqrt{(\rho_Y^{T_{\rm odd}})^\dagger \rho_Y^{T_{\rm odd}}}$$

torus state (composed of 3 cylinders) density matrix

$$\rho = \sum_{a,a'} \sum_{\overrightarrow{\mathcal{N}}_a, \overrightarrow{\mathcal{N}}'_{a'}} \frac{\psi_a^* \psi_{a'}}{Z_a^{3/2} Z_{a'}^{3/2}} \prod_i \lambda(\mathcal{N}_{a,i}) \lambda(\mathcal{N}'_{a',i}) \times$$

$$|\mathcal{N}'_{a',1}\mathcal{N}'_{a',2}\rangle\langle\mathcal{N}_{a,1}\mathcal{N}_{a,2}|_{X_1} \otimes \cdots \otimes |\mathcal{N}'_{a',3}\mathcal{N}'_{a',1}\rangle\langle\mathcal{N}_{a,3}\mathcal{N}_{a,1}|_{X_3}$$

tracing over cylinder X_2 to obtain ρ_Y sets

$$a' = a, \quad \mathcal{N}'_{a',2} = \mathcal{N}_{a,2}, \quad \mathcal{N}'_{a',3} = \mathcal{N}_{a,3}$$

as before $(\rho_Y^{T_{\rm odd}})^\dagger \rho_Y^{T_{\rm odd}}$ is diagonal, thus (with $R = 1$)

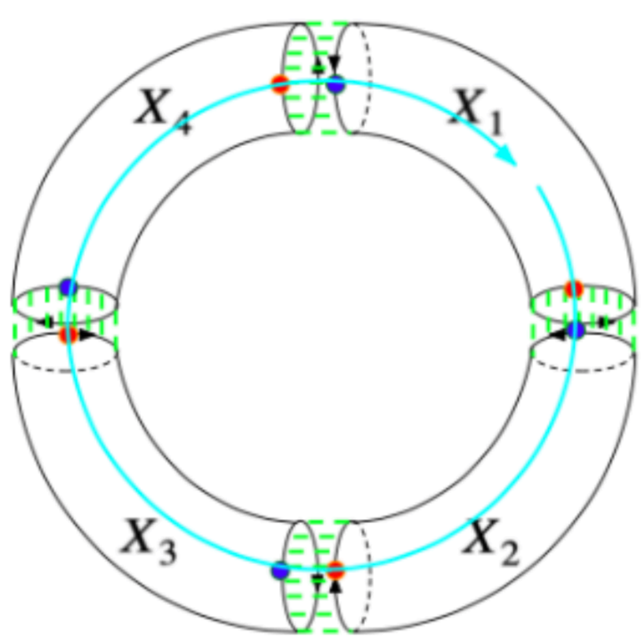
$$
\begin{aligned}
||\rho_Y^{T_{\rm odd}}||_1 &= \sum_a \sum_{\overrightarrow{\mathcal{N}}_a, \overrightarrow{\mathcal{N}'}_a} \left| \frac{\psi_a^* \psi_{a'}}{Z_a^{3/2} Z_a^{3/2}} \prod_i \lambda(\mathcal{N}_{a,i}) \lambda(\mathcal{N}'_{a,i}) \right| \\
&= \sum_a |\psi_a|^2 \left| \frac{Z_a(1/2)}{Z_a^{-1/2}(1)} \right| \left| \frac{Z_a(1/2)}{Z_a^{3/2}(1)} \right| \\
&= \sum_a \left(|\psi_a| \left(\frac{Z_a(1/2)}{\sqrt{Z_a(1)}} \right)^R \right)^2
\end{aligned}
$$

Lemma 1: Torus Geometry

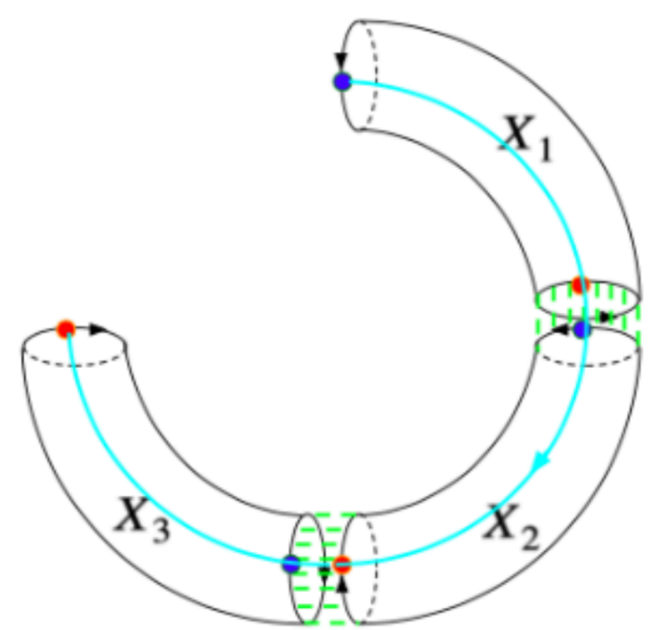
$$\mathscr{E}_{X_{\mathrm{odd}}:X_{\mathrm{even}}} = 2\log \sum_a |\psi_a| \left(\frac{Z_a(1/2)}{\sqrt{Z_a(1)}} \right)^{2M}$$

Lemma 2: Cylinder Geometry

$$\mathscr{E}_{Y_{\mathrm{odd}}:Y_{\mathrm{even}}} = \log \sum_a \left(|\psi_a| \left(\frac{Z_a(1/2)}{\sqrt{Z_a(1)}} \right)^{R} \right)^2$$



Torus Geometry



Cylinder Geometry

partition functions (Laughlin)

Jacobi θ function

$$Z_a(\beta) = \frac{\theta_0^{-a/m}(m\tau)}{\eta(\tau)}, \quad \tau = i\tau_2 = \frac{i\beta v_e}{L}$$

Dedekind η function

compute the limit $L \to \infty$ using modular transform $\tau \to -1/\tau$:

$$Z_a(\beta) \to \frac{1}{\sqrt{m}} \exp\left[\frac{\pi L}{12\beta v_e}\right] + \mathcal{O}(1/L)$$

$\Longrightarrow$

$$\mathcal{E}_{X_{\mathrm{odd}}:X_{\mathrm{even}}} = M\left(\frac{\pi}{2v_e}\right)L - M\log m + 2\log\sum_a |\psi_a|$$

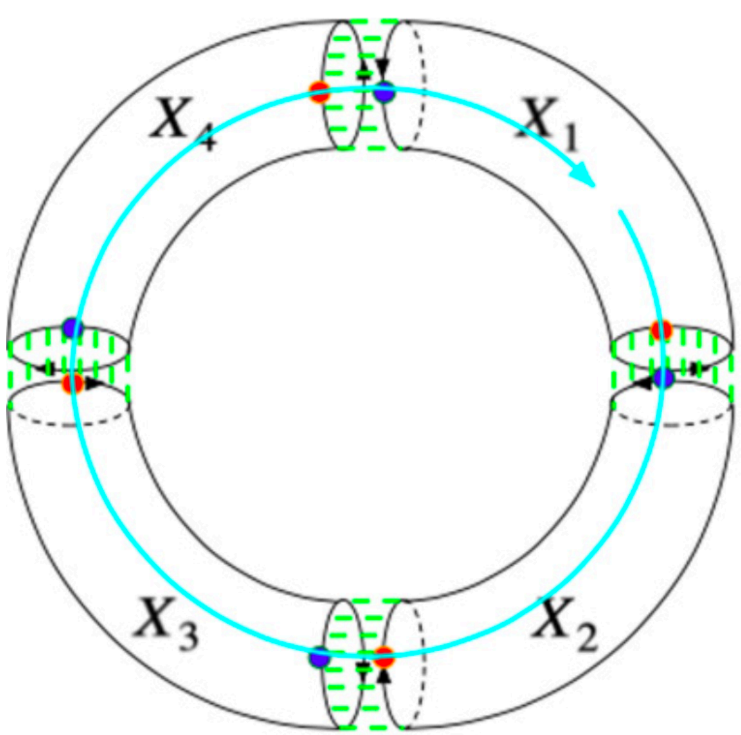
more generally,

$$\text{torus}:\ \mathscr{E}_{X_{\text{odd}}:X_{\text{even}}} = M\alpha L - M\ln\mathscr{D}^2 + 2\ln\sum_a |\psi_a|\, d_a^M$$

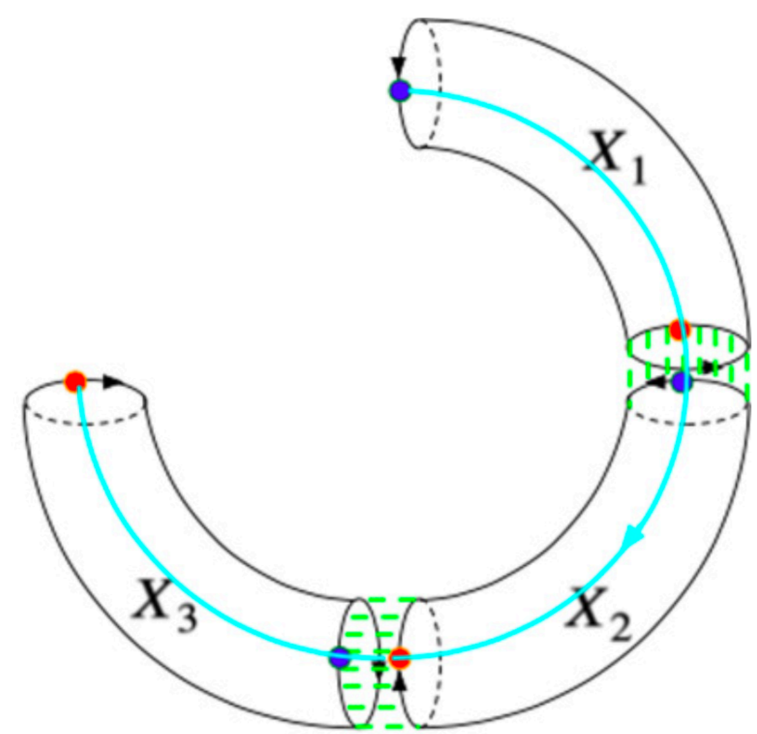
$$\text{cylinder}:\ \mathscr{E}_{Y_{\text{odd}}:Y_{\text{even}}} = \frac{R}{2}\alpha L - \frac{R}{2}\ln\mathscr{D}^2 + \ln\sum_a |\psi_a|^2 d_a^R$$

nonzero only for non-Abelian $(d_a > 1)$ states

Torus Geometry



Cylinder Geometry

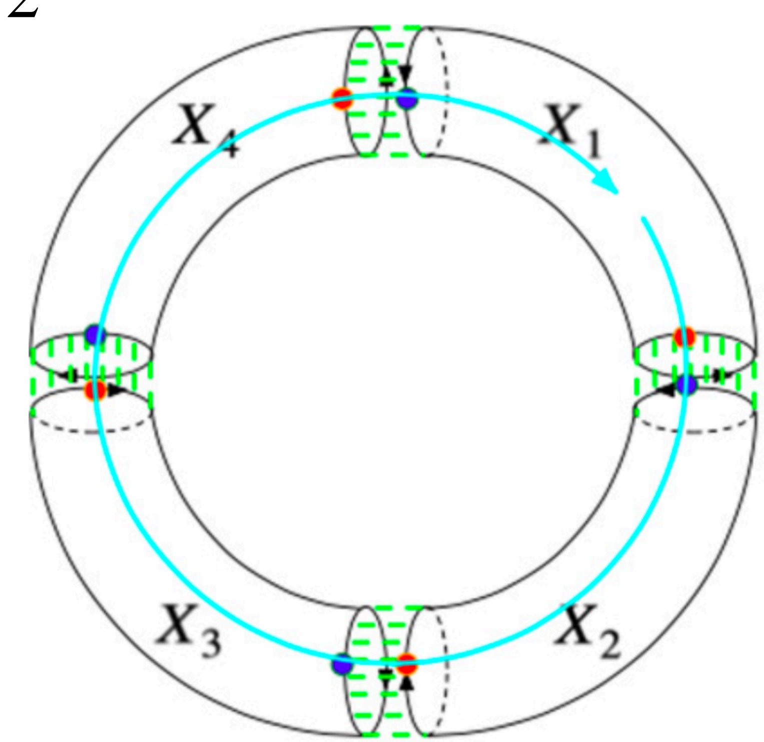


Disentanglement

disentangling condition:

$$\mathcal{E}_{A:BC}(\rho_{ABC}) - \mathcal{E}_{A:B}(\rho_{AB}) = \log \frac{\left(\sum_a |\psi_a| d_a \right)^2}{\sum_a |\psi_a|^2 d_a^2}$$

$$A = X_1, B_1 = X_2, C = X_3, B_2 = X_4, \quad B = B_1 \cup B_2$$

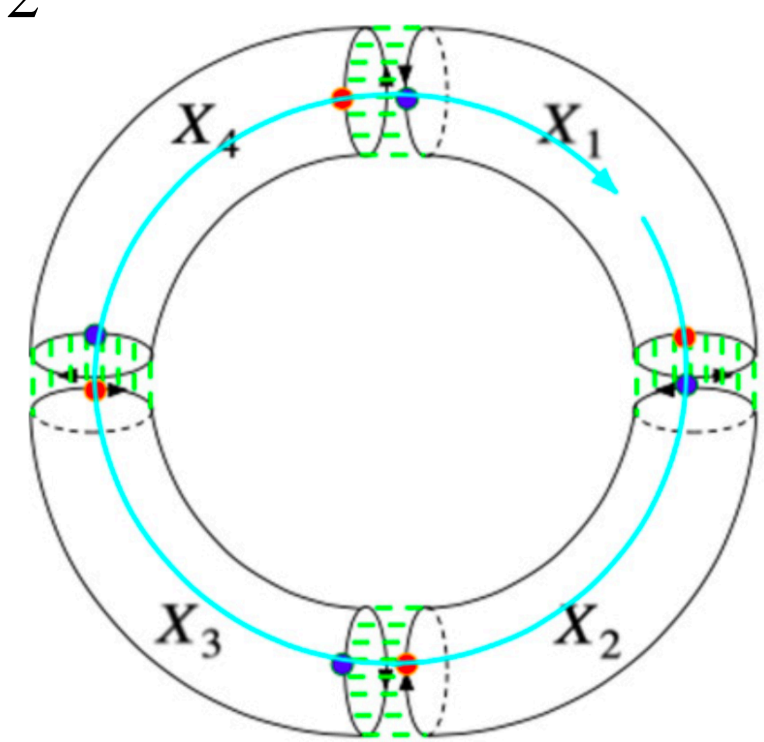


Disentanglement

disentangling condition:

$$\mathscr{E}_{A:BC}(\rho_{ABC}) - \mathscr{E}_{A:B}(\rho_{AB}) = \log \frac{\left(\sum_a |\psi_a| d_a\right)^2}{\sum_a |\psi_a|^2 d_a^2}$$

$$A = X_1, B_1 = X_2, C = X_3, B_2 = X_4, \quad B = B_1 \cup B_2$$



condition $\implies$ $\psi_a = 1$ for some a

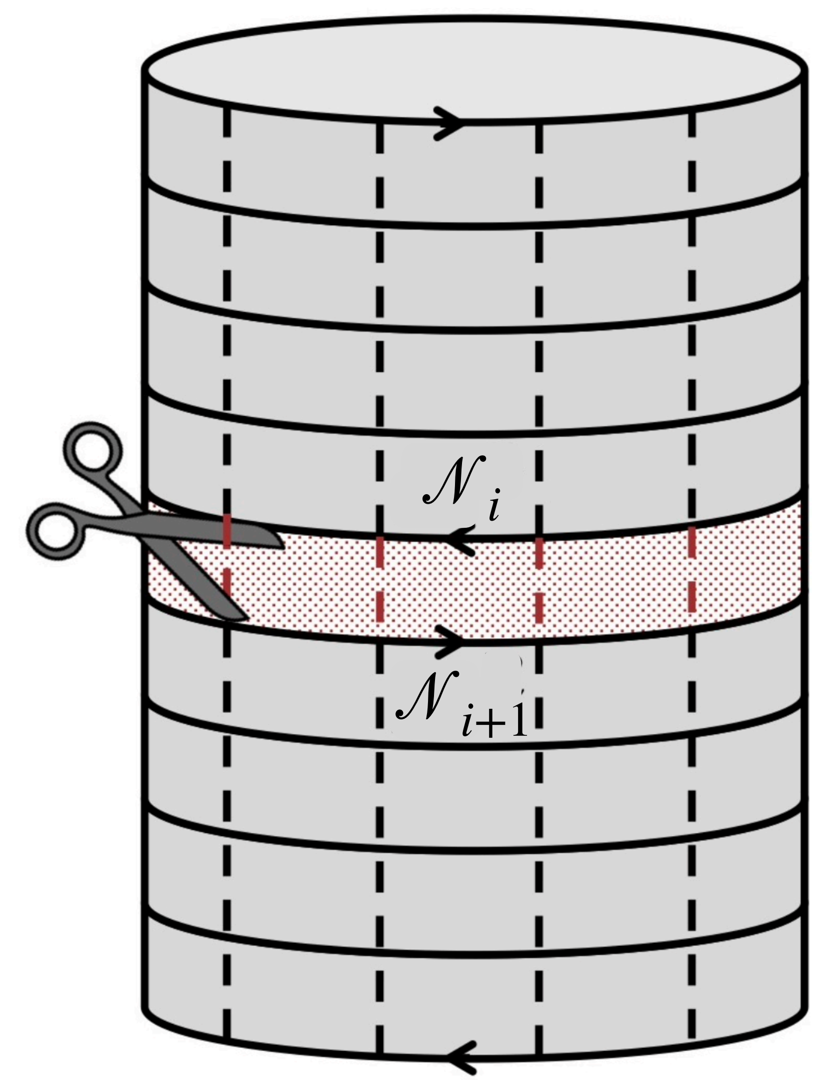
1. sufficient to disentangle Laughlin
2. only necessary for Moore-Read

in each sector a

Laughlin/untwisted Moore-Read can be disentanged

$$|\Psi_a\rangle = \bigotimes_{i=1}^{2M} \sum_{\mathcal{N}_{a,i}} \lambda(\mathcal{N}_{a,i})\,|-\mathcal{N}_{a,i}\rangle_{LX_i} \otimes |\mathcal{N}_{a,i+1}\rangle_{RX_i}$$

i.e., the cylinder state is a tensor product



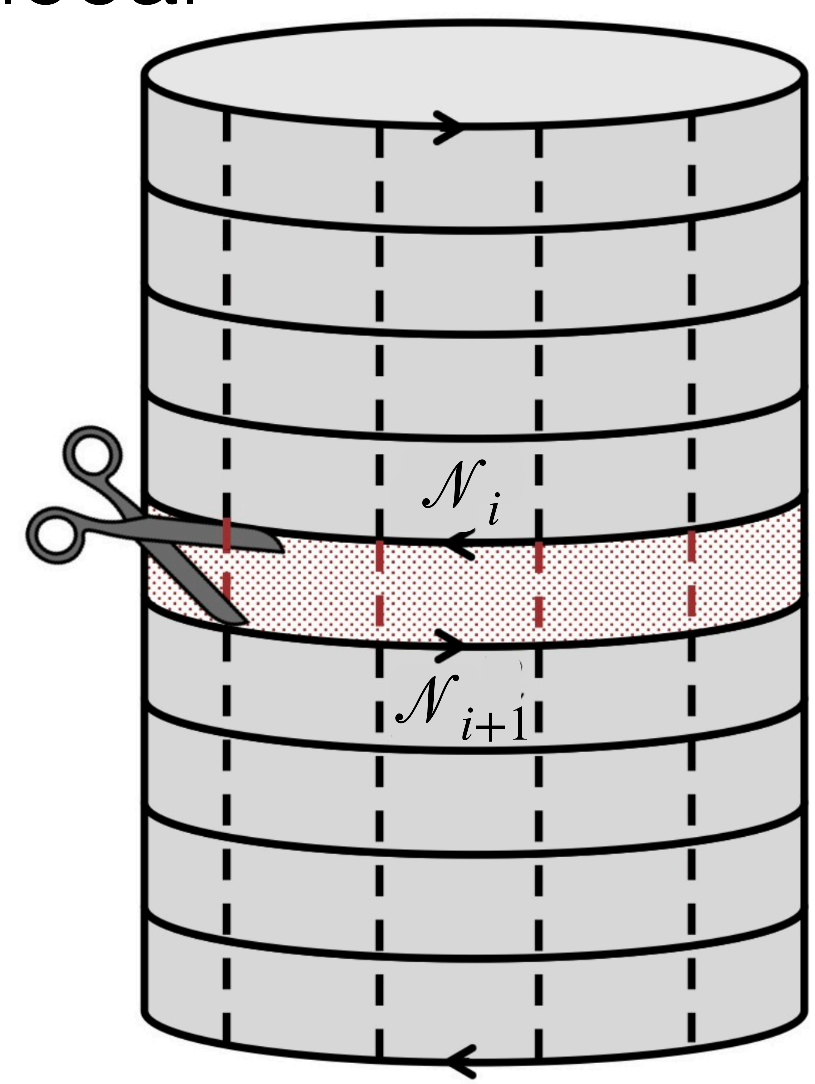
twisted Moore-Read can't be disentanged

$$|\Psi_a\rangle = \bigotimes_{i=1}^{2M} \sum_{\mathcal{N}_{a,i}} \lambda(\mathcal{N}_{a,i})\,|-\mathcal{N}_{a,i}\rangle_{LX_i} \otimes |\mathcal{N}_{a,i+1}\rangle_{RX_i}$$

Kitaev (2000)

cylinder Dirac fermion states are nonlocal

$$|\cdots\gamma_i\cdots\rangle \;=$$



Summary/Open Questions

- Topological degeneracy is distilled by the disentangling condition, as expressed by the entanglement negativity

- Is there a generalization of the disentangling condition that provides a sufficient condition for disentanglement in the Moore-Read case?

e.g., Shapourian, Mong, & Ryu (2020)

- How does a nonzero correlation length affect these results?

- What's the nature of multipartite entanglement in gapless states?

e.g., Calabrese, Cardy, & Tonni (2015)

Thank you!