Lie Superalgebra Cohomology: new insights from pseudoforms

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Motivations

- Construction of Lagrangians
- Higher Form Terms: Brane Scan
- Classification of Invariants
- Free Differential (super)Algebras Extensions: Lie n-Superalgebras?
- Pseudoforms
- ...

Outline

- Forms on Supermanifolds (Berezin, Bernstein, Leites, Manin, Penkov, Witten, etc.)
- Lie Algebra Cohomology (Chevalley-Eilenberg, Koszul, Hochschild-Serre, etc.)
- Lie Superalgebra Cohomology: what was known (Kac, Fuks, etc.)
- Lie Superalgebra Cohomology: what was not known

Forms on Supermanifolds

On a supermanifold $\mathcal{SM}$ of dimension *dim*$\mathcal{SM} = (m|n)$, *superforms*, i.e., $\left(\Omega^{(\bullet|0)}\left(\mathcal{SM}, C^{\infty}_{\mathbb{R}^m}\left[\theta^1, \ldots, \theta^n\right]\right), d\right)$, are *unbounded from above*:

$$0 \xrightarrow{\mathrm{d}} \Omega^{(0|0)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}} \Omega^{(1|0)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} \Omega^{(m|0)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}} \ldots$$

as a consequence of the commutation relations

$$dx^i dx^j = -dx^j dx^i \ , \ dx^i d\theta^\alpha = d\theta^\alpha dx^i \ , \ d\theta^\alpha d\theta^\beta = d\theta^\beta d\theta^\alpha \ .$$

Top form $\rightarrow$ *Berezinian space*, the super-analogous of the *Determinant space*.
Berezinian $\rightarrow$ *integral forms*:

$$\ldots \xrightarrow{\delta} \Omega^{(0|n)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} \Omega^{(m-1|n)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}} \Omega^{(m|n)}_{\mathcal{SM}} \xrightarrow{\mathrm{d}} 0 \ .$$

The top form can be realised as a *generalised function*:

$$\omega^{(m|n)} = \omega\left(x, \theta\right) dx^1 \ldots dx^m \delta\left(d\theta^1\right) \ldots \delta\left(d\theta^n\right) \ .$$

These distributions satisfy the relations

$$\delta\left(d\theta^{\alpha}\right)\delta\left(d\theta^{\beta}\right)=-\delta\left(d\theta^{\beta}\right)\delta\left(d\theta^{\alpha}\right)\ ,\ d\theta^{\alpha}\delta\left(d\theta^{\alpha}\right)=0\,,\ \delta\left(\lambda d\theta^{\alpha}\right)=\frac{1}{\lambda}\delta\left(d\theta^{\alpha}\right)\ .$$

We omit the function in front of the Berezinian (we are considering trivial modules), so that we can define

$$\omega^{(m|n)}\equiv\omega_{\mathfrak{g}}^{top}:=\mathcal{V}^1\dots\mathcal{V}^n\delta(\psi^1)\dots\delta(\psi^m)\ .$$

Non-zero and non-maximal number of deltas $\rightarrow$ *pseudoforms*:

$$\dots\xrightarrow{\delta}\Omega^{(0|q)}_{\mathcal{SM}}\xrightarrow{\mathrm{d}}\dots\xrightarrow{\mathrm{d}}\Omega^{(m-1|q)}_{\mathcal{SM}}\xrightarrow{\mathrm{d}}\Omega^{(m|q)}_{\mathcal{SM}}\xrightarrow{\mathrm{d}}\dots\ .$$

These forms are related to *sub-superspaces*.

Lie Algebra Cohomology

Let the supermanifold is substituted with a Lie algebra $\mathfrak{g}$:

$$[X_i, X_j] = f_{ij}^k X_k \ .$$

We can use the structure constants to define the *Maurer-Cartan differential* on forms $\Omega^p(\mathfrak{g})$:

$$dV^i = -\frac{1}{2} f^i_{jk} V^j V^k \ , \ d = -\frac{1}{2} f^i_{jk} V^j V^k \iota_i \ ,$$

and the nilpotence $d^2 = 0$ follows from Jacobi identities.

The *Chevalley-Eilenberg cohomology* groups $H^\bullet(\mathfrak{g})$ are defined as

$$H^\bullet(\mathfrak{g}) = \{\omega | d\omega = 0, \omega \neq d\eta\} \ .$$

Some facts:

- cohomology classes are given by scalars
- ring structure: products of classes are still classes
- Poincaré Duality: if ω^p is a class, then ω^{m-p} is a class
- Cartan Theorem: if G is a compact, connected Lie group with algebra $\mathfrak{g}$, its de Rham cohomology is isomorphic to the CE cohomology of the algebra

Many ways to calculate cohomology:

- brute force
- Weyl formula (localisation)
- spectral sequences (Koszul, Hochschild-Serre)

Lie Superalgebra Cohomology: what was known

The previous definitions and results hold in the case of $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ a *Lie superalgebra*: let us denote with $\mathcal{Y}^{*A} := \{\mathcal{V}^i | \psi^\alpha\}$ the basis of forms, we have

$$[\mathcal{Y}_B, \mathcal{Y}_C] = f^A_{BC} \mathcal{Y}_A \implies d = -\frac{1}{2} f^A_{BC} \mathcal{Y}^{*B} \wedge \mathcal{Y}^{*C} \iota_{\mathcal{Y}_A} \ ,$$

$$d\left(\omega \wedge \eta\right) = \left(d\omega\right) \wedge \eta + (-1)^{|\omega||\eta|} \omega \left(d\eta\right) \ ,$$

where now $[\cdot, \cdot]$ are *supercommutators*.

Key point: if we realise integral forms as generalised functions with their distributional rules, the differential for integral forms *is the same*.

Side remark: integral forms are used to define an invariant integration on supergroups $\rightarrow$ *Haar Berezinian*.

Superform cohomology of classical Lie superalgebras was classified by Fuks:

Theorem

If $m \geq n$, the natural inclusions

$$\begin{aligned} \mathfrak{gl}(m) &\to \mathfrak{gl}(m|n)_0 &\subset \mathfrak{gl}(m|n) \ , \\ \mathfrak{sl}(m) &\to \mathfrak{sl}(m|n)_0 &\subset \mathfrak{sl}(m|n) \ , \end{aligned}$$

induce an isomorphism in cohomology with trivial coefficients.

Theorem

$$H^\bullet(\mathfrak{osp}(m|n)) = \begin{cases} H^\bullet(\mathfrak{so}(m)) \ , \ \text{if } m \geq 2n \ , \\ H^\bullet(\mathfrak{sp}(n)) \ , \ \text{if } m < 2n \ . \end{cases}$$

$\Longrightarrow$ a part of the bosonic sub-algebra (hence some invariants) is lost.

Let us define the "Berezinian complement" map $\star$ as

$$\begin{aligned} \star : \Omega^{p}_{diff}(\mathfrak{g}) &\rightarrow \Omega^{m-p}_{int}(\mathfrak{g}) \\ \omega &\mapsto \star\omega^{(p)} = (\star\omega)^{(m-p)} := \left(\prod_{i=1}^{p} \iota_{\mathcal{Y}_{A_i}} \right) \omega^{top}_{\mathfrak{g}} \ , \end{aligned}$$

where $\left(\prod_{i=1}^{p} \iota_{\mathcal{Y}_{A_i}} \right) \omega = 1$. This map induces a cohomology isomorphism:

$$\star : H^{\bullet}_{diff}(\mathfrak{g}) \xrightarrow{\cong} H^{m-\bullet}_{int}(\mathfrak{g}) \ .$$

The proof of the isomorphism relies on the existence of a non-degenerate invariant bilinear form on $\mathfrak{g}$, hence it holds e.g. for "basic Lie superalgebras".

R.Catenacci, CAC, P.A.Grassi, S.Noja

Lie Superalgebra Cohomology: what was not known

Our claim is about "lost" invariants: they are smeared in the other form complexes.

Example

$\mathfrak{u}(1) \times \mathfrak{u}(1)$ vs $\mathfrak{u}(1|1)$

$$
\begin{array}{rcl}
dU = 0 \ , \ dW = 0 & \leftrightsquigarrow & dU = 0 \ , \ dW = \psi^+ \psi^- \\
H^1 \left(\mathfrak{u}(1) \times \mathfrak{u}(1) \right) = \{ U, W \} & \leftrightsquigarrow & H^{(1|0)} \left(\mathfrak{u}(1|1) \right) = \{ U \}
\end{array}
$$

BUT

$$
dW \delta \left(\psi^+ \right) \delta \left(\psi^- \right) = 0 \ \rightarrow \ H^{(1|2)} \left(\mathfrak{u}(1|1) \right) = \left\{ W \delta \left(\psi^+ \right) \delta \left(\psi^- \right) \right\}
$$

The "lost" abelian factor is found among integral forms!

In general, we expect to find invariants in all form complexes $\rightarrow$ we need a way to calculate pseudoform cohomology classes.

Problem: pseudoforms are not well defined objects.

Example

$$\psi^1 \mapsto \psi^1+\psi^2 \implies \delta\left(\psi^1\right) \mapsto \delta\left(\psi^1+\psi^2\right) = \delta\left(\psi^1\right)+\psi^2\delta'\left(\psi^1\right)+\frac{1}{2}\delta''\left(\psi^1\right)+\ldots$$

Remark: this problem does not emerge for integral forms.

Intuition: pseudoforms seem to be related to *sub-superspaces*.

Spectral Sequences

Idea: reconstruct the cohomology of a Lie (super)algebra starting from the (eventually known) cohomology of substructure.

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k} \ , \ \mathfrak{k} = \mathfrak{g}/\mathfrak{h} \ , \ \mathfrak{h} \text{ sub-algebra.}$$

$$[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h} \ , \ [\mathfrak{h}, \mathfrak{k}] \subseteq \mathfrak{h} + \mathfrak{k} \ , \ [\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{h} + \mathfrak{k} \ .$$

The idea is to calculate the cohomology via *approximations*: split the differential d as

$$d = d_0 + d_1 + d_2 + \dots$$

then, calculate the cohomology w.r.t. d_0, then d_1, then d_2 etc., up to *convergence*.

Theorem (Koszul,Hochschild-Serre)

$$\mathcal{E}_2 = H^\bullet(\mathfrak{h}) \otimes H^\bullet(\mathfrak{k}) \ .$$

Key point: we can use the same technique, but *for each picture number*.

1^{st} step: generalisation.

In the bosonic setting, forms are built starting from 1, then multiplying by dx, up to the volume form. In the super setting, we have two inequivalent ways to construct forms: start from 1, then multiply by $dx, d\theta$ or start from ω^{top}, then act with contractions.

$$\implies \mathcal{E}_p = E_p \oplus \tilde{E}_p \ .$$

2^{nd} step: idea.

Choose as sub-algebra $\mathfrak{h}$ a *sub-superalgebra* $\implies$ Pseudoforms are introduced as *integral forms of sub-structures* $(\mathfrak{h}, \mathfrak{k})$.

Remark 1: these are *well-defined objects*.

Remark 2: easy to calculate $\rightarrow$ superforms are known, integral forms are obtained from Berezinian complement map $\star$.

Theorem

$$\mathcal{E}_2 = H^{\bullet|q}(\mathfrak{h}) \otimes H^{\bullet|0}(\mathfrak{k}) \oplus H^{\bullet|0}(\mathfrak{h}) \otimes H^{\bullet|q}(\mathfrak{k}) \ .$$

Remark 1: if one restricts to "known" sectors of superforms or integral forms → correct result.

Remark 2: if one restricts to bosonic algebras, $\mathcal{E}_2$ simplifies to the one by KHS.

Result:

- non-trivial extension of bosonic theorems
- completion of cohomology to every picture number
- "lost" invariants are found

Side result:

- pseudoforms are introduced rigorously, independently from their distributional realisation → hint to correctly define pseudoforms in a more general context

Example ($\mathfrak{g} = \mathfrak{osp}(2|2)$)

4 bosons, 4 fermions

$$H^{(\bullet|0)}(\mathfrak{osp}(2|2)) = H^{\bullet}(\mathfrak{sp}(2)) = \left\{1, \omega^{(3)}\right\} \ .$$

The abelian factor $\mathfrak{so}(2)$ is the lost part. We can choose

$$\mathfrak{h} = \mathfrak{osp}(1|2) \ , \ \text{3 bosons, 2 fermions} \quad \Longrightarrow \quad \text{picture 2 integral forms;}$$
$$\mathfrak{k} = \mathfrak{g}/\mathfrak{h} \ , \ \text{1 boson, 2 fermions} \quad \Longrightarrow \quad \text{picture 2 integral forms.}$$

One finds

$$H^{(\bullet|2)}(\mathfrak{osp}(2|2)) = \left\{\omega^{(0|2)}, \omega^{(1|2)}, \omega^{(3|2)}, \omega^{(4|2)}\right\} \ ,$$

$\omega^{(1|2)}$ encodes the abelian factor.

There are other ways to approach the problem:

- brute force; but pseudoforms live in infinite dimensional spaces, computations may be arbitrarily difficult
- Molien-Weyl integral: it is possible to extend the bosonic formula to the super setting. Pseudoforms are not standard representations $\rightarrow$ infinite-dimensional representations $\rightarrow$ extremely rich, almost unexplored land

The best way is to complement every method with the other, to have a complete understanding of the problems and the results.

General Philosophy

- new point of view to approach old problems, e.g., self-duality in Physics $\implies$ possibility of introducing new objects, fields, pairings etc.
- generation of new problems and generalisation of known results, constructions, e.g., FDA, Lie$-n$ theory etc.