Hydrodynamics of Fractional Quantum Hall states as Area preserving diffeomorphismss

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Where is a place of the theory of FQHE?

For a long time the theory of FQHE survived without a Hamiltonian

- Adiabatic transport (the standard avenue of the theory): the ground state (say Laughlin wf) and the adiabatic properties (a gapped spectrum) suffices.

$$\Psi \sim \prod_{i>j}^{N} (z_i - z_j)^{\beta} e^{-\frac{B}{4}\sum_i |z_i|^2}, \quad \nu = \beta^{-1} - \text{ filling fraction}$$

- E.g., e may consider a system in multiply-connected geometry. Then the ground state is a bundle whose base is moduli space.

- A Hamiltonian is <u>not</u> necessary;

- For anything else, like optical properties of the bulk, and even dynamics on the edge one needs to know excited states.

A Hamiltonian is <u>necessary</u>!

HYDRODYNAMICS

- The Hamiltonian could be obtained from the basic physical fact:

$$\boxed{\text{Electronic states in the Quantum Hall regime is } \textit{incompressible fluid}}$$

- Incompressibility follows from the main property of the Lowest Landau Level: all states there are <u>holomorphic</u>!

- This fact alone allows to establish dynamics of the QH states beyond just the ground state!

- Incompressible flows is a geometrically governed dynamics requires minimal knowledge of microscopic

A search for a Hamiltonian is based on the basic facts

Fractional quantum Hall (interacting states on the Lowest Landau Level) form:

- **Liquid**
- **Incompressible** (*viz.* all states are *Holomorphic*)
- **Dissipation-free** liquid (inviscid), and non-resistive (at small T)
- **Ultra quantum**
- Flows are **chiral**

Quantum hydrodynamics is a natural approach.

A minor obstacle on the way is that the Hydrodynamics had never been successfully/systematically quantized

Incompressible Hydrodynamics

- Incompressible hydrodynamics is the universal theory which does not appeal to microscopic knowledge;

- Major (and remarkable) property: Once we know a stationary state and this state is adiabatic the Hamiltonian and everything else could be obtained in a unique manner.

- Hence, the Laughlin w.f. alone is sufficient to built the Hamiltonian

$$\boxed{\Psi \sim \prod_{i>j}^{N} (z_i - z_j)^{\beta} e^{-\frac{B}{4}\sum_i |z_i|^2}, \quad \nu = \beta^{-1} - \text{ filling fraction}}$$

Incompressible flows are group action

How it can be? Why just one state is sufficient?

Because,

Flow of incompressible fluid is the *Group Action*

Group is *area-preserving* diffeomorphisms *SDiff* or W_∞

$$|\Psi(t+dt)\rangle = G_{dt}|\Psi(t)\rangle \Rightarrow G_{dt} = 1 + idtH$$

Coadjoint orbit of *SDiff* - a long standing problem in mathematics.

The rest of the talk is a practical recipe how to do that

AREA-PRESERVING DIFFEOMORPHISMS

Introduced to QHE-physics by Andrea,

and also by Girvin, McDonald and Platzmann

$$\hat{\rho}_{\mathbf{k}} = \int : e^{-i\mathbf{k}\cdot\mathbf{r}} \hat{\rho}(\mathbf{r}) : \; d^2\mathbf{r}$$

$$\hat{\rho}_{\mathbf{k}} = \sum_i e^{-\frac{i}{2}\mathrm{k} z_i^\dagger} e^{-\frac{i}{2}\bar{\mathrm{k}} z_i}$$

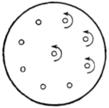
$$z_i^\dagger = 2\partial_{z_i}, \quad \mathrm{k} = k_x + ik_y$$

$$\boxed{[\hat{\rho}_{\mathbf{k}},\ \hat{\rho}_{\mathbf{k}'}] = \left(\tfrac{\nu}{2\pi}\right) i\mathbf{k} \times \mathbf{k}' \hat{\rho}_{\mathbf{k}+\mathbf{k}'}}$$

Operators acting in the Bargmann space

HYDRODYNAMICS OF FQHE

Hydrodynamics of FQH states ≡ Hydrodynamics of Fast Rotating Superfluid



Rotating superfluid is a dense array of quantum vortices (vortex matter)

In this correspondence vortices are identified with electrons (with attached magnetic flux)

Vortices ⟷ Electrons

$$\hat{\rho} = \frac{\nu}{2\pi}(\nabla \times \hat{\mathbf{v}})$$

$$H = [\text{gap}] \int \frac{\hat{\mathbf{v}}^2}{2}$$

Hamiltonian

$$H = [\text{gap}] \sum_{k \neq 0} \frac{1}{2} \hat{\mathbf{v}}_{-k} \hat{\mathbf{v}}_k = [\text{gap}] \sum_{k \neq 0} \frac{1}{2k^2} \hat{\rho}_{-k} \hat{\rho}_k,$$

$$\hat{\rho} = \frac{\nu}{2\pi} (\nabla \times \hat{\mathbf{v}}), \quad \hat{\rho}_k = \frac{\nu}{2\pi} (i\mathbf{k} \times \hat{\mathbf{v}})$$

$$[\hat{\rho}_{\mathbf{k}},\ \hat{\rho}_{\mathbf{k}'}] = \left(\frac{\nu}{2\pi} \right) \mathbf{ik} \times \mathbf{k}' \hat{\rho}_{\mathbf{k}+\mathbf{k}'}$$

$$\mathrm{u}(z,t) = u_x - \mathrm{i}u_y = -\mathrm{i}\Omega\bar{z} + \mathrm{i}\sum_{i=1}^{N} \frac{\Gamma}{z - z_i(t)}$$

Kirchhoff equations: $\mathrm{i}\dot{\bar{z}}_i = \Omega\bar{z}_i - \sum_{i\neq j}^{N} \frac{\Gamma}{z_i(t) - z_j(t)}$

Onsager quantization: $\Gamma = \frac{h}{m_{\mathrm{Atom}}}$

- Quantization $\{z_i, \bar{z}_j\}_{P.B.} \to [z_i, \bar{z}_j] = 2\ell^2\delta_{ij}, \qquad \bar{z}_i = 2\ell^2\partial_{z_i}$
- Stationary quantum flow: $-\hbar\partial_{z_i}|\Psi\rangle = \left(\Omega\bar{z}_i - x\sum_{j\neq i}\frac{\Gamma}{z_i - z_j}\right)|\Psi\rangle$
- Solution is Laughlin's w.f. $\Psi \sim \prod_{i>j}^{N}(z_i - z_j)^{2\beta} e^{-\frac{B}{4}\sum_i |z_i|^2}, \quad \beta = \frac{\Gamma}{\hbar\Omega}$

Euler Equation for incompressible rotating fluid

$$\partial_t \hat{\mathbf{v}} + (\hat{\mathbf{v}} \cdot \nabla)\,\hat{\mathbf{v}} + \nabla p = \Omega \times \hat{\mathbf{v}}, \quad \nabla \cdot \hat{\mathbf{v}} = 0,$$
$$\frac{1}{V} \int (\nabla \times \hat{\mathbf{v}}) dV = 2\Omega,$$

The stationary flow happens to be the Laughlin's wave function (PW 2012)

$$\Psi \sim \prod_{i>j}^{N} (z_i - z_j)^{2\beta} e^{-\frac{B}{4}\sum_i |z_i|^2}$$

Not particularly useful: We want to express H not in terms of operators but by their expectation values $\hat{\rho}_k \to \langle \rho_k \rangle = \rho_k$ and replace [,] by Poisson brackets { , }.

Structure factor

$$s_k := \langle 0|\hat{\rho}_k \hat{\rho}_{-k}|0\rangle = \int e^{i\mathbf{k}(\mathbf{r}_1-\mathbf{r}_2)}|\Psi(z_1, z_2, \ldots, z_N)|^2 d^2 z_3 \ldots d^2 z_N$$

$$s_k^{-1} = \frac{2}{k^2} - \left(\frac{1}{2\nu} - 1\right) + \frac{k^2}{24\nu} + \ldots,$$

$$s_k = \frac{1}{2}k^2 + \frac{1}{4\nu}\left(\frac{1}{2} - \nu\right)k^4 + \frac{1}{8\nu^2}\left(\frac{3}{4} - \nu\right)\left(\frac{1}{3} - \nu\right)k^6 + \ldots$$

Linear response theory

The Hamiltonian in harmonic approximation

$$H \approx [\text{gap}] \sum_{k\neq 0} s_k^{-1} \rho_{-k}\rho_k$$

$$s_k^{-1} = \frac{2}{k^2} - \left(\frac{1}{2\nu} - 1\right) + \frac{k^2}{24\nu} + \dots$$

Then we can write equation of motion as

$$\dot{\rho}_k = \{H, \rho_k\},$$

$$\{\rho_{\mathbf{k}},\ \rho_{\mathbf{k}'}\} = \left(\frac{\nu}{2\pi}\right)\mathbf{k}\times\mathbf{k}'\rho_{\mathbf{k}+\mathbf{k}'}$$

However, harmonic Hamiltonian changes under *SDiff* and does not generate *SDiff* flow. We need to find higher order correction in ρ_k and in gradients to make it invariant.

$$\text{Density of electrons=vorticity } \rho = \frac{\nu}{2\pi}(\nabla \times \mathrm{v})$$

$$\mathscr{H}_{\text{semi}} = \int \left(\tfrac{1}{2}\mathrm{v}^2 - \overbrace{\tfrac{\Gamma^2}{8\pi}\rho \log \rho}^{\text{odd viscosity}} \right)$$

$$\mathscr{H}_{\text{quantum}} = -\nu \int \rho \log \rho + \underbrace{\tfrac{1}{2} \log \mathrm{Det}(-\Delta_\rho)}_{\text{gravitational anomaly}}.$$

$$\Delta_\rho = \rho\, \partial_z \partial_{\bar{z}} - \text{Laplace-Beltrami operator with metric } ds^2 = \rho\, dz d\bar{z}$$

$$\text{Polyakov formula: } \log \mathrm{Det}(-\Delta_\rho) = -\frac{1}{12\pi} \int (\nabla \log \rho)^2$$

Stress, Momentum and Angular Momentum

Momentum flux and stress

$$\boxed{H \longrightarrow \Pi_{ij} = \mathrm{v}_i \mathrm{v}_j + p\delta_{ij} + \frac{\nu}{4\pi} T_{ij}}$$

Stress in complex coordinates

$$\boxed{\sigma = \log \rho}$$

$$T_{zz} = -\underbrace{\frac{1}{2}\partial_z \mathrm{v}_z}_{\text{odd-viscosity}} + \frac{1}{12}\underbrace{\left(-\frac{1}{2}(\partial_z \sigma)^2 + \partial_z^2 \sigma\right)}_{\text{Schwarzian}[\sigma]=\text{effect of grav. anomaly}}$$

$$T_{z\bar{z}} = \underbrace{-\frac{\pi}{2\nu}(\rho - \bar{\rho}) + \frac{1}{48}\Delta\sigma}_{\text{trace anomaly}}$$

Trace anomaly yields the momentum of the stationary flow (ground state)

$$\mathbf{P} = \frac{2}{\pi} \nabla \times T_{z\bar{z}}$$

and angular momentum (or spin)

$$L = \frac{2}{\ } T$$

Hamiltonian

$$\mathscr{H} = \iint \rho(r) \log \tfrac{1}{|r-r'|} \rho(r') + \int \left((\tfrac{1}{2} - \nu) \log \rho + \frac{1}{12\pi} (\nabla \log \rho)^2 \right)$$

The structure function $s_k = \langle \rho_k \rho_{-k} \rangle$ follows from the Hamiltonian $\rho = \bar{\rho} + \sum_k e^{ikr} \rho_k$

$$s_k = \frac{1}{2} k^2 + \frac{1}{4\nu} \left(\frac{1}{2} - \nu \right) k^4 + \frac{1}{8\nu^2} \left(\frac{3}{4} - \nu \right) \left(\frac{1}{3} - \nu \right) k^6 + \dots$$

$$\{\rho_{\mathbf{k}},\ \rho_{\mathbf{k}'}\} = \left(\frac{\nu}{2\pi} \right) \mathbf{k} \times \mathbf{k}' \rho_{\mathbf{k}+\mathbf{k}'}$$

Now one can compute optical properties, like inelastic light scattering as a linear response to a smooth variation of the density, etc.