CFT correlators in momentum space, conformal anomaly and anomaly-induced action

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Introduction

- Conformal invariance imposes strong constraints on correlation functions
- It determines 2- and 3-point functions of scalars, conserved vectors and the stress-energy tensor
- It determines the form of the higher functions up to functions of cross-ratios
- These results were obtained in position space and this is in contrast with general QFT were Feynman diagrams are typically computed in momentum space
- While position space methods are powerful, typically they provide results that hold only at separated points

Why study CFT directly in momentum space?

Cosmology

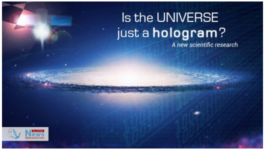
- CMB primordial power spectra and Non-Gaussianities
- Cosmological Booststrap
- Gravitational waves
- Holographic Cosmology

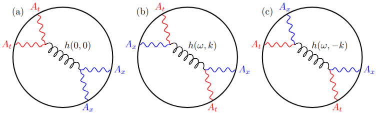
Condensed Matter

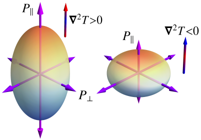
- Multipoint correlators of conformal field theories: implications for quantum critical transport
- Electromagnetic response of interacting Weyl semimetals
- Dynamical Response Near Quantum Critical Points

Anomalies

- Breaking of conformal invariance by quantum effects







Coordinate Space

Conformal transformations $x_\mu \to x'_\mu(x)$ preserve the infiniesimal length up to a local factor

$$dx_\mu dx^\mu \to dx'_\mu dx'^\mu = \Omega(x)^{-2} dx_\mu dx^\mu$$

and in the infinetisimal form

$$x'_\mu(x) = x_\mu + a_\mu + \omega_{\mu\nu} x^\nu + \lambda x_\mu + b_\mu x^2 - 2b \cdot x\, x_\mu$$

with $\Omega(x) = 1 - \sigma(x) = 1 - \lambda + 2b \cdot x.$

Isometry $x^\mu(x) \to x'^\mu(x) = x^\mu + v^\mu \implies g'_{\mu\nu}(x') = g_{\mu\nu}(x')$

$$v^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\sigma} \partial_\nu v^\sigma + g_{\sigma\nu} \partial_\mu v^\sigma = 0, \quad \text{Killing equation}$$

With the condition $g'_{\mu\nu}(x') = \Omega^{-2}(x) g_{\mu\nu}(x')$

$$v^\alpha \partial_\alpha g_{\mu\nu} + g_{\mu\sigma} \partial_\nu v^\sigma + g_{\sigma\nu} \partial_\mu v^\sigma = 2\sigma\, g_{\mu\nu}, \quad \text{Conformal Killing equation}$$

In the flat spacetime limit the conformal Killing equation has the general solution

$$v^\mu(x) = a_\mu + \omega_{\mu\nu} x^\nu + \lambda x_\mu + b_\mu x^2 - 2x_\mu\, b \cdot x$$

$$
\begin{aligned}
&\text{Translation} && \delta x^\mu = a_\mu, && && P_\mu = \partial_\mu, \\
&\text{Rotation} && \delta x^\mu = \omega_{\mu\nu} x^\nu && \Longrightarrow && L_{\mu\nu} = x_\nu \partial_\mu - x_\mu \partial_\nu, \\
&\text{Dilatations} && \delta x^\mu = \lambda x^\mu, && && D = x^\mu \partial_\mu, \\
&\text{SCT} && \delta x^\mu = b^\mu x^2 - 2x^\mu b\cdot x. && && K_\mu = 2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu.
\end{aligned}
$$

Defining the generators J_{ab}, $J_{ab} = -J_{ba}$ with $a, b = -1, 0, 1, \ldots, d$ as

$$
J_{\mu\nu} = L_{\mu\nu}, \quad J_{-1,\mu} = \frac{1}{2}\left(P_\mu - K_\mu\right),
$$
$$
J_{-1,0} = D, \quad J_{0,\mu} = \frac{1}{2}\left(P_\mu + K_\mu\right).
$$

the conformal algebra can be written as

$$
[J_{ab}, J_{cd}] = \eta_{ac} J_{bd} + \eta_{bd} J_{ac} - \eta_{ad} J_{bc} - \eta_{bc} J_{ad}
$$

with $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$. Then

$$
Conf(\mathbb{R}^d) \cong SO(1, d+1), \qquad \Rightarrow \qquad 1/2(d+2)(d+1)\,\text{parameters.}
$$

Conformal Symmetry in Coordinate Space

Coordinate Space

$$SCT = Inversion + translation + Inversion \qquad x^\mu \to \frac{x^\mu}{x^2} \to \frac{x^\mu}{x^2} - b^\mu \to \frac{\frac{x^\mu}{x^2} - b^\mu}{\left(\frac{x^\mu}{x^2} - b^\mu\right)^2}$$

$$\langle \mathcal{O}_i(x)\mathcal{O}_j(y)\rangle = \frac{\delta_{ij}}{(x-y)^{2\Delta}}, \qquad \langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{\lambda_{123}}{|x_{12}|^{\Delta_{123}}|x_{13}|^{\Delta_{132}}|x_{23}|^{\Delta_{231}}}$$

$$\langle \mathcal{O}(x_1)\dots\mathcal{O}(x_4)\rangle = \frac{f(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}}, \quad u = \frac{x_{12}^2 x_{24}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2}$$

Momentum Space

$$0 = D\,\langle \mathcal{O}^{\mu_{1_1}\mu_{1_2}\cdots\mu_{1_{r_1}}}(p_1)\dots\mathcal{O}^{\mu_{j_1}\mu_{j_2}\cdots\mu_{j_{r_j}}}(p_j)\dots\mathcal{O}^{\mu_{n_1}\mu_{n_2}\cdots\mu_{n_{r_n}}}(\bar{p}_n)\rangle$$

$$0 = K^\kappa\,\langle \mathcal{O}^{\mu_{1_1}\mu_{1_2}\cdots\mu_{1_{r_1}}}(p_1)\dots\mathcal{O}^{\mu_{j_1}\mu_{j_2}\cdots\mu_{j_{r_j}}}(p_j)\dots\mathcal{O}^{\mu_{n_1}\mu_{n_2}\cdots\mu_{n_{r_n}}}(\bar{p}_n)\rangle$$

[C. Corianó, M. M. M., Conformal Field Theory in Momentum Space and Anomaly Actions in Gravity: The Analysis of 3- and 4-Point Functions, To appear on Physics Reports, (2021)],
[C. Corianò, M. M. M. and D. Theofilopoulos,"The Conformal Anomaly Action to Fourth Order (4T) in $d = 4$ in Momentum Space," doi:10.1140/epjc/s10052-021-09523-9, (2021)]

The position space expressions are only valid at separated points and do not possess a Fourier transform prior to renormalization. For instance

$$\langle \mathcal{O}(x)\mathcal{O}(0)\rangle = \frac{C_{\mathcal{O}}}{|x|^{2\Delta}}$$

does not have a Fourier transform when

$$\Delta = \frac{d}{2} + k, \qquad k = 0, 1, 2, \ldots$$

because of short-distance singularities. In momentum space

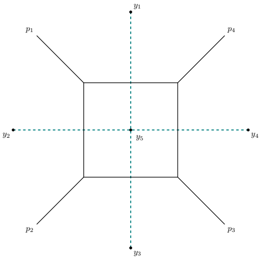
$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\rangle = (2\pi)^d \delta(p_1 + p_2) \frac{C_{\mathcal{O}}\, \pi^{d/2}\, 2^{d-2\Delta}\, \Gamma\left(\frac{d}{2} - \Delta\right)}{\Gamma(\Delta)} p_1^{2\Delta - d}, \quad \Delta \ \text{generic}$$

$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\rangle = (2\pi)^d \delta(p_1 + p_2) \left[\frac{C_{\mathcal{O}}\, \pi^{d/2}\, 2^{d-2\Delta}\, \Gamma\left(\frac{d}{2} - \Delta\right)}{\Gamma(\Delta)} \ln \frac{p_1^2}{\mu^2} + c_{\Delta'} \right] p_1^{2k}, \quad \Delta = \frac{d}{2} + k$$

with $c_{\Delta'}$ scheme dependent constant that can be absorbed by a redefinition of the scale μ.

[C. Coriano, L. Delle Rose, E. Mottola, and M. Serino, Graviton vertices and the mapping of anomalous correlators to momentum space for a general conformal field theory, JHEP 1208 (2012) 147]
[A. Bzowski, P. McFadden and K. Skenderis,"Scalar 3-point functions in CFT: renormalisation, beta functions and anomalies," JHEP **03** (2016), 066]

A particular solution of the 4 point function in momentum space using the dual conformal symmetry (DCC).



$$p_1^\mu = y_2^\mu - y_1^\mu, \quad p_2^\mu = y_3^\mu - y_2^\mu,$$
$$p_3^\mu = y_4^\mu - y_3^\mu, \quad p_4^\mu = y_1^\mu - y_4^\mu.$$

$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\mathcal{O}(\bar{p}_4)\rangle = \Phi(p_1,\dots,p_4,s,t)$$

$$0 = D(p_i)\Phi(p_1,\dots,p_4,s,t)$$
$$0 = K^\kappa(p_i)\Phi(p_1,\dots,p_4,s,t)$$

$$0 = D(y_i)\Phi(y_1,\dots,y_4)$$
$$0 = K^\kappa(y_i)\Phi(y_1,\dots,y_4)$$

$$D(y_i) = \left[\sum_{j=1}^4 \Delta_j + \sum_{j=1}^4 y_j^\alpha \frac{\partial}{\partial y_j^\alpha}\right]$$

$$K^\kappa(y_i) = \left[\sum_{j=1}^4 \left(2\Delta_j y_j^\kappa + 2 y_j^\kappa y_j^\alpha \frac{\partial}{\partial y_j^\alpha} - y_j^2 \frac{\partial}{\partial y_{j\,\kappa}}\right)\right]$$

$$\langle \mathcal{O}(p_1)\mathcal{O}(p_2)\mathcal{O}(p_3)\mathcal{O}(p_4)\rangle = C\left[I_{\frac{d}{2}-1\{\Delta-\frac{d}{2},\Delta-\frac{d}{2},0\}}(p_1\,p_3, p_2\,p_4, s\,t) + \text{perm}\right]$$

[C. Corianò, M. M. M., **JHEP 1909 (2019) 107**]

In even dimensions one observes the breaking of conformal invariance. The result is an anomalous trace for the stress energy tensor.

In $d = 2$

$$\langle T^{\mu}{}_{\mu} \rangle = -\frac{c}{24\pi} R$$

Given the conformal anomaly $\mathcal{A}$ one can integrate the equation

$$2 g_{\mu\nu} \frac{\delta \mathcal{S}_{anom}[g]}{\delta g_{\mu\nu}(x)} = \frac{\delta \mathcal{S}_{anom}[e^{2\sigma} \bar{g}]}{\delta \sigma(x)} \bigg|_{g = e^{2\sigma} \bar{g}} = \mathcal{A}$$

to obtain the anomaly action in $d = 2$

$$\mathcal{S}_{anom}[g, \sigma] = -\frac{c}{96\pi} \int d^2x \sqrt{-g}\, \sigma \Box \sigma \tag{1}$$

The conformal factor σ in $d = 2$ is uniquely determined as $\sigma = \Box_g^{-1} R$ to give the Polyakov action

$$\mathcal{S}_{anom}[g, \sigma] = -\frac{c}{96\pi} \int d^2x \sqrt{-g}\, R \,\Box_g^{-1} R \tag{2}$$

Anomaly induced Action

In $d=4$

$$\langle T^{\mu}{}_{\mu}\rangle = b\, C^2 + b'\left(E - \frac{2}{3}\Box R\right)$$

and in $d=4$

$$\mathcal{S}_{anom}[g,\sigma] = -\frac{b'}{2}\int d^4x\sqrt{-g}\Big(\sigma\Delta_4\sigma\Big) + \frac{1}{2}\int d^4x\,\mathcal{A}\,\sigma \qquad (3)$$

with

$$\Delta_4 \equiv \nabla_\mu\left(\nabla^\mu\nabla^\nu + 2R^{\mu\nu} - \tfrac{2}{3}Rg^{\mu\nu}\right)\nabla_\nu = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \tfrac{2}{3}R\Box + \tfrac{1}{3}(\nabla^\mu R)\nabla_\mu \qquad (4)$$

The conformal parameter σ can be inverted in some specific case and so far we have

$$\sigma_{FV} = 2\ln\left(1 + \frac{1}{6}\,G_{\Box - R/6}\,R\right),$$

$$\sigma_R = -\frac{1}{2}G_{\Delta_4}\left(E - \frac{2}{3}\Box R\right),$$

Anomaly Induced Actions

The Riegert actions is

$$S_{anom}[g,\sigma_R] = \frac{1}{4}\int dx\sqrt{-g_x}\left(E - \frac{2}{3}\Box R\right)_x \int dx'\,\sqrt{-g_{x'}}G_{\Delta_4}(x,x')\left[\frac{b'}{2}\left(E - \frac{2}{3}\Box R\right) + bC^2\right]_{x'}$$

whose conformal variation is the anomaly. For instance, the anomalous part of the three-point diagram $\langle TTT\rangle$ is given by

$$\langle T^{\mu_1\nu_1}(x_1)T^{\mu_2\nu_2}(x_2)T^{\mu_3\nu_3}(x_2)\rangle_{anom} = \frac{2}{\sqrt{-g_{x_1}}}\cdots\frac{2}{\sqrt{-g_{x_3}}}\frac{\delta^3 S_{anom}}{\delta g_{\mu_1\nu_1}(x_1)\ldots\delta g_{\mu_3\nu_3}(x_3)}$$

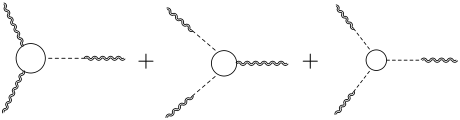
[C. Corianò, M. M. M., E. Mottola, **Nucl. Phys. B942 (2019) 303-328**]

Anomalous Conformal Ward Identities $\Longrightarrow$ Explicit breaking of Conformal Invariance

[Corianò and Maglio, The general 3-graviton vertex $\langle TTT\rangle$ of conformal field theorries in momentum space in $d=4$, Nucl.Phys. B937 (2018) 56-134]

The extra part of the TTT after the renormalization is the anomaly part

$$\langle T^{\mu_1\nu_1}T^{\mu_2\nu_2}T^{\mu_3\nu_3}\rangle^{(4)}_{extra} =$$



We have anomaly interactions!

In condensed matter systems, the "gravitational" (conformal) anomaly may be probed in an off-equilibrium regime using the Luttinger theory of thermal transport coefficients. The effect of a temperature gradient ∇T can be compensated by Φ

$$\frac{1}{T}\nabla T = -\frac{1}{c^2}\nabla\Phi.$$

For a weak gravitational field

$$g_{00} = 1 + \frac{2\Phi}{c^2},$$

and the TTT contribution to the thermal transport via Luttinger formula is

$$\langle T^{\mu\nu}(x)\rangle = \frac{1}{8}\int dx_2\, dx_3\, \langle T^{\mu\nu}(x) T^{\mu_2\nu_2}(x_2) T^{\mu_3\nu_3}(x_3)\rangle\, h_{\mu_2\nu_2}(x_2) h_{\mu_3\nu_3}(x_3). \tag{5}$$

Considering a system in a slightly off-equilibrium regime with a small temperature variation along the third direction, the anomalous part of the TTT vertex contributes as

$$\langle T^{00}\rangle_{TTT} = \frac{4b}{9}\left[3\left(\partial_3^2\Phi\right)^2 + 4\left(\partial_3^2\Phi\right)\left(\partial_3^3\Phi\right) + 2\Phi\left(\partial_3^4\Phi\right)\right]$$

$$\langle T^{11}\rangle_{TTT} = \langle T^{22}\rangle_{TTT} = \frac{4b}{9}\left[2\left(\partial_3^2\Phi\right)\left(\partial_3^3\Phi\right) + \Phi\left(\partial_3^4\Phi\right)\right]$$

[M. N. Chernodub, C. Corianò, M. M. M., **Phys.Lett.B 802 (2020) 135236**]

The TTT vertex of the conformal anomaly action leads to a qualitatively new effects: for a linearly varing temperature, there is a variation of the energy density

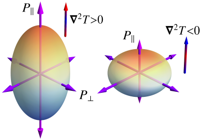
$$\delta E = \langle T^{00} \rangle_{TTT} = \frac{4b\,\hbar\,c}{3}\left(\frac{\nabla T}{T}\right)^4, \qquad \delta P = 0.$$

Assuming $\nabla^2 T \neq 0$ there is an anisotropic pressure

$$\delta P = P_{\parallel} - P_{\perp}, \qquad P_{\parallel} = \langle T^{33} \rangle, \quad P_{\perp} = \frac{\langle T^{11} \rangle + \langle T^{22} \rangle}{2}$$

and explicitly

$$\delta P = \frac{16\,b}{3}\hbar\,c\left(\frac{\nabla T}{T}\right)^2\left(\frac{\nabla^2 T}{T}\right), \qquad b = \frac{1}{320\pi^2}$$



[M. N. Chernodub, C. Corianò, M. M. M., **Phys.Lett.B 802 (2020) 135236**]

Conclusions

- Implications of conformal invariance in momentum space for 3-point function and a particular class of 4-point functions
- Exact Solutions in momentum space for the 4-point function by using the dual conformal invariance
- Anomaly action in condensed matter systems
- Structure of the anomalous part for the TTT

- How to extend the analysis to higher point functions?
- What are the implications of the dual conformal symmetry?
- What is the anomalous structure of higher point functions?