Axions as Dark Matter Candidates

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Online Event
Feb 21, 2022 - Mar 04, 2022
Overall Plan
Overall Plan

- Vacuum Structure in Quantum Chromodynamics
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- Vacuum Structure in Quantum Chromodynamics
- Strong CP Puzzle
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• Vacuum Structure in Quantum Chromodynamics
• Strong CP Puzzle
• The Axion
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- The Axion
- Axion Dark Matter
Overall Plan

- Vacuum Structure in Quantum Chromodynamics
- Strong CP Puzzle
- The Axion
- Axion Dark Matter
- Axion Experiments
Today’s Plan

- Vacuum Structure in Quantum Chromodynamics
- Strong CP Puzzle
- The Axion
Vacuum Structure in Quantum Chromodynamics

Action of Quantum Chromodynamics (QCD)
Vacuum Structure in Quantum Chromodynamics

Action of Quantum Chromodynamics (QCD)

- QCD: SU(3) gauge theory with 6 flavours of quarks (u,d,s,c,b,t) in fundamental representation
Vacuum Structure in Quantum Chromodynamics

Action of Quantum Chromodynamics (QCD)

- **QCD:** SU(3) gauge theory with 6 flavours of quarks (u,d,s,c,b,t) in fundamental representation

- Most general gauge- and Lorentz-invariant renormalizable action:

\[
S_{\text{QCD}} = \int d^4 x \left\{ -\frac{1}{4} G^a_{\mu\nu} G^{a,\mu\nu} + \bar{q} \left( i\gamma^\mu D_\mu - M_q \right) q + \theta \frac{g_5^2}{32\pi^2} G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} \right\}
\]

[Belavin et al. ’75; ’t Hooft 76; Callan et al. ’76; Jackiw, Rebbi ’76]
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  [Gross, Wilczek 73; Politzer 73; Fritzsch, Gell-Mann, Leutwyler 73]  [Belavin et al. ’75; ’t Hooft 76; Callan et al. ’76; Jackiw, Rebbi ’76 ]

- Covariant gauge kinetic term, \(-\frac{1}{4} G_{\mu\nu}^a G^{a,\mu\nu}\), build from gluonic field strength tensors, \(G_{\mu\nu}^a \equiv \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu\), in terms of gluonic gauge fields, \(G^a_\mu\).
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Vacuum Structure in Quantum Chromodynamics

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- Covariant quark kinetic term, \(\bar{q} i \gamma_\mu D^\mu q\), involving covariant derivative \(D_\mu = \partial_\mu - ig_s T^a G^a_\mu\)
Vacuum Structure in Quantum Chromodynamics

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- What about last term?
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

\[ S_{\text{QCD}} \supset \theta \frac{g_s^2}{32\pi^2} \int d^4 x \, G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} \]
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

\[ S_{\text{QCD}} \supset \theta \frac{g_s^2}{32\pi^2} \int d^4 x \, G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} \]

- Involving Hodge dual of gluonic field tensor, \( \tilde{G}^{a}_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G^{a,\rho\sigma} \)
Vacuum Structure in Quantum Chromodynamics

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\]

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- Integrand is total derivative:

\[
\theta \frac{g_s^2}{32\pi^2} \int d^4x \, G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} = \theta \frac{g_s^2}{32\pi^2} \int d^4x \, \epsilon^{\mu\alpha\beta\gamma} \partial_\mu \left( G^{a}_\alpha G^{a}_\beta \gamma - \frac{g_s}{3} f^{abc} G^{a}_\alpha G^{b}_\beta G^{c}_\gamma \right)
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Vacuum Structure in Quantum Chromodynamics

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\[ S_{QCD} \supset \theta \frac{g_s^2}{32\pi^2} \int d^4x \, G^a_{\mu\nu} \tilde{G}^{a,\mu\nu} \]

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- Divergence of Chern-Simons current
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

\[ S_{\text{QCD}} \supset \theta \frac{g_s^2}{32\pi^2} \int d^4x \, G_{\mu\nu}^a \tilde{G}^{a,\mu\nu} \]

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  - Divergence of Chern-Simons current
  - Theta-term does not contribute at the classical level and in perturbation theory
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

\[ S_{\text{QCD}} \supset \theta \frac{g_s^2}{32\pi^2} \int d^4 x \ G^a_{\mu \nu} \tilde{G}^{a,\mu \nu} \]

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  - Divergence of Chern-Simons current
- Theta-term does not contribute at the classical level and in perturbation theory
  - Depends only on boundary information (’topological’)
Vacuum Structure in Quantum Chromodynamics

**Theta-term in action of QCD**

\[
S_{\text{QCD}} \supset \theta \frac{g_s^2}{32\pi^2} \int d^4x \, G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a
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- Involving Hodge dual of gluonic field tensor, \( \tilde{G}_{\mu\nu}^a \equiv \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} G_{\rho\sigma}^a \)

- Integrand is total derivative:

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- Divergence of Chern-Simons current

- Theta-term does not contribute at the classical level and in perturbation theory
  - Depends only on boundary information (‘topological’)
  - Does not give rise to new vertices in Feynman rules
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

\[ S_{\text{QCD}} \supset \theta \frac{g_s^2}{32\pi^2} \int d^4 x \ G^{a,\mu\nu}_{\mu\nu} \]

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  - Divergence of Chern-Simons current
- Theta-term does not contribute at the classical level and in perturbation theory
  - Depends only on boundary information (‘topological’)
  - Does not give rise to new vertices in Feynman rules
- But it gives non-perturbative contribution in quantum theory because of the topologically non-trivial vacuum structure of QCD
To see this, we will use the Euclidean path integral representation of the vacuum-to-vacuum amplitude,

$$\langle 0_+ | 0_- \rangle = \int D\mathcal{G} D\overline{q} Dq \ e^{-\int d^4x \left\{ \frac{i}{4} G^a_{\mu\nu} G^a_{\mu\nu} - \overline{q} (i\gamma_\mu D_\mu - M) q \right\} + i\theta \frac{g^2}{32\pi^2} \int d^4x \ G^a_{\mu\nu} \tilde{G}^a_{\mu\nu}$$
Vacuum Structure in Quantum Chromodynamics

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• To see this, we will use the Euclidean path integral representation of the vacuum-to-vacuum amplitude,

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• Obtained from real time path integral representation by Wick rotation, \( x_0^{(M)} \rightarrow -i x_4^{(E)}, G_0^{a(M)} \rightarrow iG_4^{a(E)} \)
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

- To see this, we will use the Euclidean path integral representation of the vacuum-to-vacuum amplitude,

\[
\langle 0_{+} | 0_{-} \rangle = \int \mathcal{D}G \mathcal{D}\bar{q} \mathcal{D}q e^{-\int d^{4}x \left\{ \frac{1}{4} G_{\mu \nu}^{a} G_{\mu \nu}^{a} - \bar{q} (i\gamma_{\mu} D_{\mu} - M) q \right\} + i\theta \frac{g_{s}^{2}}{32\pi^{2}} \int d^{4}x \, G_{\mu \nu}^{a} \tilde{G}_{\mu \nu}^{a}}
\]

- Obtained from real time path integral representation by Wick rotation, \( x_{0}^{(M)} \rightarrow -ix_{4}^{(E)} \), \( G_{0}^{a(M)} \rightarrow iG_{4}^{a(E)} \), such that

\[
i \int d^{4}x \left\{ -\frac{1}{4} G_{\mu \nu}^{a} G_{\mu \nu}^{a,\mu\nu} + \theta \frac{g_{s}^{2}}{32\pi^{2}} G_{\mu \nu}^{a} \tilde{G}_{\mu \nu}^{a,\mu\nu} \right\} \Rightarrow - \int d^{4}x \left\{ \frac{1}{4} G_{\mu \nu}^{a} G_{\mu \nu}^{a} - i \theta \frac{g_{s}^{2}}{32\pi^{2}} G_{\mu \nu}^{a} \tilde{G}_{\mu \nu}^{a,\mu\nu} \right\}
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Finite Euclidean action requires that the classical gauge fields contributing to the path integral should be such that the field strength tensor vanishes at \( \ |x| \rightarrow \infty \).
Vacuum Structure in Quantum Chromodynamics

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\]

- Finite Euclidean action requires that the classical gauge fields contributing to the path integral should be such that the field strength tensor vanishes at \( |x| \to \infty \). This means that the gauge fields should approach a pure gauge form,

\[
G^a_{\mu}(x)T^a \to \frac{i}{g_s} U(x) \partial_{\mu} U^{-1}(x), \text{ for } |x| \to \infty, \text{ where } U(x) \in SU(3)
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Vacuum Structure in Quantum Chromodynamics

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\langle 0_+ | 0_- \rangle = \int D G \, D \bar{q} \, D q \, e^{-\int d^4 x \left\{ \frac{1}{4} G^a_{\mu \nu} G^a_{\mu \nu} - \bar{q} (i \gamma_\mu D_\mu - M) q \right\}} e^{+i \frac{g_s^2}{32\pi^2} \int d^4 x \, G^a_{\mu \nu} \tilde{G}^a_{\mu \nu}}
\]

• Obtained from real time path integral representation by Wick rotation, \( x^{(M)}_0 \rightarrow -i x^{(E)}_4 \), \( G^{a(M)}_0 \rightarrow i G^{a(E)}_4 \), such that

\[
i \int d^4 x \left\{ -\frac{1}{4} G^a_{\mu \nu} G^a_{\mu \nu} + \theta \frac{g_s^2}{32\pi^2} G^a_{\mu \nu} \tilde{G}^a_{\mu \nu} \right\} \Rightarrow - \int d^4 x \left\{ \frac{1}{4} G^a_{\mu \nu} G^a_{\mu \nu} - i \theta \frac{g_s^2}{32\pi^2} G^a_{\mu \nu} \tilde{G}^a_{\mu \nu} \right\}
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\]

• The theta-term can then be evaluated as

\[
\frac{g_s^2}{32\pi^2} \int d^4 x \, G^a_{\mu \nu} \tilde{G}^a_{\mu \nu} = \frac{g_s^2}{32\pi^2} \int d^4 x \, \epsilon_{\mu \alpha \beta \gamma} \partial_\mu \left( G^a_\alpha G^a_\beta \gamma - \frac{g_s}{3} f^{abc} G^a_\gamma G^b_\beta G^c_\gamma \right)
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• Obtained from real time path integral representation by Wick rotation, $x_0^{(M)} \rightarrow -ix_4^{(E)}$, $G_0^a(M) \rightarrow iG_4^a(E)$, such that

$$i \int d^4x \left\{ -\frac{1}{4} G_{\mu\nu}^a G^{a,\mu\nu} + \theta \frac{g_s^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a \right\} \Rightarrow - \int d^4x \left\{ \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a - i \theta \frac{g_s^2}{32\pi^2} G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a \right\}$$

• Finite Euclidean action requires that the classical gauge fields contributing to the path integral should be such that the field strength tensor vanishes at $|x| \rightarrow \infty$. This means that the gauge fields should approach a pure gauge form,

$$G_{\mu}^a(x) T^a \rightarrow \frac{i}{g_s} U(x) \partial_{\mu} U^{-1}(x), \text{ for } |x| \rightarrow \infty, \text{ where } U(x) \in SU(3)$$

• The theta-term can then be evaluated as

$$\frac{g_s^2}{32\pi^2} \int d^4x \ G_{\mu\nu}^a \tilde{G}_{\mu\nu}^a = \frac{g_s^2}{32\pi^2} \int d^4x \ \epsilon_{\mu\alpha\beta\gamma} \partial_{\mu} \left( G_{\alpha}^a G_{\beta}^a - \frac{g_s}{3} f^{abc} G_{\alpha}^b G_{\beta}^c G_{\gamma}^c \right)$$

$$= \frac{1}{24\pi^2} \int_{S^3} d\sigma_{\mu} \ \epsilon_{\mu\alpha\beta\gamma} \text{tr} \left( [U \partial_{\alpha} U^{-1}] [U \partial_{\beta} U^{-1}] [U \partial_{\gamma} U^{-1}] \right)$$
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

- To see this, we will use the Euclidean path integral representation of the vacuum-to-vacuum amplitude,

\[
\langle 0_+ | 0_- \rangle = \int D G D \bar{q} D q \ e^{- \int d^4 x \left\{ \frac{1}{4} G_{\mu \nu}^a G_{\mu \nu}^a - \bar{q}(i \gamma_\mu D_\mu - M)q \right\}} e^{+ i \frac{g_s^2}{32 \pi^2} \int d^4 x G_{\mu \nu}^a \tilde{G}_{\mu \nu}^a}
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- Obtained from real time path integral representation by Wick rotation, \( x_0^{(M)} \rightarrow -i x_4^{(E)} \), \( G_0^a(M) \rightarrow i G_4^a(E) \), such that

\[
i \int d^4 x \left\{ -\frac{1}{4} G_{\mu \nu}^a G_{\lambda \rho}^a + \theta \frac{g_s^2}{32 \pi^2} G_{\mu \nu}^a \tilde{G}_{\lambda \rho}^a \right\} \Rightarrow - \int d^4 x \left\{ \frac{1}{4} G_{\mu \nu}^a G_{\mu \nu}^a - i \theta \frac{g_s^2}{32 \pi^2} G_{\mu \nu}^a \tilde{G}_{\mu \nu}^a \right\}
\]

- Finite Euclidean action requires that the classical gauge fields contributing to the path integral should be such that the field strength tensor vanishes at \(|x| \rightarrow \infty\). This means that the gauge fields should approach a pure gauge form,

\[
G_\mu^a(x)T^a \rightarrow \frac{i}{g_s} U(x) \partial_\mu U^{-1}(x), \text{ for } |x| \rightarrow \infty, \text{ where } U(x) \in SU(3)
\]

- The theta-term can then be evaluated as

\[
g_s^2 \int d^4 x G_{\mu \nu}^a \tilde{G}_{\mu \nu}^a = \frac{g_s^2}{32 \pi^2} \int d^4 x \epsilon_{\mu \alpha \beta \gamma} \partial_\mu \left( G_\alpha^a G_\beta^b G_\gamma^c - \frac{g_s}{3} f^{abc} G_\alpha^a G_\beta^b G_\gamma^c \right)
= \frac{1}{24 \pi^2} \int_{S^3} d \sigma \epsilon_{\mu \alpha \beta \gamma} \text{tr} \left( [U \partial_\alpha U^{-1}] [U \partial_\beta U^{-1}] [U \partial_\gamma U^{-1}] \right)
= Q \in \mathbb{Z}
\]
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

\[ Q \equiv \frac{1}{24\pi^2} \int_{S^3} d\sigma \epsilon_{\mu\alpha\beta\gamma} \text{tr} \left( [U \partial_\alpha U^{-1}] [U \partial_\beta U^{-1}] [U \partial_\gamma U^{-1}] \right) \in \mathbb{Z} \]
The Axion

Theta-term in action of QCD

\[ Q \equiv \frac{1}{24\pi^2} \int_{S^3} d\sigma_\mu \, \epsilon_{\mu \alpha \beta \gamma} \text{tr} \left( [U \partial_\alpha U^{-1}] [U \partial_\beta U^{-1}] [U \partial_\gamma U^{-1}] \right) \in \mathbb{Z} \]

- First consider \( U \in SU(2) \subset SU(3) \)
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

\[
Q \equiv \frac{1}{24\pi^2} \int_{S^3_\infty} d\sigma_\mu \epsilon_{\mu\alpha\beta\gamma} \text{tr} \left( [U\partial_\alpha U^{-1}][U\partial_\beta U^{-1}][U\partial_\gamma U^{-1}] \right) \in \mathbb{Z}
\]

- First consider \( U \in SU(2) \subset SU(3) \)
  - Provides a mapping \( S^3_\infty \rightarrow SU(2) \cong S_3 \)
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

\[ Q \equiv \frac{1}{24\pi^2} \int_{S_3^\infty} d\sigma_\mu \epsilon_{\mu\alpha\beta\gamma} \text{tr} \left( [U \partial_\alpha U^{-1}] [U \partial_\beta U^{-1}] [U \partial_\gamma U^{-1}] \right) \in \mathbb{Z} \]

- First consider \( U \in SU(2) \subset SU(3) \)
  - Provides a mapping \( S_3^\infty \to SU(2) \cong S_3 \)
  - Pontryagin index \( Q \) counts the number of times \( S_3^\infty \) is wrapped around the group manifold \( SU(2) \sim S_3 \)
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

$$Q \equiv \frac{1}{24\pi^2} \int_{S_3^{\infty}} d\sigma_\mu \epsilon_{\mu\alpha\beta\gamma} \text{tr} \left( [U \partial_\alpha U^{-1}][U \partial_\beta U^{-1}][U \partial_\gamma U^{-1}] \right) \in \mathbb{Z}$$

• First consider $U \in SU(2) \subset SU(3)$
  • Provides a mapping $S_3^{\infty} \rightarrow SU(2) \simeq S_3$
  • Pontryagin index $Q$ counts the number of times $S_3^{\infty}$ is wrapped around the group manifold $SU(2) \sim S_3$
  • Pontryagin index aka winding number aka topological charge
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

\[ Q \equiv \frac{1}{24\pi^2} \int_{S_3^\infty} d\sigma_\mu \epsilon_{\mu\alpha\beta\gamma} \text{tr} \left( [U \partial_\alpha U^{-1}][U \partial_\beta U^{-1}][U \partial_\gamma U^{-1}] \right) \in \mathbb{Z} \]

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- In Euclidean space, SU(2) field configurations contributing to the vacuum-to-vacuum amplitude fall in homotopy classes of different topological charge
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

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Q \equiv \frac{1}{24\pi^2} \int_{S^3_\infty} d\sigma \epsilon_{\mu\alpha\beta\gamma} \text{tr} \left([U \partial_\alpha U^{-1}][U \partial_\beta U^{-1}][U \partial_\gamma U^{-1}]\right) \in \mathbb{Z}
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- In Euclidean space, SU(2) field configurations contributing to the vacuum-to-vacuum amplitude fall in homotopy classes of different topological charge
- It is not possible to continuously deform a field configuration into another of different winding number while maintaining the action finite
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

\[ Q \equiv \frac{1}{24\pi^2} \int_{S_3^\infty} d\sigma_\mu \epsilon_{\mu\alpha\beta\gamma} \text{tr} ([U \partial_\alpha U^{-1}] [U \partial_\beta U^{-1}] [U \partial_\gamma U^{-1}]) \in \mathbb{Z} \]

• First consider \( U \in SU(2) \subset SU(3) \)
  • Provides a mapping \( S_3^\infty \rightarrow SU(2) \simeq S_3 \)
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• In Euclidean space, SU(2) field configurations contributing to the vacuum-to-vacuum amplitude fall in homotopy classes of different topological charge

• It is not possible to continuously deform a field configuration into another of different winding number while maintaining the action finite

• General SU(3) gauge field configurations can be classified in the same SU(2) homotopy classes
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

\[ Q \equiv \frac{1}{24\pi^2} \int_{S_3^\infty} d\sigma_\mu \epsilon_{\mu\alpha\beta\gamma} \text{tr} \left( [U \partial_\alpha U^{-1}][U \partial_\beta U^{-1}][U \partial_\gamma U^{-1}] \right) \in \mathbb{Z} \]

- First consider \( U \in SU(2) \subset SU(3) \)
  - Provides a mapping \( S_3^\infty \to SU(2) \approx S_3 \)
  - Pontryagin index \( Q \) counts the number of times \( S_3^\infty \) is wrapped around the group manifold \( SU(2) \approx S_3 \)
  - Pontryagin index aka winding number aka topological charge
- In Euclidean space, SU(2) field configurations contributing to the vacuum-to-vacuum amplitude fall in homotopy classes of different topological charge
- It is not possible to continuously deform a field configuration into another of different winding number while maintaining the action finite
- General SU(3) gauge field configurations can be classified in the same SU(2) homotopy classes
  - The reason being that any mapping from S3 into any simple Lie group G can be deformed into a mapping to a SU(2) subgroup of G in a continuous way, hence with no change of homotopy class.
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

• Vacuum-to-vacuum amplitude can be written in terms of Fourier series

\[
\langle 0_+|0_-\rangle = \sum_{Q=-\infty}^{+\infty} Z_Q \exp[i\theta Q] \equiv Z(\theta)
\]

of Euclidean path integrals over gauge fields with fixed topological charge,

\[
Z_Q = \int_Q \mathcal{D}G \mathcal{D}\bar{q} \mathcal{D}q e^{-\int d^4x \left\{ \frac{1}{4} G^a_{\mu\nu} G^a_{\mu\nu} - \bar{q}(i\gamma_\mu D_\mu - M)q \right\}}
\]
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

- Vacuum-to-vacuum amplitude can be written in terms of Fourier series

\[
\langle 0_+ | 0_- \rangle = \sum_{Q=-\infty}^{+\infty} Z_Q \exp[i\theta Q] \equiv Z(\theta)
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of Euclidean path integrals over gauge fields with fixed topological charge,

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Z_Q = \int_Q \mathcal{D}G \mathcal{D}\bar{q} \mathcal{D}q \, e^{-\int d^4 x \left\{ \frac{1}{4} G_{\mu\nu}^a G_{\mu\nu}^a - \bar{q}(i\gamma_\mu D_\mu - M)q \right\}}
\]

- Theta is an angular parameter, \(-\pi \leq \theta \leq \pi\)
Vacuum Structure in Quantum Chromodynamics

Theta-term in action of QCD

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\[ \langle 0_+ | 0_- \rangle = \sum_{Q=-\infty}^{+\infty} Z_Q \exp[i\theta Q] \equiv Z(\theta) \]

- Theta is an angular parameter, \(-\pi \leq \theta \leq \pi\)

- Ordinary perturbative QCD resides in topologically trivial sector

\[ Z_Q = \int_Q \mathcal{D}G \mathcal{D}\bar{q} \mathcal{D}q \ e^{-\int d^4x \left\{ \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \bar{q}(i\gamma_\mu D_\mu - M)q \right\}} \]
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

- Vacuum-to-vacuum amplitude related to vacuum energy density via

\[
\langle 0_+ | 0_- \rangle = \lim_{\nu_4 \to \infty} e^{-\epsilon_0 \nu_4}
\]
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

- Vacuum-to-vacuum amplitude related to vacuum energy density via

\[ \langle 0_+ | 0_- \rangle = \lim_{\nu_4 \to \infty} e^{-\epsilon_0 \nu_4} \]

- Theta-dependence of vacuum energy density:

\[
\epsilon_0(\theta) = - \lim_{\nu_4 \to \infty} \frac{1}{\nu_4} \ln Z(\theta) = - \lim_{\nu_4 \to \infty} \frac{1}{\nu_4} \ln \left[ \sum_{Q=-\infty}^{+\infty} Z_Q e^{i\theta Q} \right]
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Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

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• Vacuum energy density is periodic: \[ \epsilon_0(\theta) = \epsilon_0(\theta + 2\pi) \]
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

- Vacuum-to-vacuum amplitude related to vacuum energy density via

\[ \langle 0_+ | 0_- \rangle = \lim_{\nu_4 \to \infty} e^{-\epsilon_0 \nu_4} \]

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- Vacuum energy density has a minimum for vanishing theta: \( \epsilon_0(0) \leq \epsilon_0(\theta) \)  

[Vafa, Witten `84]
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

• Vacuum-to-vacuum amplitude related to vacuum energy density via
  \[
  \langle 0_+ | 0_- \rangle = \lim_{\nu_4 \to \infty} e^{-\epsilon_0 \nu_4}
  \]

• Theta-dependence of vacuum energy density:
  \[
  \epsilon_0(\theta) = - \lim_{\nu_4 \to \infty} \frac{1}{\nu_4} \ln Z(\theta) = - \lim_{\nu_4 \to \infty} \frac{1}{\nu_4} \ln \left[ \sum_{Q=-\infty}^{+\infty} Z_Q e^{i\theta Q} \right]
  \]

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  • Relies on the fact that \( Z_Q \) is positive definite for a vector-like theory as QCD, which implies \( Z(\theta) \leq Z(0) \)
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

- Vacuum-to-vacuum amplitude related to vacuum energy density via

\[ \langle 0_+|0_- \rangle = \lim_{\nu_4 \to \infty} e^{-\epsilon_0 \nu_4} \]

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\[ \epsilon_0(\theta) = - \lim_{\nu_4 \to \infty} \frac{1}{\nu_4} \ln Z(\theta) = - \lim_{\nu_4 \to \infty} \frac{1}{\nu_4} \ln \left[ \sum_{Q=-\infty}^{+\infty} Z_Q e^{i\theta Q} \right] \]

- Vacuum energy density is periodic: \( \epsilon_0(\theta) = \epsilon_0(\theta + 2\pi) \)

- Vacuum energy density has a minimum for vanishing theta: \( \epsilon_0(0) \leq \epsilon_0(\theta) \) \[\text{[Vafa,Witten `84 ]}\]
  - Relies on the fact that \( Z_Q \) is positive definite for a vector-like theory as QCD, which implies \( Z(\theta) \leq Z(0) \)
  - Important for axion physics: axion field corresponds to space-time dependent theta parameter, field-dependence of axion potential is then given by theta-dependence of vacuum energy density. Result implies, axion has vanishing vacuum expectation value.

\[\text{DESY. | Axions as Dark Matter Candidates | Andreas Ringwald, Online School on Frontiers in Nuclear and Hadronic Physics, Galileo Galilei Institute, Feb 21 - Mar 04, 2022}\]
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

• Further progress can be made in the chiral limit, $\mathcal{M} \to 0$
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

- Further progress can be made in the chiral limit, $\mathcal{M} \rightarrow 0$
  - Consider first unrealistic case of one light flavour called $u$. 

Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

• Further progress can be made in the chiral limit, $\mathcal{M} \to 0$

• Consider first unrealistic case of one light flavour called $u$.

• Exploiting the chiral anomaly, that is the non-invariance of the fermionic measure in the path integral under a chiral transformation, $q \to e^{i\gamma_5 \alpha} q$, $\mathcal{D}q\mathcal{D}\bar{q} \to \left( e^{-i\alpha \frac{\epsilon_5^2}{16\pi^2} \int d^4x \, GG} \right) \mathcal{D}q\mathcal{D}\bar{q}$, one may trade the theta parameter for a phase of the quark mass.
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

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• Therefore, the vacuum-to-vacuum amplitude and thus the energy density depends on the quark mass and the theta parameter only through the product $m_u e^{i\theta}$:

$$\epsilon_0(\theta) = f(m_u e^{i\theta})$$
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

- Further progress can be made in the chiral limit, $\mathcal{M} \to 0$
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  - Therefore, the vacuum-to-vacuum amplitude and thus the energy density depends on the quark mass and the theta parameter only through the product $m_u e^{i\theta}$:
    $$\epsilon_0(\theta) = f(m_u e^{i\theta})$$
- The spectrum of the theory does not contain massless particles in the chiral limit. Expansion in powers of mass does therefore not give rise to infrared divergencies, i.e. is an ordinary Taylor expansion, [Leutwyler,Smilga `92]
  $$\epsilon_0(\theta) = \epsilon_0(0) - \Sigma m_u \cos \theta + \mathcal{O}(m_u^2)$$
  where $\Sigma = \sqrt{\langle q\bar{q}\rangle^2 + \langle qi\gamma_5 q\rangle^2} \bigg|_{m_u=0}$
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

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where

$$\Sigma = \sqrt{\langle \bar{q}q \rangle^2 + \langle \bar{q}i\gamma_5 q \rangle^2} \bigg|_{m_u=0}$$

• Vacuum energy has absolute minimum at vanishing theta angle
Further progress can be made in the chiral limit, \( \mathcal{M} \rightarrow 0 \)

Consider first unrealistic case of one light flavour called \( u \).

Exploiting the chiral anomaly, that is the non-invariance of the fermionic measure in the path integral under a chiral transformation, \( q \rightarrow e^{i\gamma_5\alpha}q, \mathcal{D}q\mathcal{D}\bar{q} \rightarrow \left( e^{-i\alpha \frac{e^{i\phi}}{16\pi^2} \int d^4x \, GG} \right) \mathcal{D}q\mathcal{D}\bar{q} \), one may trade the theta parameter for a phase of the quark mass.

Therefore, the vacuum-to-vacuum amplitude and thus the energy density depends on the quark mass and the theta parameter only through the product \( m_u e^{i\theta} \):

\[
\epsilon_0(\theta) = f(m_u e^{i\theta})
\]

The spectrum of the theory does not contain massless particles in the chiral limit. Expansion in powers of mass does therefore not give rise to infrared divergencies, i.e. is an ordinary Taylor expansion,

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\epsilon_0(\theta) = \epsilon_0(0) - \Sigma m_u \cos \theta + \mathcal{O}(m_u^2)
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where

\[
\Sigma = \sqrt{\langle \bar{q}q \rangle^2 + \langle \bar{q}i\gamma_5q \rangle^2} \bigg|_{m_u=0}
\]

Vacuum energy has absolute minimum at vanishing theta angle.

Vacuum energy independent of theta if quark mass zero.
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

• Further progress can be made in the chiral limit, \( M \to 0 \)
  • In realistic case of two light flavours, say u and d.
Further progress can be made in the chiral limit, $\mathcal{M} \to 0$

- In realistic case of two light flavours, say $u$ and $d$.
  - Exploiting the chiral anomaly, that is the non-invariance of the fermionic measure in the path integral under a common chiral transformation, $q \to e^{i\gamma_5 \alpha} q$,  
    $$DqD\bar{q} \to \left(e^{-i\alpha \frac{\delta^2}{16\pi^2} \int d^4 x \, GG} \right) DqD\bar{q},$$  
    one may trade the theta parameter for an overall phase of the quark mass.
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

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  - Therefore, the vacuum-to-vacuum amplitude and thus the energy density depends on the quark mass and the theta parameter only through the product:

$$\epsilon_0(\theta) = f(Me^{i\theta/2})$$
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

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  • Therefore, the vacuum-to-vacuum amplitude and thus the energy density depends on the quark mass and the theta parameter only through the product:
    $$\epsilon_0(\theta) = f(\mathcal{M}e^{i\theta/2})$$
  • However, in this case the spectrum of the theory does contain massless particles in the chiral limit, namely three pseudo-scalar Nambu-Goldstone bosons, the pions, originating from spontaneous breaking of the chiral symmetry by light quarks.
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

• Further progress can be made in the chiral limit, $\mathcal{M} \to 0$

  • In realistic case of two light flavours, say u and d.

  • Exploiting the chiral anomaly, that is the non-invariance of the fermionic measure in the path integral under a common chiral transformation, $q \to e^{i\gamma_5 \alpha} q$, $DqD\bar{q} \to \left( e^{-i\alpha \frac{\sqrt{2}}{16\pi^2} \int d^4 x G^2} \right) DqD\bar{q}$, one may trade the theta parameter for an overall phase of the quark mass.

  • Therefore, the vacuum-to-vacuum amplitude and thus the energy density depends on the quark mass and the theta parameter only through the product:

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  • However, in this case the spectrum of the theory does contain massless particles in the chiral limit, namely three pseudo-scalar Nambu-Goldstone bosons, the pions, originating from spontaneous breaking of the chiral symmetry by light quarks.

  • Vacuum-to-vacuum amplitude for realistic QCD at large four volume and small quark masses should be well described by the effective field theory (EFT) of pions.
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

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  - Therefore, the vacuum-to-vacuum amplitude and thus the energy density depends on the quark mass and the theta parameter only through the product:
    $$\epsilon_0(\theta) = f(\mathcal{M}e^{i\theta/2})$$
  - However, in this case the spectrum of the theory does contain massless particles in the chiral limit, namely three pseudo-scalar Nambu-Goldstone bosons, the pions, originating from spontaneous breaking of the chiral symmetry by light quarks.
  - Vacuum-to-vacuum amplitude for realistic QCD at large four volume and small quark masses should be well described by the effective field theory (EFT) of pions.
  - Exploiting chiral EFT, one finds
    $$\epsilon_0(\theta) = \epsilon_0(0) - m_\pi^2 f_\pi^2 \sqrt{1 + z^2 + 2z \cos \theta \over 1 + z}$$
    $$z \equiv m_u/m_d \approx 1/2$$

[Di Vecchia,Veneziano ’80; Leutwyler,Smilga ’92]
Vacuum Structure in Quantum Chromodynamics

Theta-dependence of vacuum energy density in QCD

\[ \epsilon_0(\theta) = \epsilon_0(0) - m_\pi^2 f_\pi^2 \frac{\sqrt{1 + z^2 + 2z \cos \theta}}{1 + z} \]

\[ \theta \]

\[ \epsilon_0(\theta) \]

\[ -3\pi \quad -2\pi \quad -\pi \quad 0 \quad \pi \quad 2\pi \quad 3\pi \]

\[ \theta \]
Strong CP Puzzle

Neutron electric dipole moment

- Theta-term violates both T and P, and thus CP
Strong CP Puzzle

Neutron electric dipole moment

- Theta-term violates both T and P, and thus CP
  - This can be understood by expressing theta term in terms of color electric and color magnetic fields,

\[
\theta \frac{g_s^2}{32\pi^2} \int d^4x \; G_{\mu\nu}^a \tilde{G}^{a,\mu\nu} = \theta \frac{g_s^2}{8\pi^2} \int d^4x \; E^a \cdot B^a
\]
Strong CP Puzzle

Neutron electric dipole moment

- Theta-term violates both T and P, and thus CP
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\[ \theta \frac{g_s^2}{32\pi^2} \int d^4x \, G^{a,\mu}_{\nu} \tilde{G}^{a,\mu\nu} = \theta \frac{g_s^2}{8\pi^2} \int d^4x \, E^a \cdot B^a \]

- Electric fields are even under T and odd under P:

\[ E^a(x, t) = E^a(x, -t), \quad E^a(x, t) = -E^a(-x, t) \]
Strong CP Puzzle

Neutron electric dipole moment

- Theta-term violates both T and P, and thus CP
  - This can be understood by expressing theta term in terms of color electric and color magnetic fields,

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- Most sensitive probe of T and P violation in flavor conserving interactions: electric dipole moment $d_n$ of neutron, corresponding to effective electric dipole interaction:

$$\mathcal{L}_{\text{eff}} \supset -\frac{i}{2} d_n \bar{\Psi}_n \sigma_{\mu\nu} \gamma_5 \Psi_n F^{\mu\nu}$$

$$d_n = C_{n\text{EDM}} e \theta$$

[ Crewther, Di Vecchia, Veneziano, Witten 79 ]
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\[\Rightarrow |\theta| \lesssim 10^{-10}\]
The Axion

Promote theta parameter to a dynamical field

- Add to the Standard Model a Nambu-Goldstone field, \( \theta(x) \equiv a(x)/f_a \in [-\pi, \pi] \), respecting, apart from an effective interaction with the gluonic topological charge density

\[
\mathcal{L} \supset \theta(x) q(x) \equiv \theta(x) \frac{\alpha_s}{8\pi} G_{\mu\nu}(x) \tilde{G}^{b,\mu\nu}(x)
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• QCD theta parameter promoted to dynamical field
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Peccei-Quinn mechanism \[\text{[Peccei, Quinn 77]}\]

- Dynamics of \(\theta(x) \equiv a(x)/f_a\), at energy scales below \(f_a\), but above \(\Lambda_{QCD}\), described by

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- At energies below $\Lambda_{\text{QCD}}$,
  \[\mathcal{L} \supset \frac{f_a^2}{2} \partial_{\mu} \theta(x) \partial^{\mu} \theta(x) - m_{\pi}^2 f_{\pi}^2 \sqrt{1 + z^2 + 2z \cos \theta} \frac{1}{1 + z} \]
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- At energies below \( \Lambda_{\text{QCD}} \),

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- Field dependence of effective potential coincides with theta-dependence of vacuum energy in QCD
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- There is no strong CP violation: $\langle \theta \rangle_0 = 0$
The Axion

PQ mechanism predicts pseudo Nambu-Goldstone boson

\[ \mathcal{L} \supset \frac{1}{2} \partial_{\mu} a(x) \partial^{\mu} a(x) - m_{\pi}^2 f_{\pi}^2 \sqrt{1 + z^2 + 2z \cos \left( \frac{a(x)}{f_{\alpha}} \right)} \frac{1}{1 + z} \]

[Weinberg 78; Wilczek 78]
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  - Precise calculation, by including \( \mathcal{O}(\alpha) \) QED corrections and NNLO corrections in chiral perturbation theory:
    \[ m_a = 5.691(51) \left( \frac{10^9 \text{ GeV}}{f_a} \right) \text{ meV} \]

[Weinberg 78; Wilczek 78]
The Axion

Coupling to the nucleon EDM

- Axion has a model-independent coupling to the EDM of the nucleon:

\[ \mathcal{L}_{aN\gamma} = -\frac{i}{2} g_{aN\gamma} a \bar{\Psi}_N \sigma_{\mu\nu} \gamma_5 \Psi_N F^{\mu\nu} \]

\[ g_{an\gamma} = -g_{ap\gamma} = e \frac{C_{\text{EDM}}}{f_a} \]

\[ C_{n\text{EDM}} = 2.4(1.0) \times 10^{-16} \text{ cm} \]

[Pospelov,Ritz 00]

- This is phenomenologically important for experiments searching for axion dark matter via oscillating nucleon electric dipole moments such as nEDM and CASPEr-electric

[AR,Rosenberg,Rybka in: 2021 Update of Review of Particle Physics]

[adapted from https://github.com/cajohare/AxionLimits]
The Axion

**Coupling to photons**

- From mixing with the pion, the axion inherits at energy scales below the QCD scale, a model-independent contribution to its coupling with two photons:

\[ \mathcal{L}_{a\gamma\gamma} \supset \frac{1}{4} g^{(\text{mi})}_{a\gamma} a F_{\mu\nu} \tilde{F}^{\mu\nu} \]

\[ g^{(\text{mi})}_{a\gamma} = \frac{\alpha}{2\pi f_a} C^{(\text{mi})}_{a\gamma} \]

- LO chiral perturbation theory:

\[ C^{(\text{mi})}_{a\gamma} \approx -\frac{2}{3} \frac{4 + z}{1 + z} \approx -2 \]

[Kaplan 85; Srednicki ́85]

- Precise NLO determination

\[ C^{(\text{mi})}_{a\gamma} = -1.92(4) \]

[Grilli di Cortona et al. ́16]

- Phenomenologically important for experiments searching for axions via coupling to photons

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