Bootstrapping Cosmological ~ Fluctuations ~

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GGI Lectures

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Based on work with...

... Daniel Baumann, Wei-Ming Chen, Austin Joyce,
Hayden Lee & Guilherme L. Pimentel

References:

1811.00024

1910.14051

2005.04234

2106.05294

Background:

0907.5424

astro-ph/0210603

1002.1416

1602.07982

1503.08043

1708.03872

& many others!

Outline

In these lectures we will learn about...

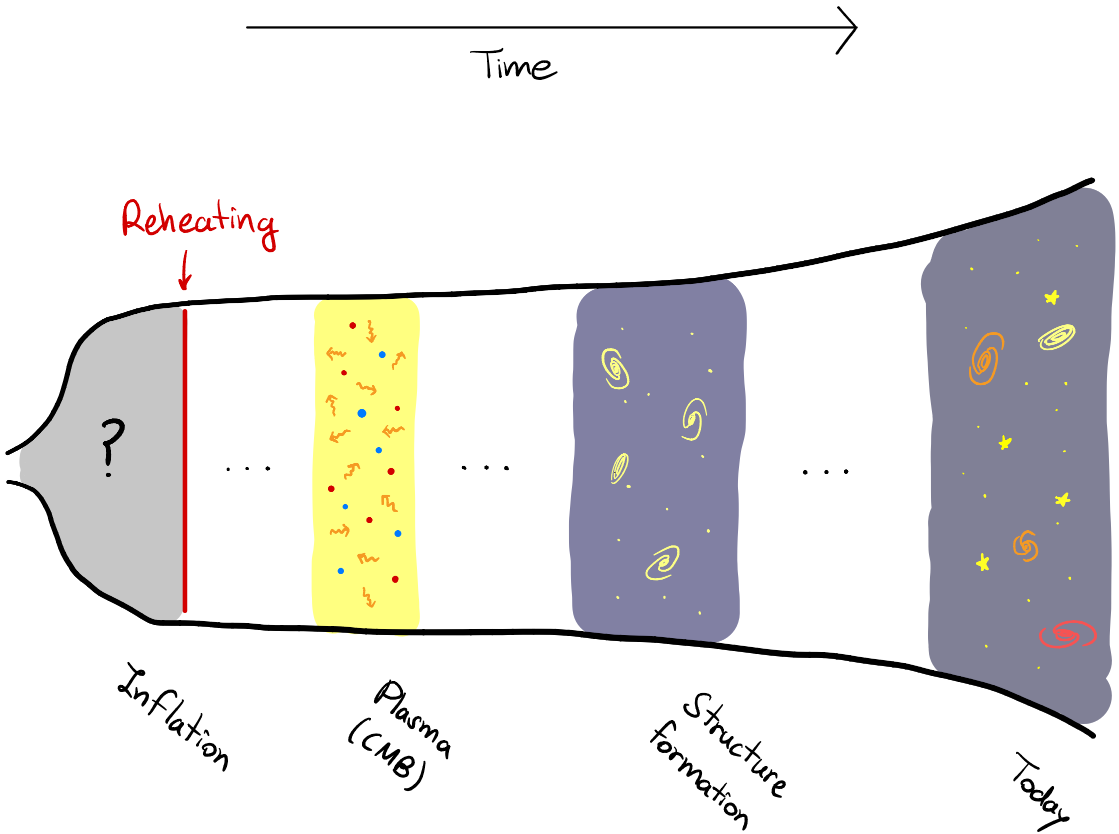
$$\ldots\ \underbrace{\text{The } \underbrace{\text{Cosmological}}_{\text{Part 1}} \ \underbrace{\text{Bootstrap}}_{\text{Part 2}}}_{\text{Part 3}}$$

1. What are cosmological correlators? Why do we care?
2. What is a bootstrap? Why to bootstrap cosmo. correlators?
3. How? What are the bootstrap techniques?

1. Cosmological correlators

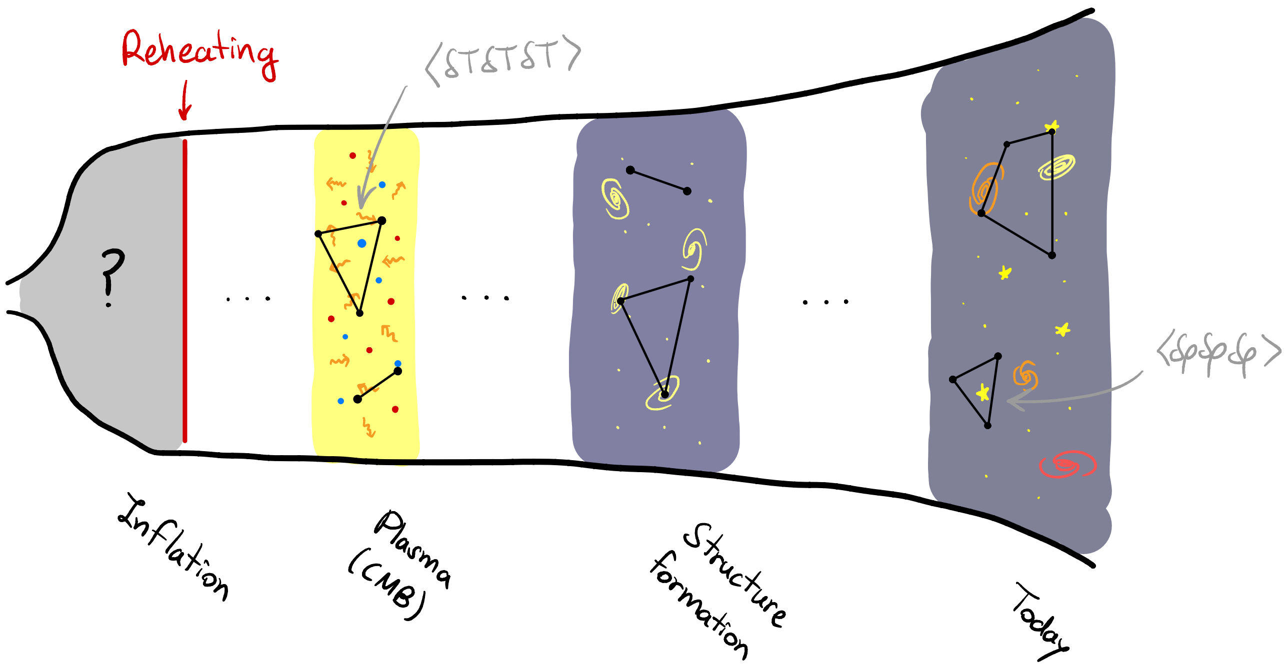
2. Bootstrap

3. Cosmological bootstrap

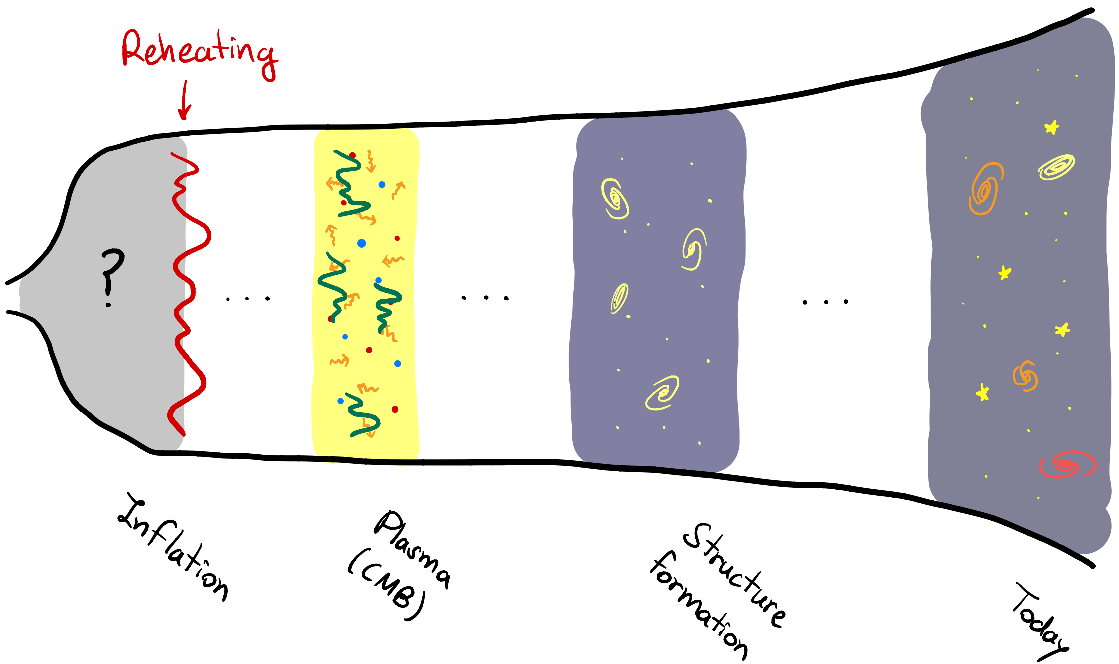


Time
Reheating
?
Inflation
Plasma (CMB)
Structure formation
Today

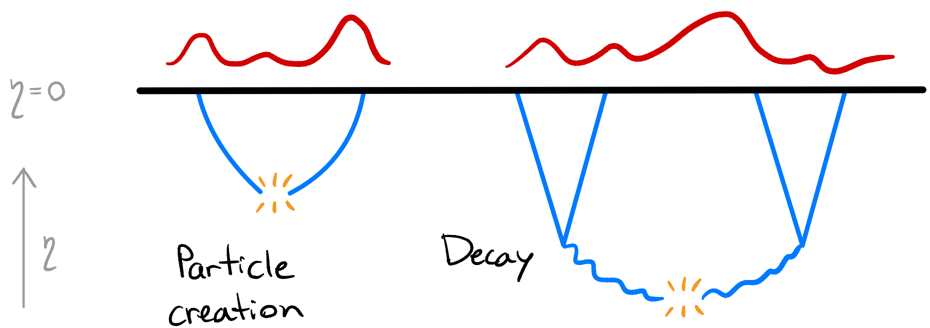
Cosmology measures spatial correlations in late time structure and tries to construct a consistent history of the universe that explains them



The primordial perturbations generated during inflation seeded the late time structure



The dynamics of inflation is encoded in these primordial perturbations



<u>Energies</u>

LHC: $\sim 10^4$ GeV

Inflation: $\lesssim 10^{14}$ GeV

$\Longrightarrow$ The early universe was a huge cosmological collider

The observables of interest are correlation functions of fields at the end of inflation

Two d.o.f. in every inflationary model:

ζ : Scalar (inflaton + graviton) ; γ_{ij} : Tensor (graviton)

- 2-point functions $\begin{cases} \langle \zeta\zeta \rangle : \text{ Measured} \\ \langle \gamma\gamma \rangle : \text{ Not measured, tells the energy scale of inflation} \end{cases}$

- 3,4,...-point functions (non-gaussianity): Not measured, contain information about the dynamics

$$\langle \zeta\zeta\zeta \rangle, \langle \gamma\zeta\zeta \rangle, \langle \gamma\gamma\gamma \rangle, \langle \zeta\zeta\zeta\zeta \rangle \ldots$$

We will focus on a more primitive object, the wave-function of the universe:

$$\Psi[\phi(\vec{x})] \equiv \langle \phi(\vec{x}) | 0 \rangle = \int_{\varphi(-\infty)=0}^{\varphi(0)=\phi} \mathcal{D}\varphi \cdot e^{iS[\varphi]}$$

Late time field profile

It computes the correlators:

$$\langle \phi(\vec{x}_1) \ldots \phi(\vec{x}_N) \rangle (\eta=0) = \int \mathcal{D}\phi \cdot \phi(\vec{x}_1) \cdots \phi(\vec{x}_N) \cdot |\Psi[\phi]|^2$$

In perturbation theory:

$$\Psi[\phi] = \int_{\varphi(-\infty)=0}^{\varphi(0)=\phi} \mathcal{D}\varphi \cdot e^{iS[\varphi]} \approx e^{iS_{cl}[\phi]}$$

We work in momentum space

If the field profiles are small:

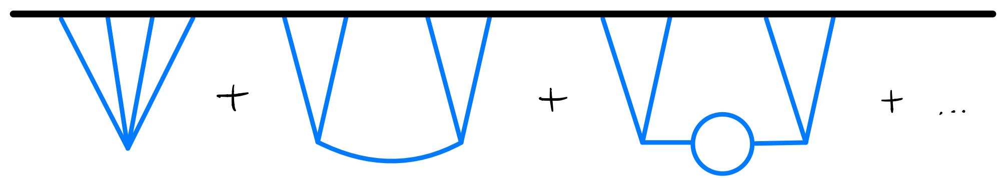
$$\Psi[\phi] \simeq \exp\left[-\sum_{n=2}^{\infty}\frac{1}{n!}\int d\vec{k}_1\cdots d\vec{k}_n \cdot \psi_n(\vec{k}\text{'s})\cdot \phi(\vec{k}_1)\cdots\phi(\vec{k}_n)\right]$$

The wavefunction coefficients contain all the relevant information, schematically:

$$\langle \phi_1 \phi_2 \rangle = \frac{1}{2\,\mathrm{Re}\ \psi_2(\vec{k}_1=-\vec{k}_2)}\ ,\qquad \langle \phi_1\phi_2\phi_3\rangle = \frac{\mathrm{Re}\ \psi_3(\vec{k}_1,\vec{k}_2,\vec{k}_3)}{4\prod_{n=1}^{3}\mathrm{Re}\ \psi_2(\vec{k}_n)}\ ,$$

$$\langle \phi_1\phi_2\phi_3\phi_4\rangle = \frac{\mathrm{Re}\ \psi_4(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) + 2\cdot\mathrm{Re}\ \psi_3(\vec{k}_1,\vec{k}_2,-\vec{k}_I)\cdot\mathrm{Re}\ \psi_3(\vec{k}_I,\vec{k}_3,\vec{k}_4)}{8\prod_{n=1}^{4}\mathrm{Re}\ \psi_2(\vec{k}_n)}\ ,\quad \ldots$$

Ψ_n's can be computed using a diagramatic (Feynman-Witten) expansion

$\Psi_4 =$ 

Propagators come from the classical solution:

$$\varphi_{cl}(\vec{k},\eta) = \phi(\vec{k}) \cdot K(\vec{k},\eta) + \int d\eta' \cdot \mathcal{G}(\vec{k},\eta,\eta') \cdot \frac{\delta S_{int}}{\delta\varphi}$$

Bulk-to-boundary propagator

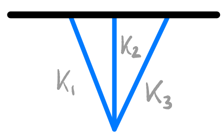
Bulk-to-bulk propagator

$$\Psi_n = \sum_{\text{diagrams}} \left(\prod_{\text{vertices}} \int d\eta \right) \cdot (\text{vertices}) \cdot (K \cdots K) \cdot (\mathcal{G} \cdots \mathcal{G})$$

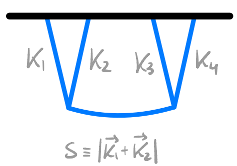
Example: Massless scalar in flat space

Enforces boundary condition at $t=0$

$$\begin{cases} K(\vec{k},t) = e^{ikt} \\ G(\vec{k},t,t') = \frac{1}{2k}\left[e^{ik(t'-t)}\cdot\theta(t-t') + e^{ik(t-t')}\cdot\theta(t'-t) - e^{ik(t+t')}\right] \end{cases}$$



$$\psi_3 = i\int_{-\infty}^{0} dt \cdot \prod_{n=1}^{3} K(\vec{k}_n,t) = i\int_{-\infty}^{0} dt\cdot e^{i\overbrace{(k_1+k_2+k_3)}^{\equiv K}t} = \frac{1}{K}\,e^{iKt}\Big]_{-\infty(1-i\varepsilon)}^{0} = \frac{1}{K}$$



$$\psi_4 = i^2\int_{-\infty}^{0} dt_L\, dt_R \cdot K(\vec{k}_1,t_L)\cdot K(\vec{k}_2,t_L)\cdot G(\vec{k}_1+\vec{k}_2, t_L, t_R)\cdot K(\vec{k}_3,t_R)\cdot K(\vec{k}_4,t_R) =$$

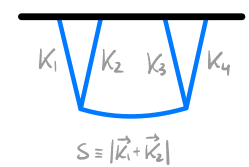
$$= 1 + 2 + 3$$

① $$\frac{-1}{2s}\int_{-\infty}^{0} dt_L\, dt_R \cdot e^{i(K_1+K_2-s)t_L} \cdot e^{i\overbrace{(K_3+K_4+s)}^{\equiv E_R}t_R} \cdot \Theta(t_L-t_R) = \frac{-1}{2s}\int_{-\infty}^{0} dt_L \cdot e^{i(K_1+K_2-s)t_L} \cdot \int_{-\infty}^{t_L} dt_R \cdot e^{iE_R t_R} =$$

$$= \frac{i}{2s}\int_{-\infty}^{0} dt_L \cdot e^{i(K_1+K_2-s)t_L} \cdot \left.\frac{e^{iE_R t_R}}{E_R}\right|_{-\infty(1-i\varepsilon)}^{t_L} = \frac{i}{2sE_R}\int_{-\infty}^{0} dt_L \cdot e^{i\overbrace{(K_1+K_2+K_3+K_4)}^{\equiv E}t_L} = \frac{1}{2sEE_R}$$

② $$\frac{-1}{2s}\int_{-\infty}^{0} dt_L\, dt_R \cdot e^{i\overbrace{(K_1+K_2+s)}^{\equiv E_L}t_L} \cdot e^{i(K_3+K_4-s)t_R} \cdot \Theta(t_R-t_L) = \frac{1}{2sEE_L}$$

③ $$\frac{1}{2s}\int_{-\infty}^{0} dt_L\, dt_R \cdot e^{iE_L t_L} \cdot e^{iE_R t_R} = -\frac{1}{2sE_L E_R}$$



Then:

$$\psi_4 = \frac{1}{2s}\left(\frac{1}{EE_R}+\frac{1}{EE_L}-\frac{1}{E_LE_R}\right) = \frac{1}{2s}\cdot\frac{E-E_L-E_R}{EE_LE_R} = \frac{1}{EE_LE_R}$$

Massive scalar in dS:

$$\begin{cases} K_\nu(\vec{k},\eta) = \left(\frac{\eta}{\eta_*}\right)^{3/2} \dfrac{H_\nu^{(2)}(-k\eta)}{H_\nu^{(2)}(-k\eta_*)} \qquad \text{with} \quad \nu \equiv \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \\[2em] G_\nu(\vec{k},\eta,\eta') = \frac{\pi}{4}(\eta\eta')^{3/2}\left[H_\nu^{(2)}(-k\eta')\cdot H^{(1)}(-k\eta)\cdot \Theta(\eta-\eta') + (\eta\leftrightarrow\eta') - \dfrac{H_\nu^{(1)}(-k\eta_*)}{H_\nu^{(2)}(-k\eta_*)} H_\nu^{(2)}(-k\eta) H_\nu^{(2)}(-k\eta')\right] \end{cases}$$

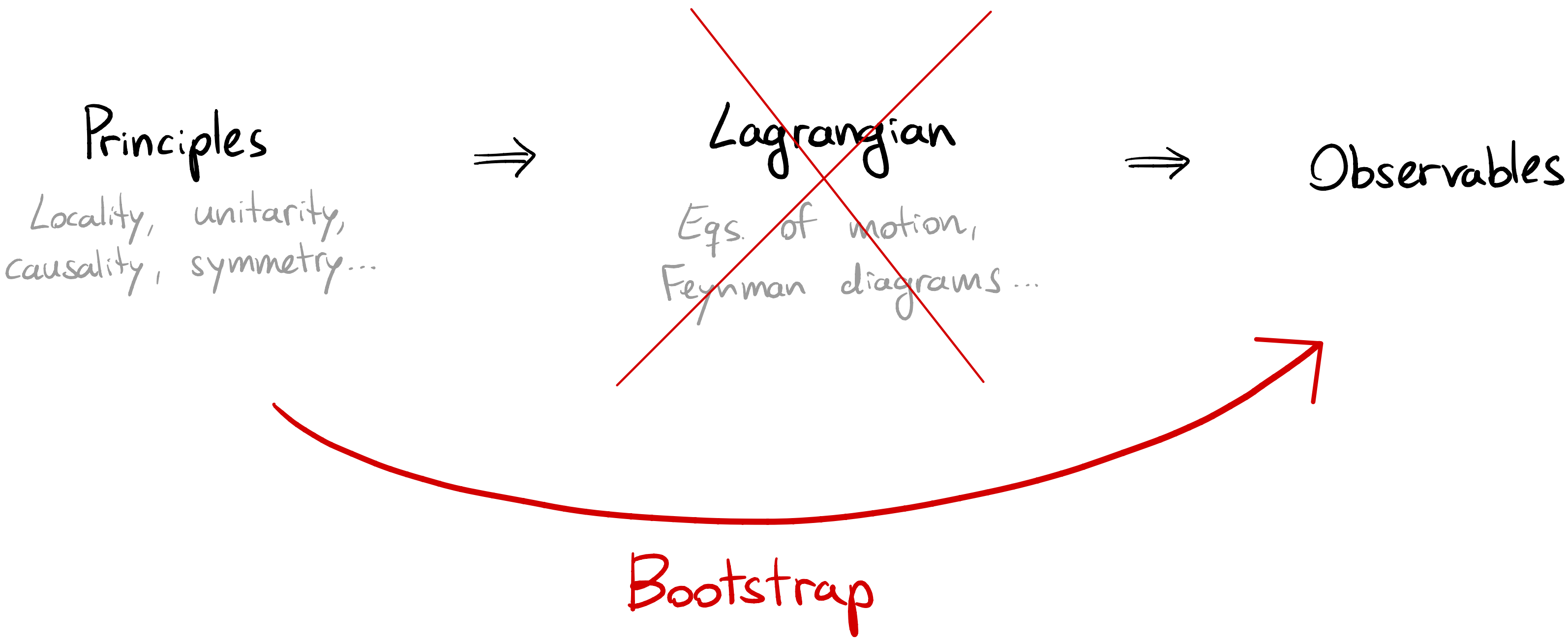
Very complicated integrals in general...

... we need some other method!

1. Cosmological correlators

2. Bootstrap

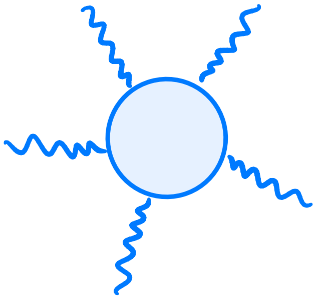
3. Cosmological bootstrap



Principles
Locality, unitarity, causality, symmetry...
Lagrangian
Eqs. of motion, Feynman diagrams...
Observables
Bootstrap

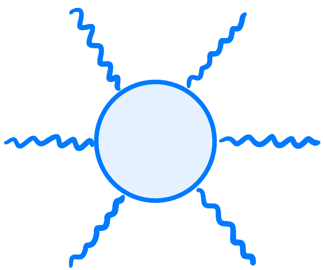
A successful story: the S-matrix

For example, gluons at tree-level:



$$= \text{25 Feynman diagrams} = \begin{cases} A(+++++) = 0 \\ A(-++++) = 0 \\ A(--+++) = \dfrac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle} \end{cases}$$

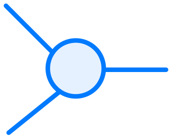
with $\langle ij\rangle = \sqrt{2 p_i p_j}\, e^{i\varphi}$



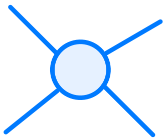
$$= \text{220 Feynman diagrams} = \begin{cases} A(++++++) = 0 \\ A(-+++++) = 0 \\ A(--++++) = \dfrac{\langle 12\rangle^3}{\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 56\rangle\langle 61\rangle} \end{cases}$$

Originally guessed, later bootstrapped

Parke, Taylor '85



3-pt. amplitudes: Poincaré invariance

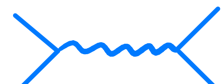


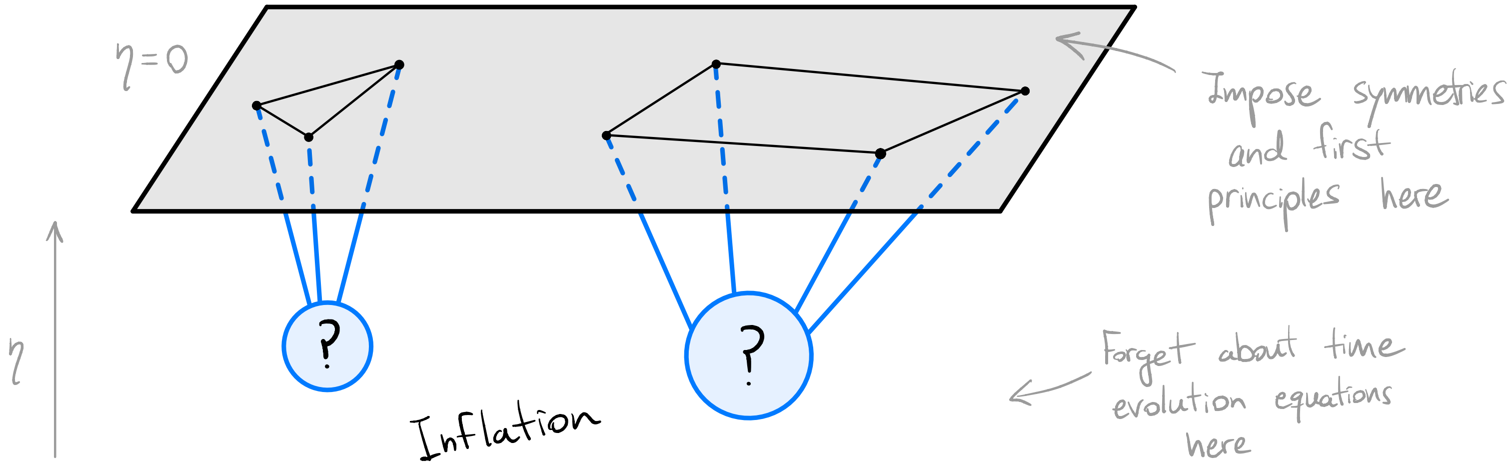
4-pt. amplitudes at tree level:

- Lorentz invariance $\longrightarrow$ Mandelstam variables s, t, u
- Locality $\longrightarrow$ Simple poles from particle exchange
- Unitarity $\longrightarrow$ Positive coefficient

$$\lim_{s \to M^2} A_4^{(\text{tree})} = \frac{A_3 \cdot A_3}{s - M^2}$$

Then: $$A_4^{(\text{tree})}(s,t) = \sum a_{nm} s^n t^m + \frac{g^2}{s - M^2} \cdot P_s\left(1 + \frac{2t}{M^2}\right) + (t\text{-ch.}) + (u\text{-ch.})$$

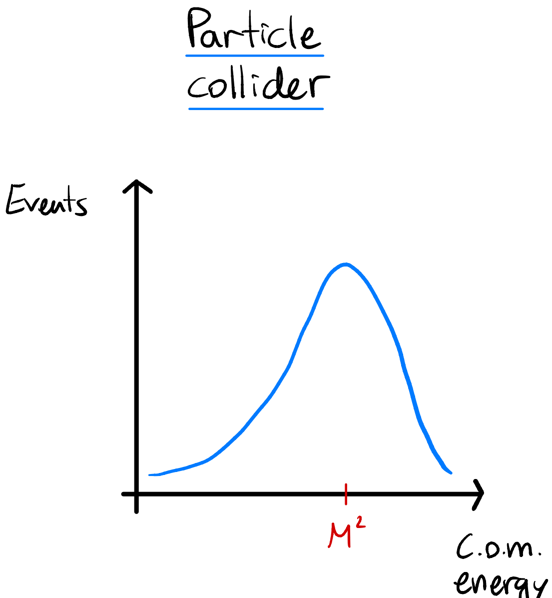


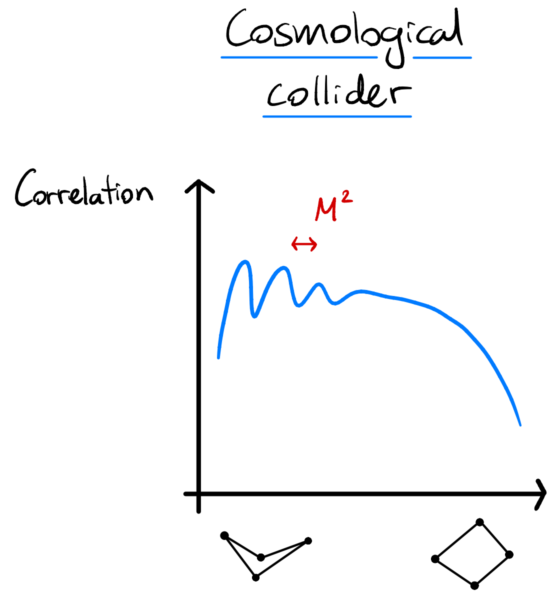


Advantages:

- More simple calculations
- Directly focuses on observables

Correlators from $M^2 > 0$ particle exchange show oscillations in certain limits, analogous to the resonances of particle colliders





1. Cosmological correlators

2. Bootstrap

3. Cosmological bootstrap

3. Cosmological bootstrap

3.1. Symmetries

3.2. Singularities

3.3. Ward identities

3.4. Unitarity

3.5. Energy deformations

3.6. Others

3. Cosmological bootstrap

3.1. Symmetries

3.2. Singularities

3.3. Ward identities

3.4. Unitarity

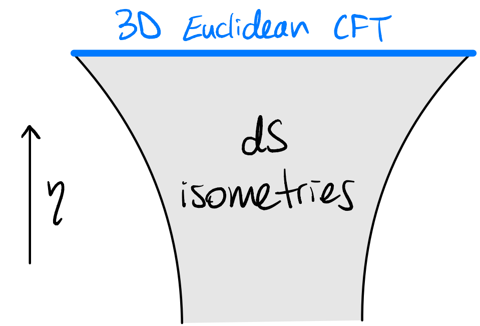
3.5. Energy deformations

3.6. Others

Inflation took place in ~dS

$$\langle \zeta \zeta \rangle \propto \frac{1}{k^{3+\underline{1-n_s}}}$$

Deviation from scale invariance: $1-n_s \simeq 0.04$



<u>dS isometries</u>

Spatial translations

Spatial rotations

Dilatation

dS boosts

$\xrightarrow{\eta \to 0}$

<u>Conformal symmetries</u>

Spatial translations

Spatial rotations

Scaling

Special conformal transf.

Ψ_n's satisfy the symmetries of 3D euclidean CFT correlators:

$$\Psi_n = \langle O_1 \cdots O_n \rangle = (2\pi)^3 \underbrace{\delta(\vec{k}_1 + \ldots + \vec{k}_n)}_{\text{Translation invariance}} \cdot \langle O_1 \cdots O_n \rangle'$$

Prime: stripped off delta function

O's are dual fields with scaling dimension:

$$\Delta = \frac{3}{2} + \sqrt{\frac{9}{4} - \frac{m^2}{H^2}} \quad (\ell = 0) \quad ; \quad \Delta = \frac{3}{2} + \sqrt{\left(\ell - \frac{1}{2}\right)^2 - \frac{m^2}{H^2}} \quad (\ell > 0)$$

Hence:

$$\langle \sigma_1 \sigma_2 \rangle' = \frac{1}{2 \mathrm{Re} \langle \phi \phi \rangle'} \quad , \quad \langle \gamma_1^{ij} \sigma_2 \sigma_3 \rangle' = \frac{\mathrm{Re} \langle T^{ij} \phi \phi \rangle'}{4 \mathrm{Re} \langle TT \rangle' (\mathrm{Re} \langle \phi \phi \rangle')^2}$$

Rotation invariance $\Rightarrow$ $\langle O_1 \cdots O_n \rangle'(\vec{k}_a) = \langle O_1 \cdots O_n \rangle'(k_a, \vec{k}_a \cdot \vec{k}_b)$

$$\left.\begin{aligned}
&\text{Scaling symmetry} \Rightarrow \left[3(n-1) - \sum_{a=1}^{n} \Delta_a + \sum_{a=1}^{n-1} \vec{k}_a \cdot \frac{\partial}{\partial \vec{k}_a}\right] \langle O_1 \cdots O_n \rangle' = 0 \\
&\text{SCTs} \Rightarrow \sum_{a=1}^{n} \left[k_a^i \frac{\partial^2}{\partial \vec{k}_a \cdot \partial \vec{k}_a} - 2 \vec{k}_a \cdot \frac{\partial^2}{\partial \vec{k}_a \partial k_a^i} + 2(\Delta_a - 3) \frac{\partial}{\partial k_a^i} \right] \langle O_1 \cdots O_n \rangle' = 0
\end{aligned}\right\}$$

Example: 2-pt. function of scalars

$k_1 = k_2 \equiv k$

$$\langle O_1 O_2 \rangle = (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2) \cdot \langle O_1 O_2 \rangle' = (2\pi)^3 \delta(\vec{k}_1 + \vec{k}_2) \cdot f(k)$$

Scaling: $\left(3 - \Delta_1 - \Delta_2 + k \frac{\partial}{\partial k}\right) f(k) = 0 \quad \Rightarrow \quad f(k) = \# \, k^{\Delta_1 + \Delta_2 - 3}$

SCTs: $\Delta_1 = \Delta_2 \equiv \Delta \quad \Rightarrow \quad f(k) = \# \, \delta_{\Delta_1 \Delta_2} \cdot k^{2\Delta - 3}$

We have then $\langle \sigma_1 \sigma_2 \rangle' = \dfrac{1}{2 \mathrm{Re} \langle O_1 O_2 \rangle'} = \dfrac{c \cdot \delta_{\Delta_1 \Delta_2}}{k^{2\Delta - 3}}$

Rotation invariance $\Rightarrow \quad \langle O_1 \cdots O_n \rangle'(\vec{k}_a) = \langle O_1 \cdots O_n \rangle'(k_a, \vec{k}_a \cdot \vec{k}_b)$

$$\left.\begin{aligned} &\text{Scaling symmetry} \Rightarrow \left[3(n-1) - \sum_{a=1}^{n} \Delta_a + \sum_{a=1}^{n-1} \vec{k}_a \cdot \frac{\partial}{\partial \vec{k}_a}\right] \langle O_1 \cdots O_n \rangle' = 0 \\ &\text{SCTs} \Rightarrow \sum_{a=1}^{n} \left[k_a^i \frac{\partial^2}{\partial \vec{k}_a \cdot \partial \vec{k}_a} - 2\vec{k}_a \cdot \frac{\partial^2}{\partial \vec{k}_a \partial k_a^i} + 2(\Delta_a - 3) \frac{\partial}{\partial k_a^i}\right] \langle O_1 \cdots O_n \rangle' = 0 \end{aligned}\right\}$$

One strategy is to try and solve these equations:

1811.00024	cosmology.amsterdam/lecce-lectures
1910.14051	sites.google.com/view/cosmologymeetscft2020
2005.04234	ggi.infn.it/ggilectures/ggilectures2020

But here we will focus on other methods which are particularly powerful for meromorphic ψ_n's

3. Cosmological bootstrap

3.1. Symmetries

3.2. Singularities

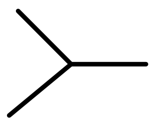
3.3. Ward identities

3.4. Unitarity

3.5. Energy deformations

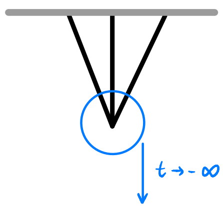
3.6. Others

Ψ_n's have singularities in certain Kinematical configurations



$$iA_3 = iV(\vec{k}) \int_{-\infty}^{\infty} dt \cdot e^{iKt} = iV(\vec{k}) \cdot 2\pi\delta(K)$$

Energy conservation

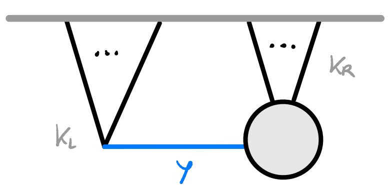


$$\Psi_3 = i\,V(\vec{k}) \int_{-\infty}^{0} dt \cdot e^{iKt} = \frac{V(\vec{k})}{K} \quad \Rightarrow \quad \lim_{K\to 0} \Psi_3 = \frac{A_3}{K}$$

1104.2846

1201.6449

Also there for more complex diagrams:

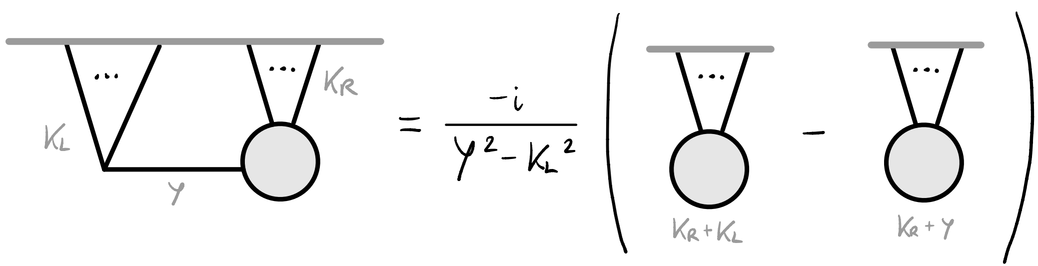


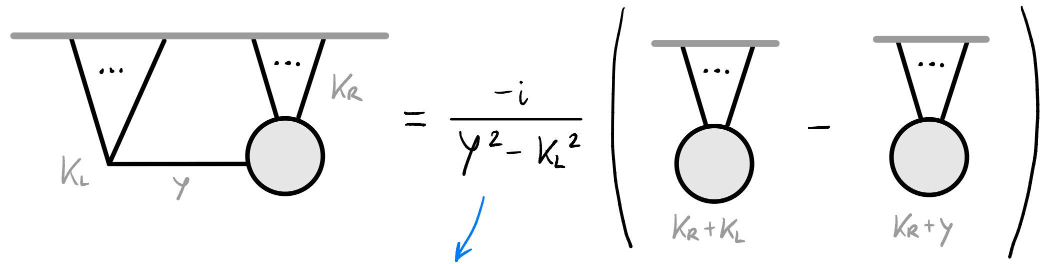
$$\Psi_n = \int_{-\infty}^{0} dt_R \cdot e^{iK_R t_R} \cdot \bigcirc \int_{-\infty}^{0} dt_L \cdot e^{iK_L t_L} \cdot G(y, t_L, t_R)$$

$$\Psi_n = i\int_{-\infty}^{0} dt_R \cdot e^{iK_R t_R} \cdot \bigcirc \int_{-\infty}^{0} dt_L \cdot e^{iK_L t_L} \cdot \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\left(\frac{e^{-i\omega(t_L-t_R)}}{\omega^2-y^2+i\varepsilon} - \frac{e^{-i\omega(t_L+t_R)}}{\omega^2-y^2+i\varepsilon}\right) =$$

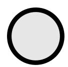
$$= i\int_{-\infty}^{0} dt_R \cdot e^{iK_R t_R} \cdot \bigcirc \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \cdot \frac{1}{K_L-\omega-i\varepsilon}\left(\frac{e^{i\omega t_R}}{\omega^2-y^2+i\varepsilon} - \frac{e^{-i\omega t_R}}{\omega^2-y^2+i\varepsilon}\right) =$$

$$= \frac{1}{y^2-K_L^2}\int_{-\infty}^{0} dt_R \cdot e^{iK_R t_R} \cdot \left(e^{iK_L t_R} - e^{iy t_R}\right) \cdot \bigcirc$$





$$y^2 - K_L^2 = |\vec{k}_1 + \ldots + \vec{k}_r|^2 - (k_1 + \ldots + k_r)^2 = K_L \cdot K_L \qquad \text{with} \quad K_L^\mu = k_1^\mu + \ldots + k_r^\mu$$

If  has a singularity in K_R with residue the scattering amplitude $\Rightarrow$ ψ_n has a singularity in $K_L + K_R = E$ with residue A_n

Since it is true for the contact diagram, we prove it recursively for all tree-level graphs

$$\boxed{\psi_n^{(\text{tree})} \xrightarrow[E \to 0]{} \frac{A_n}{E}}$$

(Flat space)

Similar arguments hold in dS because:

$$\begin{cases} K_\nu(\vec{k},\eta) \approx \eta^{3/2}\cdot H_\nu^{(2)}(-k\eta) \xrightarrow[\eta\to-\infty]{} \left(-ik\eta + \mathcal{O}(1)\right) e^{ik\eta} \\ \mathcal{G}_\nu(\vec{k},\eta,\eta') \xrightarrow[\eta,\eta'\to-\infty]{} \eta\eta'\cdot \mathcal{G}^{(\text{flat})}(\vec{k},\eta,\eta') \end{cases}$$

We have:

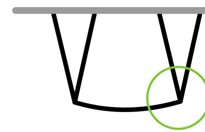
$$\boxed{\Psi_n^{(\text{tree})} \xrightarrow[E\to 0]{} \frac{\prod_a k_a^{\Delta_a-2}}{E^p} A_n}$$

with

$$p = 1 + \sum_v (\Delta_v - 4)$$

(de Sitter)

Ψ's are also singular when the sum of energies flowing into a subdiagram vanishes. For Ψ_4 for example:



In flat space:

$$\boxed{\lim_{E_R \to 0} \Psi_4 = \frac{\tilde{\Psi}_3 \times A_3}{E_R}}$$

with $\tilde{\Psi}_3 \equiv \frac{1}{2s} \left[\Psi_3(-s) - \Psi_3(+s) \right]$

In dS:

$$\boxed{\lim_{E_R \to 0} \Psi_4 = \frac{\tilde{\Psi}_3 \times \tilde{A}_3}{E_R^{\,p_L}}}$$

with
$$\begin{cases} \tilde{A}_3 \equiv \prod_a K^{\Delta_a - 2} \cdot A_3 \\ \tilde{\Psi}_3 \equiv \dfrac{(-1)^{\Delta_I}}{2s^{2\Delta_I - 3}} \left[\Psi_3(-s) - \Psi_3(+s) \right] \end{cases}$$

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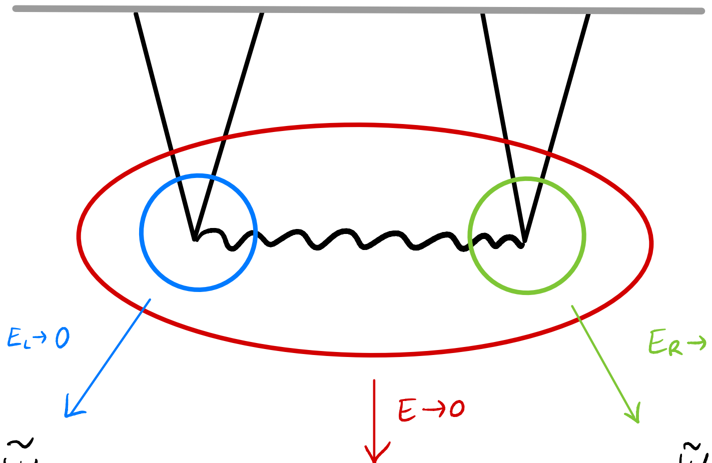
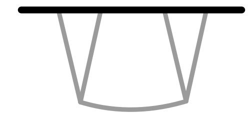
For tree-level ψ_4 :

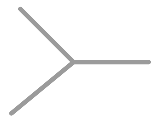
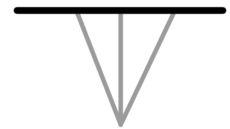


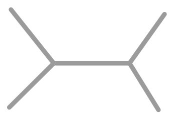
$$\frac{\tilde{A}_3 \times \tilde{\psi}_3}{E_L^{\#}} \qquad \frac{\tilde{A}_4}{E^{\#}} \qquad \frac{\tilde{\psi}_3 \times \tilde{A}_3}{E_R^{\#}}$$

- These singularities do not correspond to physical momentum configurations
- We will also require the absence of other singularities

Example: Massless scalar in flat space



 $A_3 = 1$;  $\psi_3 = \dfrac{1}{k_1 + k_2 + k_3}$



$$A_4 = -\frac{1}{s} = \frac{1}{(k_1^\mu + k_2^\mu)^2} = -\frac{1}{(k_1+k_2)^2 - \underbrace{|\vec{k}_1 + \vec{k}_2|^2}_{s^2}} = -\frac{1}{E_L\,(k_1+k_2-s)}$$

- $$\psi_4 \xrightarrow[E_L \to 0]{} \frac{1}{E_L} A_3 \times \tilde{\psi}_3 = \frac{1}{2sE_L}\left(\frac{1}{k_3+k_4-s} - \frac{1}{E_R}\right) = \frac{E_R - k_{34} + s}{2sE_L E_R (k_{34}-s)} = \frac{1}{E_L E_R\,(k_{34}-s)}$$
- $$\psi_4 \xrightarrow[E_R \to 0]{} \frac{1}{E_R} \tilde{\psi}_3 \times A_3 = \frac{1}{E_L E_R\,(k_{12}-s)}$$
- $$\psi_4 \xrightarrow[E \to 0]{} \frac{1}{E} A_4 = \frac{1}{E E_L\,(s-k_{12})}$$

The unique function that satisfies this is $\psi_4 = \dfrac{1}{E E_L E_R}$

3. Cosmological bootstrap

3.1. Symmetries

3.2. Singularities

3.3. Ward identities

3.4. Unitarity

3.5. Energy deformations

3.6. Others

Massless spinning fields: $\Psi[\gamma_{ij}, A_i, \phi]$

Graviton Vector Scalar

Gauge invariance of Ψ implies some current conservation identities:

$$\left.\begin{aligned} \delta A_i^B &= \partial_i \Lambda^B - i f^{BCD} A_i^C \Lambda^D \\ \delta \phi^a &= \Lambda_A (T^A)_{ab}\, \phi^b \end{aligned}\right\} \qquad \delta_\Lambda \Psi = \int d^3x \left[\left(\partial_i \Lambda^B - i f^{BCD} A_i^C \Lambda^D\right) \frac{\delta}{\delta A_i^B} + \Lambda_A (T^A)_{ab}\, \phi^b \frac{\delta}{\delta \phi^a} \right] \Psi = 0$$

$$\Downarrow$$

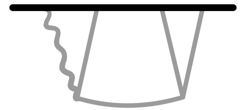
$$\vec{K}_1 \cdot \langle \vec{J}^A_{\vec{K}_1} O^{b_2}_{\vec{K}_2} \cdots O^{b_n}_{\vec{K}_n} \rangle = - \sum_{a=2}^{n} i (T^A)_{b_a c} \langle O^{b_2}_{\vec{K}_2} \cdots O^{c}_{\vec{K}_a + \vec{K}_1} \cdots O^{b_n}_{\vec{K}_n} \rangle$$

Similarly, diffeo invariance:

$$\delta_\xi \Psi = 0 \qquad \Longrightarrow \qquad K_1^i \xi_1^j \langle T^{ij}_{\vec{K}_1} O_{\vec{K}_2} \cdots O_{\vec{K}_n} \rangle = - \sum_{a=2}^{n} \chi_a (\vec{\xi}_1 \cdot \vec{K}_a) \langle O_{\vec{K}_2} \cdots O_{\vec{K}_a + \vec{K}_1} \cdots O_{\vec{K}_n} \rangle$$

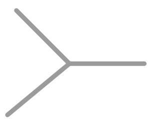
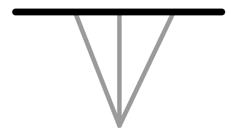
These WTIs reflect locality of the bulk physics and lead to nontrivial constraints on the couplings of gauge fields

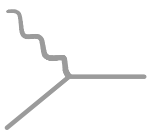
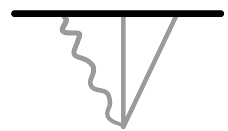
Example: Massless spin-1 and c.c. scalars in dS

$\langle J\varphi\varphi\varphi\rangle_s =$ 

$$\Delta_J = \frac{3}{2} + \sqrt{\left(1-\frac{1}{2}\right)^2 - \frac{m^2}{H^2}} = 2$$

Conformally coupled scalar φ: $m_\varphi^2 = 2H^2 \Rightarrow \Delta_\varphi = \frac{3}{2} + \sqrt{\frac{9}{4} - \frac{m_\varphi^2}{H^2}} = 2$

 $A_3^{(R)} = 1$;  $\psi_3^{(R)} = \log(K_3 + K_4 + s)$

 $A_3^{(L)} = 2(\xi_1 \cdot K_2)$;  $\psi_3^{(L)} = \frac{2}{K_1 + K_2 + s}(\vec{\xi}_1 \cdot \vec{K}_2)$

$\vec{\xi}_1 \cdot \vec{K}_1 = 0$
(transverse)

$$[\langle J\varphi\varphi\varphi\rangle_s] = \sum_{a=1}^{4} \Delta_a - 3(4-1) = 4 \cdot 2 - 9 = -1$$

- $$\lim_{E_L\to 0} \langle J\varphi\varphi\varphi\rangle_s = \frac{A_3^{(L)} \times \tilde{\psi}_3^{(R)}}{E_L} = \frac{A_3^{(L)}}{E_L} \cdot \frac{1}{2s}\left[\psi_3^{(R)}(-s) - \psi_3^{(R)}(s)\right] = -\frac{(\vec{\xi}_1 \cdot \vec{K}_2)}{sE_L} \log\left(\frac{E_R}{K_{34}-s}\right)$$

- $$\lim_{E_R\to 0} \langle J\varphi\varphi\varphi\rangle_s = \tilde{\psi}_3^{(L)} \times A_3^{(R)} \cdot \log\left(\frac{E_R}{\mu}\right) = \frac{1}{2s}\left[\psi_3^{(L)}(-s) - \psi_3^{(L)}(s)\right] \cdot A_3^{(R)} \cdot \log\left(\frac{E_R}{\mu}\right) =$$

$$= \frac{(\vec{\xi}_1 \cdot \vec{K}_2)}{s}\left(\frac{1}{K_{12}-s} - \frac{1}{K_{12}+s}\right) \cdot \log\left(\frac{E_R}{\mu}\right) = \frac{2(\vec{\xi}_1 \cdot \vec{K}_2)}{E_L(K_{12}-s)} \cdot \log\left(\frac{E_R}{\mu}\right)$$

A function that correctly factorizes is:

$$\langle J\varphi\varphi\varphi\rangle_s \stackrel{?}{=} \frac{2(\vec{\xi}_1 \cdot \vec{K}_2)}{E_L(K_{12}-s)} \cdot \log\left(\frac{E_R}{K_{34}-s}\right)$$

"Folded" singularities ($K_{12}-s=0$ and $K_{34}-s=0$) must be absent. To get rid of $K_{34}-s=0$, we can rewrite $K_{34}-s \longrightarrow K_{34}+K_{12}=E$ in the $E_L\to 0$ limit above.

Then:

$$\boxed{\langle J\varphi\varphi\varphi\rangle_s = \frac{2(\vec{\xi}_1\cdot\vec{k}_2)}{E_L(k_{12}-s)}\cdot\log\left(\frac{E_R}{E}\right)}$$

Notice that $\lim\limits_{k_{12}\to s}\langle J\varphi\varphi\varphi\rangle_s = -\dfrac{(\vec{\xi}_1\cdot\vec{k}_2)}{sE_R}$ is regular

The relevant WTI is:

$$\vec{k}_1\cdot\langle\vec{J}_{\vec{k}_1}\varphi_{\vec{k}_2}\varphi_{\vec{k}_3}\varphi_{\vec{k}_4}\rangle = -e_2\langle\varphi_{\vec{k}_2+\vec{k}_1}\varphi_{\vec{k}_3}\varphi_{\vec{k}_4}\rangle - e_3\langle\varphi_{\vec{k}_2}\varphi_{\vec{k}_3+\vec{k}_1}\varphi_{\vec{k}_4}\rangle - e_4\langle\varphi_{\vec{k}_2}\varphi_{\vec{k}_3}\varphi_{\vec{k}_4+\vec{k}_1}\rangle =$$

$$= -e_2\log\left(\frac{k_{34}+s}{\mu}\right) - e_3\cdot\log\left(\frac{k_{24}+u}{\mu}\right) - e_4\cdot\log\left(\frac{k_{23}+t}{\mu}\right) \quad \text{with} \begin{cases} t\equiv|\vec{k}_1+\vec{k}_4| \\ u\equiv|\vec{k}_1+\vec{k}_3| \end{cases}$$

Does our solution above satisfy it? We cannot compute $\vec{k}_1\cdot\langle\vec{J}\varphi\varphi\varphi\rangle_s$ directly because it misses the longitudinal term:

$$\langle J\varphi\varphi\varphi\rangle_s \sim (\dots)(\vec{z}_1\cdot\vec{k}_2) + (\dots)(\vec{z}_1\cdot\vec{k}_1)$$

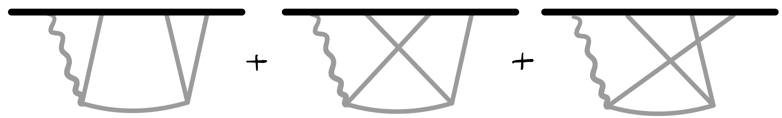
Non-transverse polariz. vector

But there's a way to compute it (using spinor helicity variables). It yields:

$$\vec{K}_1 \cdot \langle \vec{J} \varphi\varphi\varphi \rangle_s = -e_s \left[\log\left(\frac{K_{34}+s}{E}\right) + \frac{K_1}{E} \right] \quad \times$$

Not consistent with the WTI

Adding t- and u-channels:



$$\vec{K}_1 \cdot \langle \vec{J} \varphi\varphi\varphi \rangle_{s+t+u} = -e_s \cdot \log\left(\frac{K_{34}+s}{\mu}\right) - e_t \cdot \log\left(\frac{K_{23}+t}{\mu}\right) - e_u \cdot \log\left(\frac{K_{24}+u}{\mu}\right) -$$

$$-(e_s+e_t+e_u)\left[\log\left(\frac{\mu}{E}\right) + \frac{K_1}{E}\right]$$

Consistent if:

$e_s = e_2$; $e_t = e_4$; $e_u = e_3$ and $\boxed{e_2 + e_3 + e_4 = 0}$ Charge conservation!

Similarly, one can show:

- $[T^A, T^B]_{ab} = f^{ABC}\, T_{ab}{}^{C}$
- Equivalent principle
- Jacobi identity
- Massless $\ell \geqslant 3$ can't couple to matter

3. Cosmological bootstrap

We can define a conjugate wavefunction coef. $\bar{\Psi}_n$. It is computed like Ψ_n except:

$$\bar{\Psi}_n \begin{cases} G \longrightarrow \bar{G} \equiv G^* \\ iV \longrightarrow -iV \end{cases}$$

- Flat space

Let us define:

$$\tilde{G}(\vec{k},t,t') \equiv G(\vec{k},t,t') + \bar{G}(\vec{k},t,t') = \frac{1}{2k}\Big[e^{ik(t-t')}\cdot\theta(t'-t) + e^{ik(t'-t)}\cdot\theta(t-t') - e^{ik(t+t')} +$$

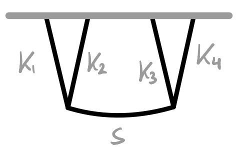
$$+ e^{-ik(t-t')}\cdot\theta(t'-t) + e^{-ik(t'-t)}\cdot\theta(t-t') - e^{-ik(t+t')}\Big] = \frac{1}{2k}\Big[e^{ik(t-t')} + e^{ik(t'-t)} -$$

$$- e^{ik(t+t')} - e^{-ik(t+t')}\Big] = -\frac{1}{2k}\left(e^{-ikt} - e^{ikt}\right)\left(e^{-ikt'} - e^{ikt'}\right) =$$

$$= -\frac{1}{2k}\left[K(-k,t) - K(k,t)\right]\cdot\left[K(-k,t') - K(k,t')\right]$$

We call this a cut propagator:

$$\tilde{G}(\vec{k},t,t') \equiv G(\vec{k},t,t') + \overline{G}(\vec{k},t,t') = -\frac{1}{2k}\left[K(-k,t)-K(k,t)\right]\cdot\left[K(-k,t')-K(k,t')\right]$$



In the tree-level 4-pt. case:

$$\psi_4 + \overline{\psi_4} = (iV)^2\int_{-\infty}^{0} dt_L dt_R \cdot K_1 \cdots K_4\, G_s + (-iV)^2\int_{-\infty}^{0} dt_L dt_R \cdot K_1 \cdots K_4\, \overline{G}_s = (iV)^2\int_{-\infty}^{0} dt_L dt_R \cdot K_1 \cdots K_4 \cdot \overbrace{(G_s+\overline{G}_s)}^{\tilde{G}_s} =$$

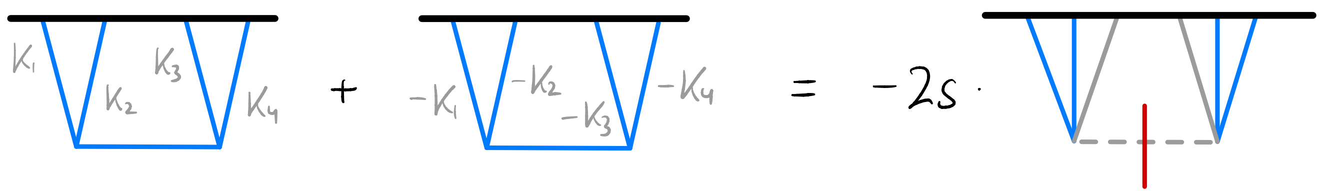
$$= -\frac{(iV)^2}{2s}\int_{-\infty}^{0} dt_L \cdot K(k_1)\cdot K(k_2)\cdot\left[K(-s)-K(s)\right]\cdot\int_{-\infty}^{0} dt_R \cdot K(k_3)\cdot K(k_4)\cdot\left[K(-s)-K(s)\right] = -2s\,\tilde{\psi}_3 \times \tilde{\psi}_3$$

$$\psi_4 + \overline{\psi_4} = -2s\, \tilde{\psi}_3 \times \tilde{\psi}_3$$

$$\Downarrow$$

Unitarity implies $\overline{\psi}_n(\underbrace{k_a}_{\text{External}}, \overbrace{y_a}^{\text{Internal}}) = \psi_n^*(-k_a, y_a)$

$$\boxed{\psi_4(k_a) + \psi_4^*(-k_a) = -2s\, \tilde{\psi}_3 \times \tilde{\psi}_3}$$



- de Sitter

Analogously:

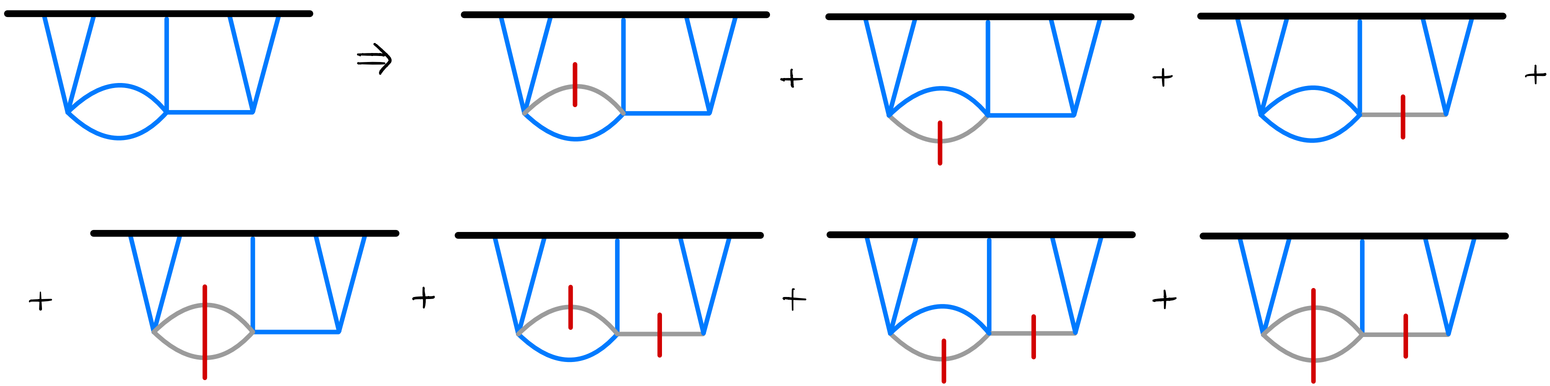
$$\boxed{\psi_4(k_a) + \psi_4^*(-k_a) = -P_\nu(s) \cdot \tilde{\psi}_3 \times \tilde{\psi}_3}$$

2009.02898

2009.07874

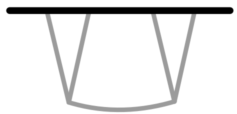
These cutting rules exist for any diagram in any FLRW background

$$\boxed{\psi_n(k\text{'s}) + \psi_n^*(-k\text{'s}) = -\sum_{\text{cuts}} \psi_n}$$

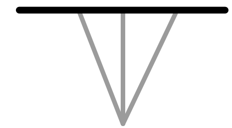


2103.09832

Example: Massless scalar in flat space



$$\psi_4 = \frac{1}{E E_L E_R} \quad ; $$



$$\psi_3 = \frac{1}{K}$$

$$\boxed{\psi_4(K_a) + \psi_4^*(-K_a) = -2s\, \tilde{\psi}_3 \times \tilde{\psi}_3}$$

with $\tilde{\psi}_3 = \frac{1}{2s}\left(\psi_3(-s) - \psi_3(s)\right)$

Let's check:

$$\psi_4(K_a) + \psi_4^*(-K_a) = \frac{1}{E E_L E_R} + \frac{1}{(-E)(-K_{12}+s)(-K_{34}+s)} = \frac{1}{E} \cdot \frac{(K_{12}-s)(K_{34}-s) - E_L E_R}{(K_{12}^2 - s^2)(K_{34}^2 - s^2)} =$$

$$= \frac{1}{E} \cdot \frac{-2sE}{(K_{12}^2 - s^2)(K_{34}^2 - s^2)} = -\frac{2s}{(K_{12}^2 - s^2)(K_{34}^2 - s^2)}$$

$$-2s\, \tilde{\psi}_3 \times \tilde{\psi}_3 = \frac{-1}{2s} \cdot \left(\frac{1}{K_{12}-s} - \frac{1}{K_{12}+s}\right) \cdot \left(\frac{1}{K_{34}-s} - \frac{1}{K_{34}+s}\right) = \frac{-1}{2s} \cdot \frac{2s}{K_{12}^2 - s^2} \cdot \frac{2s}{K_{34}^2 - s^2} = -\frac{2s}{(K_{12}^2 - s^2)(K_{34}^2 - s^2)}$$

3. Cosmological bootstrap

3.1. Symmetries

3.2. Singularities

3.3. Ward identities

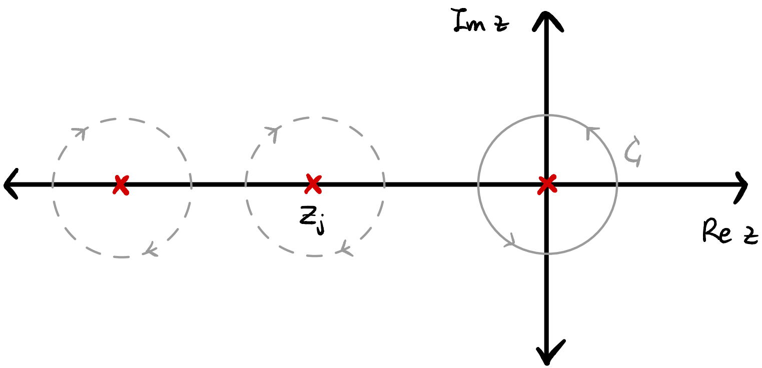
3.4. Unitarity

3.5. Energy deformations

3.6. Others

Inspired by the BCFW recursion of scattering amplitudes, we can shift:

$$\left.\begin{array}{l} K_a \longrightarrow K_a + C_a z \\ |\vec{K}_a+\vec{K}_b| \longrightarrow |\vec{K}_a+\vec{K}_b| + d_{ab}\, z \end{array}\right\} \psi_n(z)$$



Then

$$\psi_n = \psi_n(z=0) = \frac{1}{2\pi i}\oint_C dz\, \frac{\psi_n(z)}{z}$$

$$\boxed{\psi_n = -\sum_j \operatorname*{Res}_{z=z_j}\left(\frac{\psi_n(z)}{z}\right) + B_\infty}$$

Poles of $\psi(z)$ Pole at $z=\infty$

Example: 2103.08649

$$\psi_4 = \frac{1}{E E_L E_R} \xrightarrow{K_1 \longrightarrow K_1+z} \psi_4(z) = \frac{1}{(E+z)(E_L+z)E_R}$$

Poles of $\psi_4(z)$ at $z_j = \{-E, -E_L\}$

For simple poles, the residues are given by lower-point ψ's and amplitudes

For poles of order $m \geq 2$ we need subleading terms of the Laurent expansion of ψ around the singularity:

$$\operatorname*{Res}_{z=z_j}\left(\frac{\psi_n(z)}{z}\right) = -\sum_{\ell=1}^{m} \frac{R^{(\ell)}}{(-z_j)^\ell}$$

with:

$$\lim_{z\to z_j} \psi_n(z) = \sum_{\ell=1}^{m} \frac{R^{(\ell)}}{(z-z_j)^\ell} + \text{finite}$$

For $E \to 0$ we don't have all of these terms. Luckily, for partial energy singularities we have them from:

$$\psi_n(K\text{'s}) + \psi_n^*(-K\text{'s}) = -\sum_{\text{cuts}} \psi_n$$

Example: Massless scalar with $\dot{\phi}^3$ in dS

$$S = \int dt\, d^3\vec{x}\, a^3(t) \left[\frac{1}{2}\dot{\phi}^2 - \frac{1}{a^2(t)}(\vec{\nabla}\phi)^2 + \frac{g}{3!}\dot{\phi}^3 \right]$$

- Breaks dS boosts
- Produces rational ψ's

3-pt. vertex gives

$$\psi_3(k_1, k_2, k_3) = -\frac{2(k_1 k_2 k_3)^2}{(k_1 + k_2 + k_3)^3}$$

so:

$$-2s^3 \tilde{\psi}_3 \times \tilde{\psi}_3 = -8(k_1 k_2 k_3 k_4)^2 s^3 \frac{(3k_{12}^2 + s^2)(3k_{34}^2 + s^2)}{E_L^3 (k_{12} - s)^3 E_R^3 (k_{34} - s)^3}$$

We will shift:

$$k_{12} \longrightarrow k_{12} + z$$

so that $\psi_4(z)$ will have poles at

$$z_T = -E \quad ; \quad z_L = -E_L$$

Now we use:

$$\Psi_4(k's) + \Psi_4^*(-k's) = -2s^3\,\tilde{\Psi}_3 \times \tilde{\Psi}_3$$

to get:

$$-\operatorname*{Res}_{z=-E_L}\left(\frac{\Psi_4(z)}{z}\right) = \lim_{k_{12}\to -s} \Psi_4\Big|_{\text{singular}} = \lim_{k_{12}\to -s} -2s^3\,\tilde{\Psi}_3 \times \tilde{\Psi}_3\Big|_{\text{singular}} =$$

$$= \lim_{k_{12}\to -s} -8(k_1k_2k_3k_4)^2 s^3 \frac{(3k_{12}^2+s^2)(3k_{34}^2+s^2)}{E_L^3(k_{12}-s)^3E_R^3(k_{34}-s)^3}\Bigg|_{\text{singular}} = \frac{4(k_1k_2k_3k_4 s)^2(3k_{34}^2+s^2)}{E_L^3E_R^3(k_{34}-s)^3}$$

The other residue is:

$$-\operatorname*{Res}_{z=-E}\left(\frac{\Psi_4(z)}{z}\right) = \lim_{k_{12}\to -k_{34}} \Psi_4\Big|_{\text{singular}} = -\frac{24(k_1k_2k_3k_4)^2k_{34}^2}{E^5E_R(k_{34}-s)} + \frac{R_E}{E^4}$$

$$R_E \equiv R^{(4)} + E\cdot R^{(3)} + E^2\cdot R^{(2)} + E^3\cdot R^{(1)}$$

$$\psi_4 = - \sum_{z_j = -E_L, -E} \operatorname{Res}_{z_j} \left(\frac{\psi_4(z)}{z} \right) =$$

$$= \frac{4 (K_1 K_2 K_3 K_4 s)^2 (3 K_{34}^2 + s^2)}{E_L^3 E_R^3 (K_{34} - s)^3} - \frac{24 (K_1 K_2 K_3 K_4)^2 K_{34}^2}{E^5 E_R (K_{34} - s)} + \frac{R_E}{E^4}$$

To fix R_E, we require 1. the $K_{34} - s = 0$ singularity to be absent and 2. the correct factorization at $E_R = 0$:

1. $$\lim_{K_{34} \to s} E^4 \cdot \psi_4 \Big|_{\text{singular}} = \frac{2 s E_L (K_1 K_2 K_3 K_4)^2}{(K_{34} - s)^3} + \frac{8 s (K_1 K_2 K_3 K_4)^2}{(K_{34} - s)^2} + \lim_{K_{34} \to s} R_E \Big|_{\text{singular}} = 0$$

$$\Downarrow$$

$$\lim_{K_{34} \to s} R_E \Big|_{\text{singular}} = 2 s (K_1 K_2 K_3 K_4)^2 \left[\frac{E_L}{(K_{34} - s)^3} + \frac{4}{(K_{34} - s)^2} \right]$$

2. $$\lim_{K_{34}\to -s} E^4 \cdot \Psi_4 \Big|_{\text{singular}} = \lim_{K_{34}\to -s} E^4\left(-2s^3 \tilde{\Psi}_3 \times \tilde{\Psi}_3\right)\Big|_{\text{singular}}$$

$$\Downarrow$$

$$\lim_{K_{34}\to -s} R_E \Big|_{\text{singular}} = 2s\,(K_1K_2K_3K_4)^2 \cdot \frac{E+3E_R}{E_R^{\,3}}$$

Adding up these series gives:

$$\frac{R_E}{E^4} = -\frac{4(K_1K_2K_3K_4 s)^2}{E^3 E_R^2 (K_{34}-s)^2}\left[\frac{6K_{34}}{E} + \frac{3K_{34}^2+s^2}{E_R(K_{34}-s)}\right]$$

Sanity check: no E_L or $K_{12}-s$ poles

And finally:

$$\Psi_4 = (K_1K_2K_3K_4)^2\left\{2s\left[\frac{6}{E^5}\left(\frac{1}{E_L}+\frac{1}{E_R}\right)+\frac{3}{E^4}\left(\frac{1}{E_L^2}+\frac{1}{E_R^2}\right)+\frac{1}{E^3}\left(\frac{1}{E_L^3}+\frac{1}{E_R^3}\right)-\frac{1}{E_L^3E_R^3}\right]-\frac{24}{E^5}\right\}$$

3. Cosmological bootstrap

3.1. Symmetries

3.2. Singularities

3.3. Ward identities

3.4. Unitarity

3.5. Energy deformations

3.6. Others

Manifestly local test

For ψ_n's of massless scalars and gravitons, if the interactions of the dynamical fields are local:

$$\left.\frac{\partial}{\partial K_a}\psi_n\right|_{K_a=0} = 0 \qquad \forall\, a \text{ corresponding to those fields}$$

2103.08649

Soft limits

Shift symmetry of the interactions implies:

$$\lim_{\vec{K}_a \to 0} \psi_n = 0$$

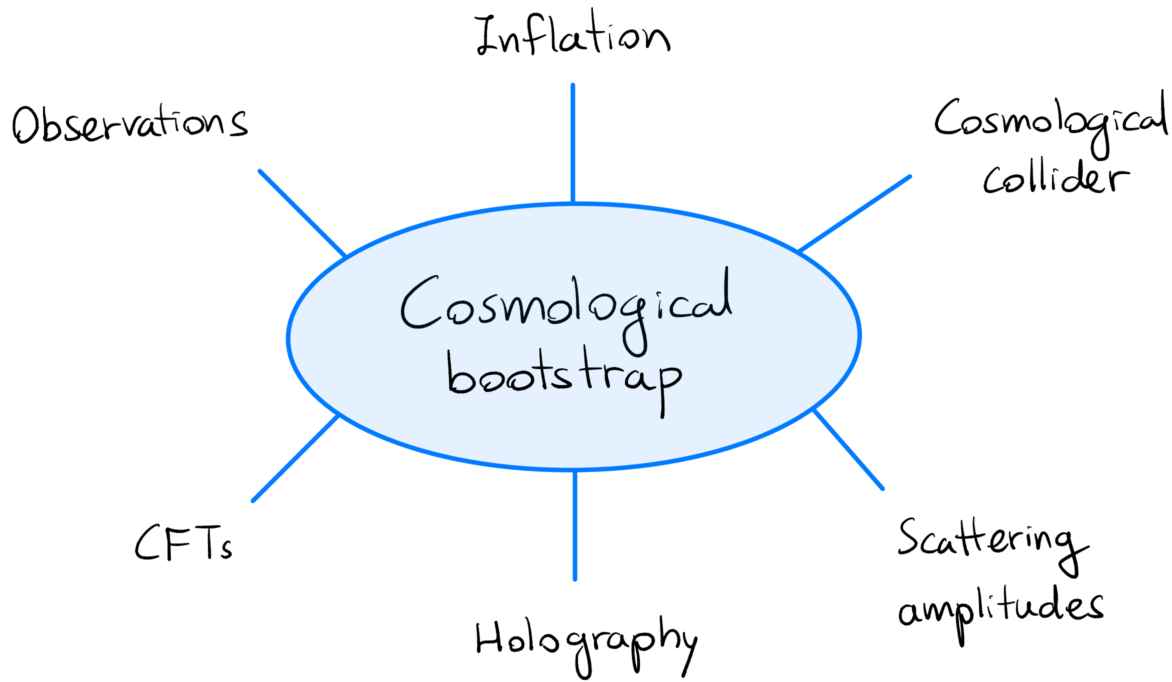
Transmutation

There exist integral/differential operators that transmute:

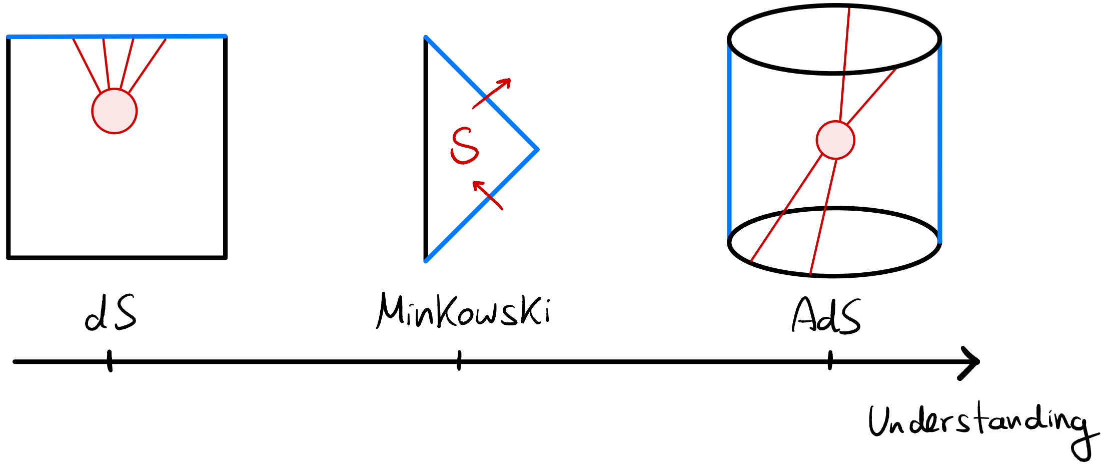
$$\psi_n^{(\text{flat space})} \Longrightarrow \psi_n^{(dS)}$$

(simple) (complex)

Outlook


Inflation
Observations
Cosmological collider
Cosmological bootstrap
CFTs
Holography
Scattering amplitudes

The closer to the real world, the less we understand:



Many challenges: On-shell formulation of Ψ's, go beyond diagrams, non-perturbative rules, causality, positivity bounds, double copy, loops, UV-completion...

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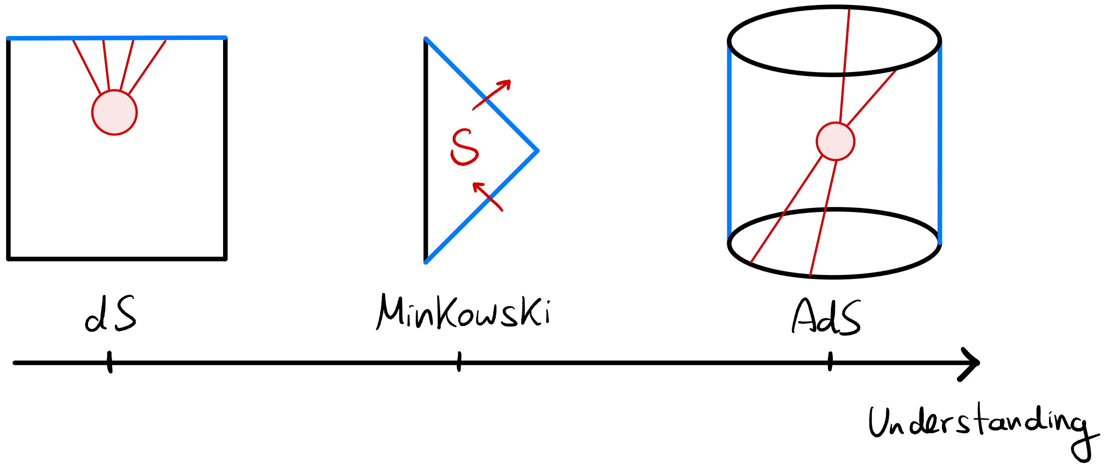

Many challenges: On-shell formulation of ψ's, go beyond diagrams, non-perturbative rules, causality, positivity bounds, double copy, loops, UV-completion...

...Thanks!