Scattering Amplitudes in Maximally Supersymmetric Gauge Theory and a New Duality

Lance Dixon (SLAC)

LD, Ö. Gürdoğan, A. McLeod, M. Wilhelm, 2112.06243

GGI Tea Break
9 February 2022
LHC is a QCD Machine

- Copious production of quarks and gluons which materialize as collimated jets of hadrons

- Confrontation between experiment and theory at high precision requires taking into account higher order corrections in the strong coupling $\alpha_s$
Example: Total cross section for producing Higgs boson at LHC via gluon fusion

Leading Order (LO)

- Higgs production at LHC is dominantly via gluon fusion, mediated by a top quark loop.
- Since $2m_{\text{top}} = 350 \text{ GeV}$
  $\gg m_{\text{Higgs}} = 125 \text{ GeV}$,
we can integrate out the top quark to get a leading operator $H G_\mu^a G^{\mu \nu} a$
Perturbative Short-Distance Cross Section

\[ \tilde{\sigma}(\alpha_s, \mu_F, \mu_R) = [\alpha_s(\mu_R)]^{n\alpha} \left[ \tilde{\sigma}^{(0)} + \frac{\alpha_s}{2\pi} \tilde{\sigma}^{(1)}(\mu_F, \mu_R) + \left(\frac{\alpha_s}{2\pi}\right)^2 \tilde{\sigma}^{(2)}(\mu_F, \mu_R) + \cdots \right] \]

Leading-order (LO) predictions qualitative: poor convergence of expansion in \( \alpha_s(\mu) \)

Uncertainty bands from varying \( \mu_R = \mu_F = \mu \)

Example: Higgs gluon fusion cross section at LHC vs. CM energy \( \sqrt{s} \)

LO \( \rightarrow \) NNNLO \( \rightarrow \) factor of 2.7 increase!

L. Dixon  N=4 Amps

Anastasiou, Duhr, Dulat, Herzog, Mistlberger, 1503.06056

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NLO QCD topologies

virtual \( gg \rightarrow H \)

real, \( gg \rightarrow Hg \)
NNNLO QCD topologies

\[ gg \rightarrow Hg \]
@ 2 loops, state of art in QCD

+ \ldots 
+ quarks 
+ operator renormalization 
+ \( 1/m_t^2 \) corrections 
+ parton distributions
Multi-loop complexities

- Multi-loop multiscale integrals typically very difficult to evaluate

- All 1 loop integrals with external legs in $D=4$ are reducible to scalar box integrals + simpler

$\to$ combinations of

\[ \text{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t) \]


- At $L$ loops, special functions with up to $2L$ integrations

  Hannesdottir, McLeod, Schwartz, Vergu, 2109.09744

- Weight $2L$ iterated integrals, generalized polylogarithms, or worse
Planar N=4 SYM, toy model for QCD amplitudes

- QCD’s maximally supersymmetric cousin, N=4 super-Yang-Mills theory (SYM), gauge group $SU(N_c)$, in the large $N_c$ (planar) limit
- Structure very rigid:
  \[
  \text{Amplitudes} = \sum_i \text{rational}_i \times \text{transcendental}_i
  \]
- For planar N=4 SYM, we understand rational structure quite well, focus on the transcendental functions.
- Furthermore, at least three dualities hold:
  1. AdS/CFT
  2. Amplitudes dual to Wilson loops
  3. New “antipodal” duality between amplitudes and form factors
Finite radius of convergence

- Planar N=4 SYM has no renormalons ($\beta(g) = 0$) and no instantons ($e^{-1/g_{YM}^2} = e^{-N_c/\lambda}$).
- Its perturbative expansion can have a finite radius of convergence, unlike QCD, QED, whose perturbative series are asymptotic.
- For cusp anomalous dimension, using coupling $g^2 \equiv \frac{N_c g_{YM}^2}{16 \pi^2} = \frac{\lambda}{16 \pi^2}$, the radius is $\frac{1}{16}$.
  
  Beisert, Eden, Staudacher (BES), 0610251

- Ratio of successive loop orders $\frac{\Gamma_{cusp}^{(L)}}{\Gamma_{cusp}^{(L-1)}} \rightarrow -16$

- See same radius of convergence in high-loop-order behavior of amplitudes and form factors, in suitable kinematic regions.
N=4 SYM particle content

Brink, Schwarz, Scherk; Gliozzi, Scherk, Olive (1977)

massless spin 1 gluon
4 massless spin 1/2 gluinos
6 massless spin 0 scalars

\[ G = SU(N_c) \]

<table>
<thead>
<tr>
<th>( N )</th>
<th>1</th>
<th>4</th>
<th>6</th>
<th>4</th>
<th>1</th>
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</thead>
<tbody>
<tr>
<td>helicity</td>
<td>-1</td>
<td>-( \frac{1}{2} )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
</tr>
</tbody>
</table>

SUSY
\( Q_a, a=1,2,3,4 \)
shifts helicity
by 1/2

all in adjoint representation of \( G \)
QCD vs. N=4 SYM

- QCD has gluons and quarks in fundamental rep. of SU($N_c$)
- Replace quarks with 4 copies of fermions in adjoint rep. (gluinos) and add 6 real adjoint scalars
- Feynman vertices:
QCD vs. N=4 SYM at tree level

At tree-level essentially identical

Consider a tree amplitude for \( n \) gluons. Fermions and scalars cannot appear because they are produced in pairs.

Hence the amplitude is the same in QCD and N=4 SYM. The QCD tree amplitude “secretly” obeys all identities of N=4 supersymmetry:

\[
\begin{align*}
\text{QCD} & = 0 \\
\text{N=4 SYM} & = \frac{1}{\langle ij \rangle^4}
\end{align*}
\]

No longer true at quantum (loop) level
N=4 SYM very special

- At one loop, cancellation of loop momenta in numerator $\rightarrow$ only scalar box integrals
  Bern, LD, Dunbar, Kosower, hep-ph/9403226

- Weight 2 functions – dilogs. E.g., $gg \rightarrow Hg$ @ 1 loop $\supset$

\[
\begin{align*}
H \quad &g_1
\end{align*}
\]

\[
\begin{align*}
g_3 \quad &g_2
\end{align*}
\]

\[
\text{\Large \hspace{1cm}}
\]

\[
\text{\Large \hspace{1cm}}
\]

\[
\begin{align*}
= \text{Li}_2 \left( 1 - \frac{s_{123}}{s_{12}} \right) + \text{Li}_2 \left( 1 - \frac{s_{123}}{s_{23}} \right) + \frac{1}{2} \ln^2 \left( \frac{s_{12}}{s_{23}} \right) + \ldots
\end{align*}
\]

- QCD results also contain single log’s and rational parts from (tensor) triangle + bubble integrals

\[
\begin{align*}
\text{\Large \hspace{1cm}}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{\epsilon} - \ln(s_{123})
\end{align*}
\]
Higher loops

• Much evidence that N=4 SYM amplitudes have “uniform weight (transcendentality)” \(2L\) at loop order \(L\)

• Weight \(\sim\) number of integrations, e.g.

\[
\ln(s) = \int_1^s \frac{dt}{s} = \int_1^s d\ln t
\]

\[
\text{Li}_2(x) = -\int_0^x \frac{dt}{t} \ln(1-t) = \int_0^x d\ln t \cdot [-\ln(1-t)]
\]

\[
\text{Li}_n(x) = \int_0^x \frac{dt}{t} \text{Li}_{n-1}(t)
\]
AdS/CFT

Conformal field theory (like N=4 SYM) is dual to a theory of gravity in anti-de Sitter space (like strings in $AdS_5 \times S^5$)

A weak-strong duality

$SO(4,2)$ isometry of 5 dimensional space-time

$\leftrightarrow$ 4d conformal symmetry

Maldacena (1997)
T-duality symmetry of string theory

- Exchanges string world-sheet variables $\sigma, \tau$
- $X^\mu(\tau, \sigma) = x^\mu + k^\mu \tau + \text{oscillators}$
  $\Rightarrow X^\mu(\tau, \sigma) = x^\mu + k^\mu \sigma + \text{oscillators}$
- Strong coupling limit of planar N=4 SYM is semi-classical limit of string theory: world-sheet stretches tight around minimal area surface in AdS.
- Boundary determined by momenta of external states: light-like polygon with null edges = momenta $k^\mu$

Alday, Maldacena, 0705.0303
Amplitudes = Wilson loops

- Polygon vertices $x_i$ are not positions but dual momenta, $x_i - x_{i+1} = k_i$
- Transform like positions under dual conformal symmetry

Duality verified to hold at weak coupling too

weak-weak duality, holds order-by-order

Alday, Maldacena, 0705.0303
Drummond, Korchemsky, Sokatchev, 0707.0243
Brandhuber, Heslop, Travaglini, 0707.1153
Drummond, Henn, Korchemsky, Sokatchev, 0709.2368, 0712.1223, 0803.1466;
Bern, LD, Kosower, Roiban, Spradlin, Vergu, Volovich, 0803.1465
Dual conformal invariance

- Wilson $n$-gon invariant under inversion:

$$x_{ij}^2 = (k_i + k_{i+1} + \cdots + k_{j-1})^2 \equiv s_{i,i+1,\ldots,j-1}$$

- Fixed, up to functions of invariant cross ratios:

$$u_{ijkl} \equiv \frac{x_{ij}^2 x_{kl}^2}{x_{ik}^2 x_{jl}^2}$$

- $x_{i,i+1}^2 = k_i^2 = 0 \rightarrow$ no such variables for $n = 4,5$

\[ n = 6 \rightarrow \text{precisely 3 ratios:} \]

\[ n = 7 \rightarrow 6 \text{ ratios.} \]

In general, $3n-15$ ratios.
6-gluons: already rich kinematical playground

Multi-particle factorization $u, w \to \infty$

(near) collinear
$v = 0, u + w = 1$

multi-Regge
$(1,0,0)$

spurious pole $u = 1$
Hexagon function bootstrap

Loops

3
4,5
6,7

LD, Drummond, Henn, 1108.4461, 1111.1704;
Caron-Huot, LD, Drummond, Duhr, von Hippel, McLeod, Pennington,
1308.2276, 1402.3300, 1408.1505, 1509.08127; 1609.00669;
Caron-Huot, LD, Dulat, von Hippel, McLeod, Papathanasiou,
1903.10890, 1906.07116; LD, Dulat, 22mm.nnnnn (NMHV 7 loop)

• Use analytical properties of perturbative (six point) amplitudes in planar N=4 SYM to determine them directly, without ever peeking inside the loops
• Step toward doing this nonperturbatively (no loops to peek inside) for general kinematics

\[ \text{Loops} = \text{Hexagon} + \sim 10^9 \text{ more} \]
Rich theoretical “data” mine

- Rare to have perturbative results to 7 loops
- Usually high loop order → single numbers such as $\beta$ functions or anomalous dimensions
- Here we have analytic functions of 3 variables
- Many limits to study (and exploit)
Example: MHV finite remainder $R_6^{(L)}$ on $(u,u,u)$

- Amazing proportionality of each perturbative coefficient at small $u$, and also with the strong coupling result

Alday, Gaiotto, Maldacena, 0911.4708

### Diagram

**Graph**: $R_6^{(L)}(u,u,u)/R_6^{(L)}(1,1,1)$

- **Legend**:
  - AGM
  - L=7
  - L=6
  - L=5
  - L=4
  - L=3
  - L=2
Origin at weak coupling

- Remarkably, MHV remainder $R_6$ and closely-related quantity $\ln \mathcal{E}$ are quadratic in logarithms through 7 loops. CDDvHMP, 1903.10890
- Previously observed through 2 loops, and at strong coupling, on the diagonal $(u,u,u)$. AGM, 0911.4708

\[ \ln \mathcal{E}(u_i) \approx -\frac{\Gamma_{\text{oct}}}{24} \ln^2(u_1u_2u_3) - \frac{\Gamma_{\text{hex}}}{24} \sum_{i=1}^{3} \ln^2 \frac{u_i}{u_{i+1}} + C_0 \]

<table>
<thead>
<tr>
<th></th>
<th>$L = 1$</th>
<th>$L = 2$</th>
<th>$L = 3$</th>
<th>$L = 4$</th>
<th>$L = 5$</th>
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</thead>
<tbody>
<tr>
<td>$\Gamma_{\text{oct}}$</td>
<td>4</td>
<td>-16$\zeta_2$</td>
<td>256$\zeta_4$</td>
<td>-3264$\zeta_6$</td>
<td>$\frac{126976}{3}\zeta_8$</td>
</tr>
<tr>
<td>$\Gamma_{\text{cusp}}$</td>
<td>4</td>
<td>-8$\zeta_2$</td>
<td>88$\zeta_4$</td>
<td>-876$\zeta_6 - 32\zeta_3^2$</td>
<td>$\frac{28384}{3}\zeta_8 + 128\zeta_2\zeta_3^2 + 640\zeta_3\zeta_5$</td>
</tr>
<tr>
<td>$\Gamma_{\text{hex}}$</td>
<td>4</td>
<td>-4$\zeta_2$</td>
<td>34$\zeta_4$</td>
<td>$-\frac{603}{2}\zeta_6 - 24\zeta_3^2$</td>
<td>$\frac{18287}{6}\zeta_8 + 48\zeta_2\zeta_3^2 + 480\zeta_3\zeta_5$</td>
</tr>
<tr>
<td>$C_0$</td>
<td>-3$\zeta_2$</td>
<td>$\frac{77}{4}\zeta_4$</td>
<td>$-\frac{4463}{24}\zeta_6 + 2\zeta_3^2$</td>
<td>$\frac{67645}{32}\zeta_8 + 6\zeta_2\zeta_3^2 - 40\zeta_3\zeta_5$</td>
<td>$-\frac{1484281}{160}\zeta_{10} - 65\zeta_4\zeta_3^2 - 120\zeta_2\zeta_3\zeta_5 + 228\zeta_5^2 + 420\zeta_3\zeta_7$</td>
</tr>
</tbody>
</table>

- Coefficients involve same BES kernel as for cusp, but “tilted” by angle $\alpha$,
  \[ \Gamma_{\text{cusp}} = \Gamma_{\alpha=\pi/4}, \quad \Gamma_{\text{oct}} = \Gamma_{\alpha=0}, \quad \Gamma_{\text{hex}} = \Gamma_{\alpha=\pi/3} \]


L. Dixon    N=4 Amps    GGI Tea Break - 2022/02/09
Solving for Planar N=4 SYM Amplitudes

't Hooft coupling $\lambda$

minimal surface

flux tube

perturbative gluons

collinear limit

Kinematical variables

Images: A. Sever, N. Arkani-Hamed
“Higgs” amplitudes and N=4 SYM form factors

LD, A. McLeod, M. Wilhelm, 2012.12286 3,4,5 loops
+ Ö. Gürdoğan, to appear 6,7,8 loops

- At leading order in $1/m_{top}$, Higgs boson couples to gluons via the operator $H G_{\mu\nu}^a G^{\mu\nu} a$
Form factors (cont.)

- Higgs is a scalar, color singlet. In QCD its amplitudes with gluons are matrix elements of $G^a_{\mu\nu} G^{\mu\nu} G^a$ with on-shell gluons: "form factors"
- In N=4, this operator is part of the (BPS-protected) stress tensor supermultiplet, which also includes for example $\phi_1^+ \phi_1 - \phi_2^+ \phi_2 (\in 20$ of $SU(4)_R$)
- $H_{gg}$ "Sudakov" form factor is "too simple"; it has no kinematic dependence beyond overall $(-s_{12})^{-L\epsilon}$
- $H_{ggg}$ is "just right", depends on 2 dimensionless ratios
Hggg kinematics is two-dimensional

\[ k_1 + k_2 + k_3 = -k_H \]
\[ s_{123} = s_{12} + s_{23} + s_{31} = m_H^2 \]

\[ s_{ij} = (k_i + k_j)^2 \quad k_i^2 = 0 \]

\[ u = \frac{s_{12}}{s_{123}} \quad v = \frac{s_{23}}{s_{123}} \quad w = \frac{s_{31}}{s_{123}} \]

\[ u + v + w = 1 \]

I = decay / Euclidean
IIa,b,c = scattering / spacelike operator
IIIa,b,c = scattering / timelike operator

\[ D_3 \equiv S_3 \text{ dihedral symmetry generated by:} \]

a. cycle: \( i \to i + 1 \) (mod 3), or \( u \to v \to w \to u \)
b. flip: \( u \leftrightarrow v \)

\[ N=4 \text{ amplitude is } S_3 \text{ invariant} \]
One loop integrals/amplitudes

\[ H \]

\[ g_1 \]

\[ g_2 \]

\[ g_3 \]

\[ = \text{Li}_2 \left( 1 - \frac{s_{123}}{s_{12}} \right) + \text{Li}_2 \left( 1 - \frac{s_{123}}{s_{23}} \right) + \frac{1}{2} \ln^2 \left( \frac{s_{12}}{s_{23}} \right) + \cdots \]

\[ = \text{Li}_2 \left( 1 - \frac{1}{u} \right) + \text{Li}_2 \left( 1 - \frac{1}{v} \right) + \frac{1}{2} \ln^2 \left( \frac{u}{v} \right) + \cdots \]
A two-loop story

- Gehrmann et al. computed $H_{ggg}$ in QCD at 2 loops
  Gehrmann, Jaquier, Glover, Koukoutsakis, 1112.3554
- Soon after, Brandhuber et al. computed stress tensor
  3-point form factor $F_3$ in N=4 SYM,
  Brandhuber, Travaglini, Yang, 1201.4170
  saw that “maximally transcendental part” of QCD
  result (both (+++) and (-++)) was same as N=4 result!!
- This “principle of maximal transcendentality”
  was known to work for DGLAP and BFKL anomalous
  dimensions, but not for generic scattering amplitudes, so
  this one is very special
2d HPLs

Space graded by weight $n$. Every function $F$ obeys:

\[
\frac{\partial F(u, v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w}
\]

\[
\frac{\partial F(u, v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{w} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{1-w}
\]

where $F^u, F^v, F^w, F^{1-u}, F^{1-v}, F^{1-w}$ are weight $n-1$ 2d HPLs.

To bootstrap $Hggg$ amplitude beyond 2 loops, find as small a subspace of 2d HPLs as possible, construct it to high weight.
Generalized polylogarithms

Chen, Goncharov, Brown,…

- Can be defined as iterated integrals, e.g.

\[ G(a_1, a_2, \ldots, a_n, x) = \int_0^x \frac{dt}{t - a_1} G(a_2, \ldots, a_n, t) \]

- Or define differentially:

\[ dF = \sum_{s_k \in S} F^{s_k} \ d \ln s_k \]

- There is a Hopf algebra that “co-acts” on the space of polylogarithms, \( \Delta: F \to F \otimes F \)
- The derivative \( dF \) is one piece of \( \Delta: \Delta_{n-1,1}F = \sum_{s_k \in S} F^{s_k} \otimes \ln s_k \)
- so we refer to \( F^{s_k} \) as a \( \{n-1,1\} \) coproduct of \( F \)
- \( s_k \) are letters in the symbol alphabet \( S \)
Generalized polylogarithms (cont.)

• The \( \{n-1,1\} \) coaction can be applied iteratively.

• Define the \( \{n-2,1,1\} \) double coproducts, \( F^{s_k,s_j} \), via the derivatives of the \( \{n-1,1\} \) single coproducts \( F^{s_j} \):

\[
d F^{s_j} \equiv \sum_{s_k \in S} F^{s_k,s_j} \ d \ln s_k
\]

• And so on for the \( \{n-m,1,\ldots,1\} \) \( m \)th coproducts of \( F \).

• The maximal iteration, \( n \) times for a weight \( n \) function, is the symbol,

\[
S[F] = \sum_{s_{i_1},\ldots,s_{i_n} \in S} F^{s_{i_1},\ldots,s_{i_n}} \ d \ln s_{i_1} \ldots d \ln s_{i_n} \equiv \sum_{s_{i_1},\ldots,s_{i_n} \in S} F^{s_{i_1},\ldots,s_{i_n}} s_{i_1} \otimes \ldots \otimes s_{i_n}
\]

where now \( F^{s_{i_1},\ldots,s_{i_n}} \) are just rational numbers.

Goncharov, Spradlin, Vergu, Volovich, 1006.5703
Example: The classical polylogarithms

\[ \text{Li}_1(x) = -\ln(1 - x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \]

\[ \text{Li}_n(x) = \int_0^x \frac{dt}{t} \text{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{x^k}{k^n} \]

- Regular at \( x = 0 \), branch cut starts at \( x = 1 \).
- Iterated differentiation gives the symbol:
  \[ S[\text{Li}_n(x)] = S[\text{Li}_{n-1}(x)] \otimes x \]
  \[ = \ldots = -(1 - x) \otimes x \otimes \ldots \otimes x \]
- Branch cut discontinuities displayed in first entry of symbol, e.g. clip off leading \( (1 - x) \) to compute discontinuity at \( x = 1 \).
- Derivatives computed from symbol by clipping last entry, multiplying by that \( d \ln(...) \).
**Example:** Harmonic Polylogarithms in one variable (HPL\{0,1\})

Remiddi, Vermaseren, hep-ph/9905237

- Generalize the classical polylogs
- Define HPLs by iterated integration:
  \[
  H_{0,\overrightarrow{w}}(x) = \int_0^x \frac{dt}{t} H_{\overrightarrow{w}}(t), \quad H_{1,\overrightarrow{w}}(x) = \int_0^x \frac{dt}{1-t} H_{\overrightarrow{w}}(t)
  \]
- Or by derivatives:
  \[
  dH_{0,\overrightarrow{w}}(x) = H_{\overrightarrow{w}}(x) d\ln x, \quad dH_{1,\overrightarrow{w}}(x) = -H_{\overrightarrow{w}}(x) d\ln(1-x)
  \]
- Symbol alphabet: \( \mathcal{S} = \{x, 1-x\} \)
- Weight \( n = \) length of binary string \( \overrightarrow{w} \)
- Number of functions at weight \( n = 2L \) is number of binary strings: \( 2^{2L} \)
- Branch cuts dictated by first integration/entry in symbol
- Derivatives dictated by last integration/entry in symbol
Symbol alphabet $\mathcal{S}$ for $Hggg$

Comparing

$$\frac{\partial F(u, v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w}$$

$$\frac{\partial F(u, v)}{\partial v} = \frac{F^v}{v} - \frac{F^w}{w} - \frac{F^{1-v}}{1-v} + \frac{F^{1-w}}{1-w}$$

with

$$dF = \sum_{S_k \in \mathcal{S}} F^{S_k} d \ln s_k$$

alphabet is $\mathcal{S} = \{u, v, w, 1-u, 1-v, 1-w\}$
Heuristic view of function space

weight

... 

4

3

Li₃(1-1/uᵢ), true 2D HPLs, ...

2

Li₂(1-1/uᵢ) ln²uᵢ lnuᵢ ln(uᵢ₊₁) - ζ₂

1

lnu lnv lnw

1

0

\[ \frac{\partial F(u, v)}{\partial u} = \frac{F^u}{u} - \frac{F^w}{w} - \frac{F^{1-u}}{1-u} + \frac{F^{1-w}}{1-w} \]
Symbol alphabets for $n$-gluon amplitudes

$n = 6$ has 9 letters: $\mathcal{S}_6 = \{\hat{u}, \hat{v}, \hat{w}, 1 - \hat{u}, 1 - \hat{v}, 1 - \hat{w}, \hat{y}_u, \hat{y}_v, \hat{y}_w\}$

Goncharov, Spradlin, Vergu, Volovich, 1006.5703;
LD, Drummond, Henn, 1108.4461; Caron-Huot,
LD, von Hippel, McLeod, 1609.00669

$n = 7$ has 42 letters

Golden, Goncharov, Paulos, Spradlin, Volovich, Vergu, 1305.1617, 1401.6446, 1411.3289; Drummond, Papathanasiou, Spradlin 1412.3763

$n = 8$ has at least 222 letters, could even be infinite as $L \to \infty$

Arkani-Hamed, Lam, Spradlin, 1912.08222;
Drummond, Foster, Kalousios, 1912.08217, 2002.04624;
Henke, Papathanasiou 1912.08254, 2106.01392;
Z. Li, C. Zhang, 2110.00350
Beyond $n = 8$

sum all planar Feynman graphs with $L$ loops and $n$ external lines.
3-gluon form factor: better alphabet

- Motivated by 6 gluon case, switch to equivalent alphabet

\[ S' = \{ a = \frac{u}{vw}, b = \frac{v}{wu}, c = \frac{w}{uv}, d = \frac{1-u}{u}, e = \frac{1-v}{v}, f = \frac{1-w}{w} \} \]

- Symbols of (suitably normalized) form factor \( F_3^{(L)} \) simplify remarkably at 1 and 2 loops, just 1 and 2 terms, plus \( D_3 \) dihedral images(!!!):

\[ S \left[ F_3^{(1)} \right] = (-1) \ b \otimes d + \text{dihedral} \]
\[ S \left[ F_3^{(2)} \right] = 4 \ b \otimes d \otimes d \otimes d + 2 \ b \otimes b \otimes b \otimes d + \text{dihedral} \]

dihedral cycle: \( a \rightarrow b \rightarrow c \rightarrow a, \quad d \rightarrow e \rightarrow f \rightarrow d \)
dihedral flip: \( a \leftrightarrow b, \quad d \leftrightarrow e \)
Simplest analytic form is for \( \nu \to \infty \)

\[ \Rightarrow \text{Harmonic polylogarithms } H_{\nu} \equiv H_{\nu}(1 - \frac{1}{u}) \]

\[
F_3^{(1)}(\nu \to \infty) = 2H_{0,1} + 6\zeta_2 \\
F_3^{(2)}(\nu \to \infty) = -8H_{0,0,0,1} - 4H_{0,1,1,1} + 12\zeta_2 H_{0,1} + 13\zeta_4 \\
F_3^{(3)}(\nu \to \infty) = 96H_{0,0,0,0,0,1} + 16H_{0,0,0,1,0,1} + 16H_{0,0,0,1,1,1} + 16H_{0,0,1,0,0,1} + 8H_{0,0,1,0,1,1} + 8H_{0,0,1,1,0,1} + 16H_{0,1,0,0,0,1} + 8H_{0,1,0,0,1,1} + 12H_{0,1,0,1,0,1} + 4H_{0,1,0,1,1,1} + 8H_{0,1,1,0,0,1} + 4H_{0,1,1,0,1,1} + 4H_{0,1,1,1,0,1} + 24H_{0,1,1,1,1,1} - \zeta_2(32H_{0,0,0,1} + 8H_{0,0,1,1} + 4H_{0,1,0,1} + 52H_{0,1,1,1}) - \zeta_3(8H_{0,0,1} - 4H_{0,1,1}) - 53\zeta_4 H_{0,1} - \frac{167}{4}\zeta_6 + 2(\zeta_3)^2 \]

8 loop result has \( \sim 2^{2\times8-2} = 16,384 \) terms
6-gluon amplitude is simplest for \((\hat{u}, \hat{v}, \hat{w}) = (1, \hat{v}, \hat{v})\)

- Let \(H_{\hat{w}} \equiv H_{\hat{w}}(1 - \frac{1}{\hat{v}})\)

\[
\begin{align*}
A_6^{(1)}(1, \hat{v}, \hat{v}) &= 2H_{0,1} \\
A_6^{(2)}(1, \hat{v}, \hat{v}) &= -8H_{0,1,1,1,1} - 4H_{0,0,0,1} - 4\zeta_2H_{0,1} - 9\zeta_4 \\
A_6^{(3)}(1, \hat{v}, \hat{v}) &= 96H_{0,1,1,1,1,1} + 16H_{0,1,0,1,1,1} + 16H_{0,0,0,1,1,1} + 16H_{0,1,1,0,1,1} + 8H_{0,0,1,0,1,1} \\
&\quad + 8H_{0,1,0,0,1,1} + 16H_{0,1,1,1,0,1} + 8H_{0,0,1,1,0,1} + 12H_{0,1,0,1,0,1} + 4H_{0,0,0,1,0,1} \\
&\quad + 8H_{0,1,1,0,0,1} + 4H_{0,0,1,0,0,1} + 4H_{0,1,0,0,0,1} + 24H_{0,0,0,0,0,1} \\
&\quad + \zeta_2(8H_{0,0,0,1} + 8H_{0,1,0,1} + 48H_{0,1,1,1}) \\
&\quad + 42\zeta_4H_{0,1} + 121\zeta_6
\end{align*}
\]

There’s an exact map at symbol level, with \(\frac{1}{\hat{v}} = 1 - \frac{1}{u}\), \(0 \leftrightarrow 1\), if you also reverse the order of the symbol entries!!!

Works to 7 loops, where \(\sim 2^{2 \times 7^2 - 2} = 4,096\) terms agree
Antipodal duality in 2d

Strong-weak duality

\[ F_3^{(L)}(u, v, w) = S \left( A_6^{(L)}(\hat{u}, \hat{v}, \hat{w}) \right) \]

Antipode map \( S \), at symbol level, reverses order of all letters:

\[ S(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = (-1)^m x_m \otimes \cdots \otimes x_2 \otimes x_1 \]

Kinematic map is

\[ \hat{u} = \frac{vw}{(1 - v)(1 - w)}, \quad \hat{v} = \frac{wu}{(1 - w)(1 - u)}, \quad \hat{w} = \frac{uv}{(1 - u)(1 - v)} \]

Maps \( u + v + w = 1 \) to parity-preserving surface

\[ \Delta \equiv (1 - \hat{u} - \hat{v} - \hat{w})^2 - 4\hat{u}\hat{v}\hat{w} = 0 \]

also corresponds to "twisted forward scattering":

\[ \hat{k}_i^\mu = -\hat{k}_i^\mu, \quad i = 1, 2, \ldots, n \quad (n = 3 \text{ here}) \]
6-gluon alphabet and symbol map

- $S_6 = \{ \hat{u}, \hat{v}, \hat{w}, 1 - \hat{u}, 1 - \hat{v}, 1 - \hat{w}, \hat{a}, \hat{b}, \hat{c} \}$

$\rightarrow S'_6 = \{ \hat{a} = \frac{\hat{u}}{\hat{v} \hat{w}}, \hat{b} = \frac{\hat{v}}{\hat{w} u}, \hat{c} = \frac{\hat{w}}{\hat{u} v}, \hat{d} = \frac{1-\hat{u}}{\hat{u}}, \hat{e} = \frac{1-\hat{v}}{\hat{v}}, \hat{f} = \frac{1-\hat{w}}{\hat{w}} \}$

- Kinematic map on letters:

$\sqrt{\hat{a}} = d, \quad \hat{d} = a, \quad plus \ cyclic \ relations$

$S\left[A_6^{(1)}\right] = (-\frac{1}{2})\hat{b} \otimes \hat{d} + \text{dihedral}$

$S\left[A_6^{(2)}\right] = \hat{b} \otimes \hat{d} \otimes \hat{d} \otimes \hat{d} + \frac{1}{2} \hat{b} \otimes \hat{b} \otimes \hat{b} \otimes \hat{d} + \text{dihedral}$

- Works through 7 loops!

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<th>number of terms</th>
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</thead>
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<tr>
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<td>4,916,466</td>
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<td>7</td>
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<tr>
<td>8</td>
<td>1,671,656,292</td>
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</table>
Map covers entire phase space for 3-gluon form factor

- Soft is dual to collinear; collinear is dual to soft
- White regions in \((u, \nu)\) map to some of \(\hat{u}, \hat{\nu}, \hat{w} > 1\)
Many special dual points

There is an “f” alphabet at all of these points, which is a way of writing multiple zeta values (MZV’s) so that the coaction is manifest.

F. Brown, 1102.1310; O. Schnetz, HyperlogProcedures
The simplest point

• \((\hat{u}, \hat{v}, \hat{w}) = (1,1,1) \iff u, v \to \infty\)

• At this point,

\[
\begin{align*}
A_6^{(1)}(\cdot) &= 0 & F_3^{(1)}(\cdot) &= 8\zeta_2 \\
A_6^{(2)}(\cdot) &= -9\zeta_4 & F_3^{(2)}(\cdot) &= 31\zeta_4 \\
A_6^{(3)}(\cdot) &= 121\zeta_6 & F_3^{(3)}(\cdot) &= -145\zeta_6 \\
A_6^{(4)}(\cdot) &= 120f_{3,5} - 48\zeta_2f_{3,3} - \frac{6381}{4}\zeta_8 & F_3^{(4)}(\cdot) &= 120f_{5,3} + \frac{11363}{4}\zeta_8 \\
A_6^{(5)}(\cdot) &= -2688f_{3,7} - 1560f_{5,5} + \mathcal{O}(\pi^2) & F_3^{(5)}(\cdot) &= -2688f_{7,3} - 1560f_{5,5} + \mathcal{O}(\pi^2) \\
A_6^{(6)}(\cdot) &= 48528f_{3,9} + 37296f_{5,7} + 21120f_{7,5} + \mathcal{O}(\pi^2) & F_3^{(6)}(\cdot) &= 48528f_{9,3} + 37296f_{7,5} + 21120f_{5,7} + \mathcal{O}(\pi^2)
\end{align*}
\]

• Reversing ordering of words in \(f\)-alphabet, the blue values show that antipodal duality holds at these points beyond symbol level, modulo \(i\pi\)

• modulo \(i\pi\) seems to be the best we can get from the antipode
Euclidean Region numerics

For $L > 3$, ratio at $u = \frac{1}{3}$ is within 3% of cusp anomalous dimension ratio, $\frac{\Gamma^{(L)}_{\text{cusp}}}{\Gamma^{(L-1)}_{\text{cusp}}}$

→ same finite radius of convergence…
Bootstrap boundary data: 
Flux tubes at finite coupling

Alday, Gaiotto, Maldacena, Sever, Vieira, 1006.2788; 
Basso, Sever, Vieira, 1303.1396, 1306.2058, 1402.3307, 1407.1736, 1508.03045 
BSV+Caetano+Cordova, 1412.1132, 1508.02987

- Tile \( n \)-gon with pentagon transitions.
- Quantum integrability \( \rightarrow \) compute pentagons exactly in \( 't \) Hooft coupling
- 4d S-matrix as expansion (OPE) in number of flux-tube excitations = expansion around near collinear limit
The new FFOPE

- Form factors are Wilson loops in a periodic space, due to injection of operator momentum
  Alday, Maldacena, 0710.1060; Maldacena, Zhiboedov, 1009.1139; Brandhuber, Spence, Travaglini, Yang, 1011.1899

- Besides pentagon transitions $\mathcal{P}$, this program needs an additional ingredient, the form factor transition $\mathcal{F}$
  Sever, Tumanov, Wilhelm, 2009.11297, 2105.13367, 2112.10569
OPE representation

• 6-gluon amplitude:

\[ \mathcal{W}_{\text{hex}} = \sum_a \int du \, P_a(0|u)P_a(\bar{u}|0) \, e^{-E(u)\tau + ip(u)\sigma + im\phi} \]

\[ T = e^{-\tau}, \ S = e^{-\sigma}, \ F = e^{i\phi}. \quad \nu = \frac{T^2}{1+T^2} \rightarrow 0, \]

weak-coupling, \( E = k + \mathcal{O}(g^2) \rightarrow \) expansion in \( T^k \)

• 3-gluon form factor:

\[ \mathcal{W}_3 = \sum_{\psi} \int e^{-E\psi\tau + ip\psi\sigma} \mathcal{P}(0|\psi)\mathcal{F}(\psi) \]

weak-coupling \( \rightarrow \) expansion in \( T^{2k} \)  \( \) (no azimuthal angle \( \phi \))

\[ \psi = \text{helicity 0 pairs of states} \]
OPE parametrizations

- Amplitude:
  \[ \hat{u} = \frac{1}{1 + (\hat{T} + \hat{S}\hat{F})(\hat{T} + \hat{S}/\hat{F})}, \]
  \[ \hat{v} = \hat{u}\hat{w}\hat{S}^2/\hat{T}^2, \quad \hat{w} = \frac{\hat{T}^2}{1 + \hat{T}^2} \]

- Form factor:
  \[ u = \frac{1}{1 + S^2 + T^2}, \quad v = \frac{T^2}{1 + T^2}, \]
  \[ w = \frac{1}{(1 + T^2)(1 + S^{-2}(1 + T^2))} \]

- Apply the kinematic map \( \hat{T} = \frac{T}{S}, \quad \hat{S} = \frac{1}{TS} \)

- There is apparently a correspondence between single flux tube excitations for the amplitude \( T^1 \) and double (or bound state) excitations for the form factor \( T^2 \)

\( \hat{F} = 1 \) for \( \Delta = 0 \)
8-gluon Amp $\leftrightarrow$ 4-gluon FF

LD, Ö. Gürdoğan, Y.-T. Liu A. McLeod, M. Wilhelm, in progress

- We have a candidate kinematic map for a 4-dimensional surface (4-gluon FF is 5d).
- $S[R_8^{(2)}]$ is known S. Caron-Huot, 1105.5606
- The kinematic+antipodal maps take it to a symbol with 40 letters, the first 8 of which are “right”: $u_i = \frac{s_{i,i+1}}{s_{1234}}$, $v_i = \frac{s_{i,i+1,i+2}}{s_{1234}}$
- But we still have to run more checks on this candidate 2-loop 4-gluon form factor
8-4 Kinematic Map in OPE Parametrization

- 8-point amplitude has $D_8$ dihedral symmetry; change it to $D_4$ of the form factor by requiring
  \[ \hat{T}_3 = \hat{T}_1, \quad \hat{S}_3 = \hat{S}_1, \quad \hat{F}_3 = \hat{F}_1 \]

- To get $S[R_8^{(2)}]$ to have only 8 final entries, we also fix $\hat{F}_1 = \hat{F}_2 = 1$.
- The kinematic map becomes
  \[ \hat{T}_1 = \frac{T}{S}, \quad \hat{S}_1 = \frac{1}{TS}, \]
  \[ \hat{T}_2 = \frac{T_2}{S_2}, \quad \hat{S}_2 = \frac{1}{T_2S_2} \]
  and requires $F_2 = i$
Summary & Outlook

- Form factors as well as scattering amplitudes in planar N=4 SYM can now be **bootstrapped** to high loop order
- Remarkably simple behavior at “origin”
- Comparing the 3-gluon form factor to the 6-gluon amplitude, a **strange new antipodal duality** swaps the role of **branch cuts** and **derivatives**, and may map single flux-tube excitations (amplitude) to doubles (form factor).
- What is the underlying **physical reason** for this duality?
- (How) does it hold at strong coupling?
- (How much) can we verify of it at the 8-4 level, and beyond?
- How much can we exploit it to learn more about both amplitudes and form factors?
Extra Slides
Removing Amplitude (or Form Factor) Infrared Divergences

- On-shell amplitudes IR divergent due to long-range gluons

- Polygonal Wilson loops UV divergent at cusps, anomalous dimension $\Gamma_{\text{cusp}}$
  – known to all orders in planar N=4 SYM: Beisert, Eden, Staudacher, hep-th/0610251

- Both removed by dividing by BDS-like ansatz
  Bern, LD, Smirnov, hep-th/0505205, Alday, Gaiotto, Maldacena, 0911.4708

- Normalized [MHV] amplitude is finite, dual conformal invariant, also uniquely (up to constant) maintains important symbol adjacency relations due to causality (Steinmann relations for 3-particle invariants):

$$\mathcal{E}(u_i) = \lim_{\epsilon \to 0} \frac{\mathcal{A}_6(s_{i,i+1}, \epsilon)}{\mathcal{A}_6^{\text{BDS-like}}(s_{i,i+1}, \epsilon)} = \exp\left[\frac{\Gamma_{\text{cusp}}}{4} \mathcal{E}^{(1)} + R_6\right]$$

remainder function
BDS & BDS-like normalization for $F_3$

\[
\frac{F_3}{F_{3,\text{MHV, tree}}} = \exp\left\{ \sum_{L=1}^{\infty} g^{2L} \left[ \left( \frac{\Gamma_{\text{cusp}}}{4} + O(\epsilon) \right) M^{1-\text{loop}}(L\epsilon) + C^{(L)} + R^{(L)}(u, v, w) \right] \right\}
\]

BDS ansatz

split 1-loop amplitude judiciously:

\[
\frac{F^{1-\text{loop}}_3}{F_{3,\text{MHV, tree}}} \equiv M^{1-\text{loop}}(\epsilon) = M(\epsilon) + \mathcal{E}^{(1)}(u, v, w)
\]

\[
M(\epsilon) = -\frac{1}{\epsilon^2} \sum_{i=1}^{3} \left( \frac{\mu^2}{-s_{i,i+1}} \right)^\epsilon - \frac{7}{2} \zeta_2 + \sum_{i=1}^{3} \left( 1 - \frac{1}{v} \right) + \text{Li}_2 \left( 1 - \frac{1}{w} \right)
\]

$\mathcal{E}^{(1)}(u, v, w)$ obeys “adjacency constraints”

\[
\mathcal{E}^{(1)}, u + \mathcal{E}^{(1)}, 1-u = 0
\]

Now divide by:

\[
\frac{F_{3,\text{BDS-like}}}{F_{3,\text{MHV, tree}}} = \exp\left\{ \sum_{L=1}^{\infty} g^{2L} \left[ \left( \frac{\Gamma_{\text{cusp}}}{4} + O(\epsilon) \right) M(L\epsilon) + C^{(L)} \right] \right\} \Rightarrow \mathcal{E} = \exp \left[ \frac{\Gamma_{\text{cusp}}}{4} \mathcal{E}^{(1)} + R \right]
\]
Number of (symbol-level) linearly independent \( \{n, 1, \ldots, 1\} \) coproducts (\( 2L - n \) derivatives)

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</table>

- Properly normalized \( L \) loop N=4 form factors \( \mathcal{E}^{(L)} \) belong to a small space \( \mathcal{C} \), dimension saturates on left
- \( \mathcal{E}^{(L)} \) also obeys multiple-final-entry relations, saturation on right
Some numerics

I = decay / Euclidean
IIa,b,c = scattering / spacelike operator
IIIa,b,c = scattering / timelike operator
Real “impact factor” appears in space-like Regge limit, $\nu \to \infty$

Remainder function $R$ is nontrivial function of $u = \frac{s_{12}}{m_H^2}$ as $s_{23} \to \infty$
Numerical implications of antipodal duality?
Values of HPLs \{0,1\} at \( u = 1 \)

- Classical polylogs
  
  \[
  \text{Li}_n(u) = \int_0^u \frac{dt}{t} \text{Li}_{n-1}(t) = \sum_{k=1}^{\infty} \frac{u^k}{k^n}
  \]

  \[
  \text{Li}_n(1) = \sum_{k=1}^{\infty} \frac{1}{k^n} = \zeta(n) \equiv \zeta_n
  \]

- HPL’s evaluate to nested sums called multiple zeta values (MZVs):
  \[
  \zeta_{n_1,n_2,\ldots,n_m} = \sum_{k_1 > k_2 > \ldots > k_m > 0} \frac{1}{k_1^{n_1} k_2^{n_2} \cdots k_m^{n_m}}
  \]

  Weight \( n = n_1 + n_1 + \ldots + n_m \)

- MZV’s obey many identities, e.g. stuffle
  \[
  \zeta_{n_1} \zeta_{n_2} = \zeta_{n_1,n_2} + \zeta_{n_2,n_1} + \zeta_{n_1+n_2}
  \]

- All reducible to Riemann zeta values until weight 8.

Irreducible MZVs: \( \zeta_{5,3}, \zeta_{7,3}, \zeta_{5,3,3}, \zeta_{9,3}, \zeta_{6,4,1,1}, \ldots \)
Many “empirical” adjacency constraints

\[ F^{d,e} = F^{e,d} = F^{e,f} = F^{f,e} = F^{f,d} = F^{d,f} = 0 \]

Hold for 2 loop QCD amplitudes too, planar and nonplanar!

LD, Mcleod, Wilhelm, 2012.12286

\[ F^{a,d} = F^{d,a} = F^{b,e} = F^{e,b} = F^{c,f} = F^{f,c} = 0 \]

Latter are NEW: Hold for planar N=4 SYM to 8 loops!

Mnemonic for dihedral symmetry;
6 dashed lines indicate 12 forbidden pairs.
Empirical multi-final entry relations

1. $\mathcal{E}^a = 0$ (plus dihedral images)

2. $\mathcal{E}^{a,e} = \mathcal{E}^{a,f}$ (plus ...)

3. $\mathcal{E}^{a,b,d} = 0, \quad \mathcal{E}^{a,e,e} = - \mathcal{E}^{a,f,f}, \quad \mathcal{E}^{e,a,f} = \mathcal{E}^{f,a,f} - \mathcal{E}^{a,f,f}$

4. ...
Number of remaining parameters in form-factor ansatz after imposing constraints

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</tr>
<tr>
<td>OPE $T^2 \ln^{L-3} T$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>OPE $T^2 \ln^{L-4} T$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>OPE $T^2 \ln^{L-5} T$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 4: Number of parameters left when bootstrapping the form factor $\mathcal{E}^{(L)}$ at $L$-loop order in the function space $\mathcal{C}$ at symbol level, using all the conditions on the final $(L - 1)$ entries, which can be deduced at $(L - 1)$ loops.
The [Dual] Conformal Group

\[ \text{SO}(4,2) \supset \text{SO}(3,1) \ [\text{rotations}+\text{boosts}] + \text{translations}+\text{dilatations} + \text{special-conformal} \]

\[ 15 = 3 + 3 + 4 + 1 + 4 \]

- The nontrivial generators are special conformal \( K^\mu \)
- Correspond to inversion \( \cdot \) translation \( \cdot \) inversion
- To obtain a [dual] conformally invariant function \( f(x_{ij}^2) \)
  just have to check invariance under inversion,

\[ x_i^\mu \rightarrow x_i^\mu / x_i^2 \]
Different routes to perturbative amplitudes

- Draw all Feynman graphs $G_i$
- Evaluate all Feynman rules: $I_i$
- Perform all loop integrations: $A_i$

\[ A = \sum_i A_i \]

- Evaluate all unitarity cuts $C_\alpha$
- Construct local integrand $I$
- Perform all loop integrations: $A_\alpha$

\[ A = \sum_\alpha A_\alpha \]

Bootstrap: Guess
\[ A = \sum_m c_m F_m \]
$F_m$ known functions
$c_m \in \mathbb{Q}$ unknown constants

- Solve constraints, linear equations for $c_m \rightarrow r_m$
- $A = \sum_m r_m F_m$
Beyond 8-4

• The map \( \hat{T}_1 = \frac{T}{S}, \hat{S}_1 = \frac{1}{TS}, \hat{T}_2 = \frac{T_2}{S_2}, \hat{S}_2 = \frac{1}{T_2S_2} \)

seems likely to generalize to give rise to a \(2(n - 2)\) parameter subspace of the full \(3n - 7\) dimensional \(n\)-point form factor kinematics, presumably from setting \(F_2 = \cdots = F_{n-2} = i\)

• We can conjecture that antipodal duality applies on this subspace

• But there is still a lot to be checked!