Inverse scattering method

$$\partial_t u = F(u, \partial_z u, \partial_{zz} u, \ldots) \qquad u(t,z)$$

$$D\psi = \lambda \psi \qquad D = \frac{d^2}{dz^2} + u$$

- Belinski, Zakharov JETP 48: 985 – 994, '78
- " " " 50: 1 – 9, '79
- Belinski, Verdaguer "Gravitational solitons", CUP

$$\boxed{\text{BZ} \Rightarrow \text{pure gravity} \qquad R_{\mu\nu} = 0}$$

Alekseev $\Rightarrow$ Einstein – Maxwell

$$S = \int d^4x \sqrt{-g}\, R \quad \Rightarrow \quad R_{\mu\nu} = 0$$

$$ds^2 = g_{ab}(\rho, z)\, dx^a dx^b + f(\rho, z)(d\rho^2 + dz^2)$$

$$a, b = t, \phi \qquad g = \begin{pmatrix} g_{tt} & g_{t\phi} \\ g_{t\phi} & g_{\phi\phi} \end{pmatrix}$$

$$\boxed{\det g = -\rho^2}$$

$$R^t{}_t = 0, \qquad R^\phi{}_t = 0, \qquad R^t{}_\phi = 0 \qquad \begin{pmatrix} R^t{}_t & R^\phi{}_t \\ R^t{}_\phi & \cdots \end{pmatrix} = 0$$

$$\boxed{\partial_\rho(\rho\, \partial_\rho g \cdot g^{-1}) + \partial_z(\rho\, \partial_z g \cdot g^{-1}) = 0} \qquad (E1)$$

$$R_{\rho\rho} - R_{zz} = 0$$

$$R_{\rho z} = 0$$

$$\Longleftrightarrow \quad \boxed{\begin{aligned} \partial_\rho \log f &= -\frac{1}{\rho} + \frac{1}{4\rho}\,\mathrm{Tr}(U^2 - V^2) \\ \partial_z \log f &= \frac{1}{2\rho}\,\mathrm{Tr}(UV) \end{aligned}} \qquad (E2)$$

$$U := \rho\,(\partial_\rho g)\cdot g^{-1}\,, \qquad V := \rho\,(\partial_z g)\cdot g^{-1}$$

$$\boxed{\partial_\rho U + \partial_z V = 0} \qquad (1)$$

$$\boxed{\partial_z U - \partial_\rho V + \frac{1}{\rho}[U, V] + \frac{1}{\rho} V = 0} \qquad (2)$$

(1), (2) are compatibility conditions of an overdetermined eigenvalue system (Lax pair)

System depends on ρ, z, λ, $\lambda \in \mathbb{C}$

g, U, V will depend on the analytic structure of the eigenfunctions

$$D_1 := \partial_z - \frac{2\lambda^2}{\lambda^2+\rho^2}\partial_\lambda , \qquad D_2 := \partial_\rho + \frac{2\lambda\rho}{\lambda^2+\rho^2}\partial_\lambda \qquad (\lambda \text{ not depend } \rho, z)$$

$$[D_1, D_2] = 0$$

Define $\Psi(\lambda, \rho, z)$ (2x2 matrix)

$$\boxed{D_1\Psi = \frac{\rho V - \lambda U}{\lambda^2+\rho^2}\Psi , \qquad D_2\Psi = \frac{\rho U + \lambda V}{\lambda^2+\rho^2}\Psi} \qquad (S1), \quad (S2)$$

U, V, ρ ..

$$D_2(S_1) - D_1(S_2) \longrightarrow 0 = \lambda(\ldots) + \lambda^2(\ldots) + \ldots$$

$\Rightarrow$ (1), (2)

$(S1), (S2) \underset{\lambda=0}{\Rightarrow} \quad U = \rho(\partial_\rho \Psi)\Psi^{-1}, \qquad V = \rho(\partial_z \Psi)\Psi^{-1}$

$$\Rightarrow \quad \boxed{g(\rho, z) = \Psi(0, \rho, z)}$$

Seed: $g_0, f_0 \to U_0, V_0 \to \Psi_0(\lambda, \rho, z)$

$\Psi = \chi \cdot \Psi_0$, χ 2x2 matrix

$(S1), (S2) \Rightarrow$

$$\boxed{\begin{aligned} D_1 \chi &= \frac{\rho V - \lambda U}{\lambda^2 + \rho^2} \chi - \chi \frac{\rho U_0 - \lambda V_0}{\lambda^2 + \rho^2} \\ D_2 \chi &= \frac{\rho U + \lambda V}{\lambda^2 + \rho^2} \chi - \chi \frac{\rho U_0 + \lambda V_0}{\lambda^2 + \rho^2} \end{aligned}}$$

$$\chi^*(\lambda^*) = \chi(\lambda), \qquad \psi^*(\lambda^*) = \psi(\lambda) \qquad \leftarrow \text{ reality}$$

$$g = \chi(\lambda)\, g_0\, \chi^T(-\rho^2/\lambda) \qquad \leftarrow \text{ symmetry}$$

$$\chi(\lambda = \infty) = \mathbb{1} \qquad \Leftrightarrow \qquad g = \chi(0)\, g_0$$

$$\left(\det g = -\rho^2 \qquad \Leftrightarrow \qquad \det \chi(0) = 1 \right)$$

$$\chi = \mathbb{1} + \sum_{k=1}^{n} \frac{R_k}{\lambda - \mu_k}$$ $R_k(\rho, z)$, $\mu_k(\rho, z)$ poles trajectories

χ simple poles $\Rightarrow$ (g, f) soliton solution

$[\mu_1, \dots, \mu_4$ $\mu_1 \to \mu_2$, $\mu_3 \to \mu_4$ pole fusion$]$

$$\partial_z \mu_k = - \frac{2\mu_k^2}{\mu_k^2 + \rho^2}, \qquad \partial_\rho \mu_k = \frac{2\rho\mu_k}{\mu_k^2 + \rho^2}$$

$$\Leftrightarrow \quad \mu_k^2 + 2(z - w_k)\mu_k - \rho^2 = 0, \qquad w_k \in \mathbb{C}$$

$$\boxed{\begin{aligned} \mu_k &= \sqrt{\rho^2 + (z-w_k)^2} - (z-w_k) \\ \bar{\mu}_k &= -\sqrt{\rho^2 + (z-w_k)^2} - (z-w_k) \end{aligned}}$$

w_k poles, μ_k solitons, $\bar{\mu}_k$ anti-solitons

$k = 1, \dots, n$

R_k degenerate $\Rightarrow$ $(R_k)_{ab} = n_a^{(k)} m_b^{(k)}$

$a, b = 0, 1 \quad (t, \phi)$

$$\boxed{m_a^{(k)} = m_{0,b}^{(k)} \left[\psi_0^{-1}(\mu_k, \rho, z) \right]_{ba}}$$

$m_0^{(k)} = (m_{0,0}^{(k)}, m_{0,1}^{(k)})$

constant vector

$\lambda = \mu_k$

BZ vectors

$$\sum_{l=1}^{n} \Gamma_{kl}\, n_a^{(k)} = \mu_k^{-1}\, m_c^{(k)}(g_0)_{ca} \quad \longleftarrow \quad \chi(\infty) = \mathbb{1}$$

$$\Gamma_{kl} = \frac{m_c^{(k)}(g_0)_{cb}\, m_b^{(l)}}{\rho^2 + \mu_k \mu_l}$$

$$\Rightarrow n_a^{(k)} = \sum_{l=1}^{k} (\Gamma^{-1})_{lk}\, \mu_k^{-1} L_a^{(k)}$$

$$L_a^{(k)} = m_c^{(k)}(g_0)_{ca}$$

Now I can find g:

$$g = \Psi(\lambda=0) \underset{\substack{\uparrow \\ \Psi = \chi \Psi_0}}{=} \chi(\lambda=0)\cdot\Psi_0(\lambda=0) \underset{\substack{\uparrow \\ \Psi_0(0) = g_0}}{=} \chi(0)\, g_0 = \left(1 - \sum_{k=1}^{n} R_k/\mu_k\right) g_0$$

$$\boxed{g_{ab} = (g_0)_{ab} - \sum_{k,e=1}^{n} (T^{-1})_{ke} \frac{L_a^{(k)} L_b^{(e)}}{\mu_k \mu_e} \qquad \left(L_a^{(k)} = m_c^{(k)} (g_0)_{ca} \right)}$$

$$\det g = (-1)^n \rho^{2n} \left(\prod_{k=1}^{n} \mu_k^{-2} \right) \underbrace{\det g_0}_{= -\rho^2}$$

$\boxed{n \text{ even}}$

$$g^{(ph)} = \pm \rho \sqrt{-\det g} \cdot g$$

$$\boxed{g^{(ph)} = \pm \rho^{-n} \left(\prod_{k=1}^{n} \mu_k^{-2} \right) g, \qquad \det g^{(ph)} = -\rho^2}$$

$$\boxed{f^{(ph)} = 16 C_f f_0 \rho^{-\frac{n^2}{2}} \left(\prod_{k=1}^{n} \mu_k^{-2} \right)^{n+1} \left[\prod_{k>e=1}^{n} (\mu_k - \mu_e)^{-2} \right] \det T_{ke}}$$

Kerr-NUT

$$ds_0^2 = -dt^2 + \rho^2 d\phi^2 + d\rho^2 + dz^2$$

$$g_0 = \begin{pmatrix} -1 & 0 \\ 0 & \rho^2 \end{pmatrix}, \qquad f_0 = 1, \qquad \det g_0 = -\rho^2$$

$$U_0 = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \qquad V_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\psi_0 = \begin{pmatrix} -1 & 0 \\ 0 & \rho^2 - 2\lambda z - \lambda^2 \end{pmatrix} \qquad \left(\psi_0(\lambda = 0) = \begin{pmatrix} -1 & 0 \\ 0 & \rho^2 \end{pmatrix} = g_0 \right)$$

$$m^{(k)} = \left(-m^{(k)}_{0,0}, \frac{m^{(k)}_{0,1}}{2w_k \mu_k} \right) \equiv \left(C_0^{(k)}, \frac{C_1^{(k)}}{\mu_k} \right)$$

$$\Gamma_{kl} = (\rho^2 + \mu_k \mu_l)^{-1} \left(-C_0^{(k)} C_0^{(l)} + C_1^{(k)} C_1^{(l)} \mu_k^{-1} \mu_l^{-1} \rho^2 \right) \qquad n = ?$$

$n = 2 \Rightarrow \lambda = \mu_1, \quad \lambda = \mu_2 \quad \to \quad w_1, w_2$

$w_1 = \tilde{z}_1 - \sigma, \qquad w_2 = \tilde{z}_1 + \sigma$

$\tilde{z}_1 \in \mathbb{R} \qquad (\sigma \in \mathbb{R})$

$\rho = \sqrt{(r-m)^2 - \sigma^2}\,\sin\vartheta\,, \qquad z = \tilde{z}_1 - (r-m)\cos\vartheta$

$\mu_1 = \sqrt{\rho^2 + (z-w_1)^2} - (z-w_1) = (r-m-\sigma)(1-\cos\vartheta)$

$\mu_2 = (r-m+\sigma)(1-\cos\vartheta)$

$C_1^{(1)} C_0^{(2)} - C_0^{(1)} C_1^{(2)} = \sigma, \qquad C_1^{(1)} C_0^{(2)} + C_0^{(1)} C_1^{(2)} = -m$

$C_0^{(1)} C_0^{(2)} - C_1^{(1)} C_1^{(2)} = n, \qquad C_0^{(1)} C_0^{(2)} - C_1^{(1)} C_1^{(2)} = a$

$C_a^{(k)} \mapsto \gamma^{(k)} C_a^{(k)}. \qquad \sigma^2 = m^2 - a^2 + n^2$

$$d\rho^2 + dz^2 = \left[(r-m)^2 - \sigma^2 \cos^2\vartheta \right] \left[\frac{dr^2}{(r-m)^2 - \sigma^2} + d\theta^2 \right]$$

$$ds^2 = - \frac{\Delta - a^2 \sin^2\vartheta}{\Sigma} dt^2 + \frac{\Delta (a \sin^2\vartheta + 2n \cos\vartheta)^2 - \sin^2\vartheta (r^2 + a^2 + n^2)^2}{\Sigma} d\phi^2 +$$

$$+ \frac{4n\Delta \cos\vartheta - 4a \sin^2\vartheta (mr + n^2)}{\Sigma} dt\, d\phi - C_f \left(\frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 \right)$$

$\Sigma = r^2 + a^2 \cos^2\vartheta$, $\quad \Delta = r^2 - 2mr + a^2 - n^2 \quad (t \to t + a\phi)$

$\Rightarrow$ Kerr-NUT $\quad C_f = -1 \quad \sigma^2 > 0 \rightarrow m^2 > a^2 - n^2$

$m_0^{(1)} = (0, 1)$

Schwarzschild : $a = n = 0 \quad C_0^{(1)} = C_1^{(2)} = 0$

$C_0^{(2)} = C_1^{(1)} = 1 \quad m_0^{(2)} = (1, 0)$

Multi-black hole solutions

Double-Schwarzschild

$n=4 \qquad \mu_1, \dots, \mu_4 . \qquad C_0^{(1)} = C_1^{(2)} = C_0^{(3)} = C_1^{(4)} = 0$

$$g^{(ph)} = \mathrm{diag}\left(-\frac{\mu_1\mu_3}{\mu_2\mu_4}, \ \rho^2 \frac{\mu_2\mu_4}{\mu_1\mu_3}\right)$$

$$f^{(ph)} = \dots$$

N-Schwarzschild $\qquad g^{(ph)} = \mathrm{diag}\left(-\frac{\mu_1\mu_3\dots\mu_{2k+1}}{\mu_2\mu_4\dots\mu_{2k}}, \ \rho^2 \frac{\mu_2\dots}{\mu_1\dots}\right)$