Solution generation techniques in gravitational theories

Stationary and axisymmetric $\rightarrow$ $\boxed{\frac{\partial}{\partial t}\,,\ \frac{\partial}{\partial \phi}}$

I) Ernst method

II) Inverse scattering method

ERNST EQUATIONS

Ernst, Phys. Rev. 167 : 1175 - 1179, '68

Phys. Rev. 168 : 1415 - 1417, '68

Stephani et al., "Exact solutions of Einstein's field equations", CUP

$$S = \int d^4x \sqrt{-g} \left(R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

$$\nabla_\mu F^{\mu\nu} = 0$$

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \qquad T_{\mu\nu} = \frac{1}{4\pi} \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} F^2 g \right)$$

$$R_{\mu\nu} = 8\pi T_{\mu\nu}$$

$$ds^2 = -f\,(dt - \omega\, d\phi)^2 + f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right]$$

Lewis - Weyl - Papapetrou $\quad (t, \phi, \rho, z) \to$ Weyl

$f(\rho, z), \quad \omega(\rho, z), \quad \gamma(\rho, z) \quad \Rightarrow \quad \partial_t, \partial_\phi$

$$A = A_t\, dt + A_\phi\, d\phi,$$

$A_t(\rho, z), \quad A_\phi(\rho, z)$

$(\hat{\rho}, \hat{\phi}, \hat{z})$

$$\nabla f = \frac{\partial f}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\hat{\varphi} + \frac{\partial f}{\partial z}\hat{z}$$

Maxwell equations

$$M^{\nu} := \nabla_{\mu} F^{\mu\nu}$$

$$M^{\phi} = e^{-2\gamma} f \Big[\rho^{-2} f \left(\nabla^2 A_\phi + \omega \nabla^2 A_t\right) - 2\rho^{-3} f \left(\frac{\partial A_\phi}{\partial \rho} + \omega \frac{\partial A_t}{\partial \rho} \right)$$
$$+ \rho^{-2} f \nabla\omega \cdot \nabla A_t + \rho^{-2} \nabla f \cdot \left(\nabla A_\phi + \omega \nabla A_t \right) \Big] \overset{!}{=} 0$$

$$\boxed{\nabla \cdot \left[\rho^{-2} f \left(\nabla A_\phi + \omega \nabla A_t \right) \right] = 0} \qquad (1)$$

$$-2\rho^{-3} \hat{\rho} \cdot f \left(\nabla A_\phi + \omega \nabla A_t\right) + \rho^{-2} \nabla f \cdot \left(\nabla A_\phi + \omega \nabla A_t\right) +$$
$$+ \rho^{-2} f \left(\nabla^2 A_\phi + \nabla\omega \cdot \nabla A_t + \omega \nabla^2 A_t \right) =$$
$$= -2\rho^{-3} f \left(\frac{\partial A_\phi}{\partial \rho} + \omega \frac{\partial A_t}{\partial \rho} \right) + \dots$$

$$M^t = e^{-2\gamma} f \Big[\rho^{-2} f \omega \left(\nabla^2 A_\phi + \omega \nabla^2 A_t \right) - f^{-1} \nabla^2 A_t +$$

$$+ \rho^{-2} \omega \nabla f \cdot \left(\nabla A_\phi + \omega \nabla A_t \right) + f^{-2} \nabla f \cdot \nabla A_t + \rho^{-2} f \nabla \omega \cdot \left(\nabla A_\phi + 2\omega \nabla A_t \right)$$

$$- 2\rho^{-3} f \omega \left(\frac{\partial A_\phi}{\partial \rho} + \omega \frac{\partial A_t}{\partial \rho} \right) \Big] \overset{!}{=} 0$$

$$\boxed{\nabla \cdot \left[- f^{-1} \nabla A_t + \rho^{-2} f \omega \left(\nabla A_\phi + \omega \nabla A_t \right) \right] = 0} \qquad (2)$$

Einstein equations

$$E_{\mu\nu} := R_{\mu\nu} - 8\pi T_{\mu\nu}$$

$$E_{tt} = - \frac{e^{-2\gamma}}{2\rho^2} \Big\{ 2\rho^{-2} f^3 \left[(\nabla A_\phi)^2 + \omega^2 (\nabla A_t)^2 + 2\omega \nabla A_\phi \cdot \nabla A_t \right] +$$

$$+ 2f (\nabla A_t)^2 + (\nabla f)^2 - \rho^{-2} f^4 (\nabla \omega)^2 - f \nabla^2 f \Big\} \overset{!}{=} 0$$

$$\boxed{f \nabla^2 f = \rho^{-2} f^4 (\nabla \omega)^2 + 2f (\nabla A_t)^2 + 2\rho^{-2} f^3 (\nabla A_\phi + \omega \nabla A_t)} \qquad (3)$$

$$E_{t\phi} = \ldots$$

$$\omega E_{tt} + E_{t\phi} = 0$$

$$\boxed{\nabla \cdot \left[\rho^{-2} f^2 \nabla \omega + 4\rho^{-2} f A_t (\nabla A_\phi + \omega \nabla A_t) \right] = 0} \qquad (4)$$

$$E_{\rho z} = 0$$

$$E_{\rho\rho} - E_{zz} = 0$$

$h(\rho, z) \Rightarrow \nabla \cdot (\rho^{-1} \hat{\varphi} \times \nabla h) = 0 \quad (*) \qquad (\hat{\rho}, \hat{\varphi}, \hat{z})$

$\tilde{A}_\phi$ s.t.

$\hat{\varphi} \times \nabla \tilde{A}_\phi = \rho^{-1} f (\nabla A_\phi + \omega \nabla A_t)$

$0 \overset{(*)}{=} \nabla \cdot (\rho^{-1} \hat{\varphi} \times \nabla \tilde{A}_\phi) = \nabla \cdot [\rho^{-2} f (\nabla A_\phi + \omega \nabla A_t)] \quad (\square)$

$A_\phi \longleftrightarrow \tilde{A}_\phi$

$\hat{\varphi} \times (\square) \longrightarrow \rho^{-1} \hat{\varphi} \times \nabla A_\phi = - f^{-1} \nabla \tilde{A}_\phi - \rho^{-1} \omega \hat{\varphi} \times \nabla A_t$

$\xrightarrow{\nabla \cdot (\,)} \nabla \cdot (f^{-1} \nabla \tilde{A}_\phi + \rho^{-1} \omega \hat{\varphi} \times \nabla A_t) = \nabla \cdot (\rho^{-1} \hat{\varphi} \times \nabla A_\phi) \overset{(*)}{=} 0$

$(2) \rightarrow \nabla \cdot (- f^{-1} \nabla A_t + \rho^{-1} \omega \hat{\varphi} \times \nabla \tilde{A}_\phi) = 0$

$$\boxed{\begin{aligned} &\nabla \cdot \left(f^{-1} \nabla \tilde{A_\phi} + \rho^{-1} \omega \hat{\varphi} \times \nabla A_t \right) = 0 \\ &\nabla \cdot \left(-f^{-1} \nabla A_t + \rho^{-1} \omega \hat{\varphi} \times \nabla \tilde{A_\phi} \right) = 0 \end{aligned}} \qquad (M)$$

$\Phi := A_t + i \tilde{A_\phi}$ $\Leftarrow$ Ernst potential

$$\boxed{\nabla \cdot \left(f^{-1} \nabla \Phi + i \rho^{-1} \omega \hat{\varphi} \times \nabla \Phi \right) = 0}$$

$$(4) \iff \nabla \cdot \left[\rho^{-2} f^{2} \nabla \omega + 2 \rho^{-1} \hat{\varphi} \times \mathrm{Im} \left(\Phi^{*} \nabla \Phi \right) \right] = 0 \qquad (4')$$

χ s.t. $\hat{\varphi} \times \nabla \chi = -\rho^{-1} f^{2} \nabla \omega - 2 \hat{\varphi} \times \mathrm{Im} \left(\Phi^{*} \nabla \Phi \right)$

$$\nabla \cdot \left[f^{-2} \nabla \chi + 2 f^{-2} \operatorname{Im}(\Phi^* \nabla \Phi) \right] = 0 \qquad (4'')$$

$$f \nabla^2 f = (\nabla f)^2 - \left[\nabla \chi + 2 \operatorname{Im}(\Phi^* \nabla \Phi) \right]^2 + 2 f \, \nabla \Phi \cdot \nabla \Phi^* \qquad (3')$$

$$\mathcal{E} := f - |\Phi|^2 + i\chi \qquad \leftarrow \quad \text{Ernst potential}$$

$$\boxed{\begin{aligned} (\operatorname{Re}\mathcal{E} + |\Phi|^2)\, \nabla^2 \mathcal{E} &= \nabla \mathcal{E} \cdot (\nabla \mathcal{E} + 2 \Phi^* \nabla \Phi) \\ (\operatorname{Re}\mathcal{E} + |\Phi|^2)\, \nabla^2 \Phi &= \nabla \Phi \cdot (\nabla \mathcal{E} + 2 \Phi^* \nabla \Phi) \end{aligned}} \qquad \begin{aligned} &(E1) \\ &(E2) \end{aligned}$$

$$S = \int d^3x \; \frac{(\nabla \mathcal{E} + 2\Phi^* \nabla \Phi)\cdot(\nabla \mathcal{E}^* + 2\Phi \nabla \Phi^*) - \nabla\Phi\cdot\nabla\Phi^*}{(\mathrm{Re}\,\mathcal{E} + |\Phi|^2)}$$

$\mathcal{E} = x + iy$, $\quad \Phi = u + iv$

$\nabla \mathcal{E} \to dx + i\,dy$, $\quad \nabla \Phi \to du + i\,dv$, $\quad \mathrm{Re}\,\mathcal{E} = x$

$$dS^2 = \frac{1}{4}(u^2+v^2+x)^{-2}\,[\,dx^2 + dy^2 - 4x(du^2 + dv^2) - 4v\,dy\,du + 4u\,dy\,dv + 4v\,dx\,dv + 4u\,dx\,du\,]$$

$$\nabla_i \xi_j + \nabla_j \xi_i = 0$$

$\xi_1, \ldots, \xi_8$

$$\boxed{\begin{array}{llll}
\mathrm{I}) & \mathcal{E}' = |\lambda|^2 \mathcal{E}, & \Phi' = \lambda \Phi & \times \\[1em]
\mathrm{II}) & \mathcal{E}' = \mathcal{E} + ib, & \Phi' = \Phi & \times \\[1em]
\mathrm{III}) & \mathcal{E}' = \dfrac{\mathcal{E}}{1+ic\mathcal{E}}, & \Phi' = \dfrac{\Phi}{1+ic\mathcal{E}} & \\[1.5em]
\mathrm{IV}) & \mathcal{E}' = \mathcal{E} - 2\beta^* \Phi - |\beta|^2, & \Phi' = \Phi + \beta & \times \\[1.5em]
\mathrm{V}) & \mathcal{E}' = \dfrac{\mathcal{E}}{1 - 2\alpha^2 \Phi - |\alpha|^2 \mathcal{E}}, & \Phi' = \dfrac{\alpha \mathcal{E} + \Phi}{1 - 2\alpha^2 \Phi - |\alpha|^2 \mathcal{E}} &
\end{array}}$$

$\alpha, \beta, \lambda \in \mathbb{C}$, $\quad b, c \in \mathbb{R}$

III) = Ehlers, $\quad$ V) = Harrison

$$\boxed{SU(2,1)}$$

$t \to i\chi, \quad \phi \to i\tau$

$$ds_m^2 = f(d\chi - \omega\, d\tau)^2 + f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) - \rho^2 d\tau^2\right]$$

$$ds_e^2 = -f(dt - \omega\, d\phi)^2 + f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right]$$

Ehlers

$$\mathcal{E}' = \frac{\mathcal{E}}{1 + ic\,\mathcal{E}} \,, \qquad \Phi' = \frac{\Phi}{1 + ic\,\mathcal{E}}$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \,, \qquad A = 0$$

$$ds_e^2 = -f\,(dt - \cancel{\chi}\, d\phi)^2 + f^{-1}\left[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\phi^2\right]$$

$$f = 1 - \frac{2M}{r} \,, \qquad \omega = 0 \;\Rightarrow\; \chi = 0 \,, \qquad A_t = \tilde{A}_\phi = 0$$

$$\mathcal{E} = f - |\Phi|^2 + i\chi = 1 - \frac{2M}{r} \,, \qquad \Phi = 0$$

$$\mathcal{E}' = \frac{r^2 - 2Mr}{r^2 + c^2 (r - 2M)^2} - i\,\frac{c\,(r - 2M)^2}{r^2 + c^2 (r - 2M)^2} \,, \qquad \Phi' = 0 \,, \qquad \gamma' = \gamma$$

$$f' = \frac{r^2 - 2Mr}{r^2 + c^2 (r-2M)^2}, \qquad \chi' = - \frac{c(r-2M)^2}{r^2 + c^2 (r-2M)^2}$$

$$\omega' = 4cM\cos\vartheta$$

$$ds'^2 = - \frac{r^2 - 2Mr}{r^2 + c^2(r-2M)^2} \left(dt - 4Mc\cos\vartheta \, d\phi \right)^2 + \frac{r^2 + c^2 (r-2M)^2}{r^2 - 2Mr} dr^2 +$$

$$+ \left(r^2 + c^2 (r-2M)^2 \right) \left(d\vartheta^2 + \sin^2\vartheta \, d\phi^2 \right)$$

$$R := r\sqrt{1+c^2} - \frac{2Mc^2}{\sqrt{1+c^2}}, \qquad T = \frac{t}{\sqrt{1+c^2}}$$

$$M = -\frac{n}{2c}\sqrt{1+c^2}, \qquad c = \frac{m - \sqrt{m^2+n^2}}{n}$$

$$ds'^2 = -\frac{R^2-2mR-n^2}{R^2+n^2}\left(dT - 2n\cos\theta\, d\phi\right)^2 +$$

$$+ \frac{R^2+n^2}{R^2-2mR-n^2}\, dR^2 + (R^2+n^2)\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

Taub-NUT metric

Harrison

$$\mathcal{E}' = \frac{\mathcal{E}}{1-2\alpha^2\Phi - |\alpha|^2\mathcal{E}} , \qquad \Phi' = \frac{\alpha\mathcal{E} + \Phi}{1-2\alpha^2\Phi - |\alpha|^2\mathcal{E}}$$

$$ds_m^2 = f(d\phi - \omega\, dt)^2 + \cdots$$

$$f = r^2\sin^2\theta , \qquad \omega = 0 , \qquad \chi = 0$$

$$\mathcal{E} = r^2\sin^2\theta , \qquad \Phi = 0$$

$$\mathcal{E}' = \frac{r^2\sin^2\theta}{1-|\alpha|^2 r^2\sin^2\theta} , \qquad \Phi' = \frac{\alpha r^2\sin^2\theta}{1-|\alpha|^2 r^2\sin^2\theta}$$

$$f', \ \cancel{\chi'}, \ A_t', \ \breve{A}_\phi'$$

$$\alpha = -\frac{B}{2} + i\frac{E}{2}$$

$$ds'^2 = \Lambda^2\left[-\left(1-\frac{2M}{r}\right)dt^2 + \frac{dr^2}{1-\frac{2M}{r}} + r^2 d\theta^2\right] + \Lambda^{-1} r^2 \sin^2\theta$$

$$A' = E(r-2M)\cos\theta\, dt - \frac{B}{2}\Lambda^{-1} r^2 \sin^2\theta\, d\phi$$

$$\Lambda(r,\theta) = 1 - \frac{E^2+B^2}{4} r^2 \sin^2\theta$$