Lecture 7. Multiquark states in N colours, in the limit $N \to \infty$

- ***Summary***

1. The overall panorama ✓
2. Constituent Quark Model and masses of conventional mesons and baryons ✓
3. Light and Heavy Tetraquarks. First comparison with hadron molecules ✓
4. Tetraquarks and the EightFold Way. Di-J/Ψ resonances ✓
5. $X(3872)$ and its missing partners ✓
6. Born-Oppenheimer approximation for double charm baryons and tetraquarks ✓
7. Multiquark states in N colours, in the $N \to \infty$ limit ✓
8. Tetraquarks vs. molecules: the Weinberg criterium for $X(3872)$ and the double charm $\mathcal{T}_{cc}^{+}(3875)$

1. QCD at large N in a nutshell

Quark-guon coupling

$$\mathcal{L}_{QCD} = g_{\rm QCD}\ \bar{q}\frac{\lambda^A}{2}g^A_\mu\gamma^\mu q$$

’t-Hooft shows that in the limit $N \to \infty$ with

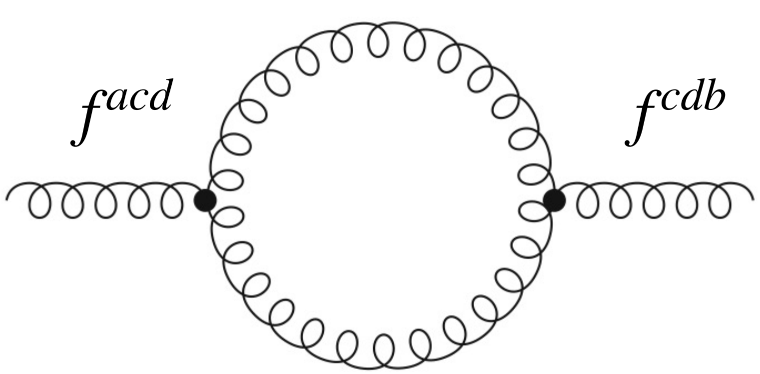
$$g_{\rm QCD} \to 0, g_{\rm QCD} \cdot \sqrt{N} = \text{fixed}$$

only planar diagrams with quarks at the edge survive.

Planar diagrams with L quark loops are of order N^{-L} with a coefficient which is a non trivial, non perturbative function of the reduced coupling, sometime referred to as the ’t-Hooft coupling

$$\lambda = g^2_{\rm QCD} \cdot N$$

Example:



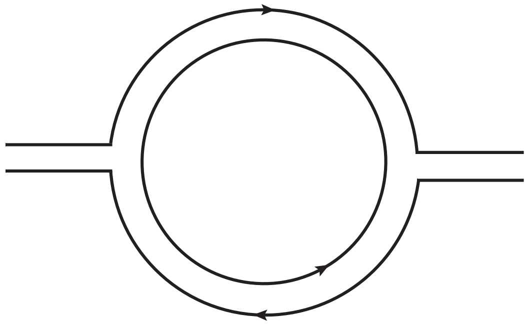
$$A \propto g^2_{QCD}\ \mathrm{Tr}(T^a T^b) = g^2_{QCD}\ 2N$$

remains finite for:

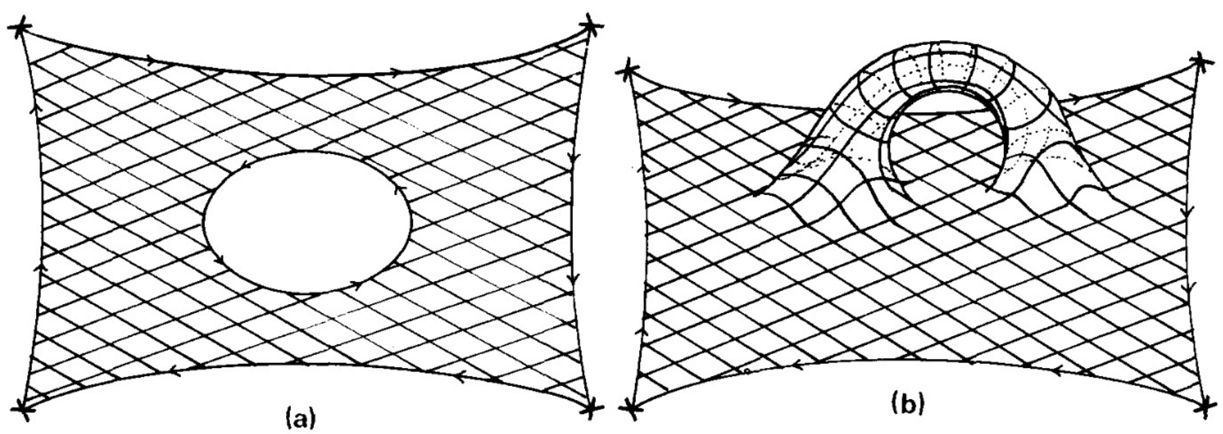
$$N \to +\infty, g_{QCD} \to 0,\ g_{QCD}\sqrt{N} \to \text{fixed} = \lambda$$

- Another way to get to the same result is to rewrite the gluon field in terms of the N x N matrices

$$(A_\mu)^c_d = (T^a)^c_d A^a_\mu$$



- In the large-N limit we neglect the tracelessness condition and treat all N^2 components as independent.
- With this parametrization we replace the gluon line by two lines running in opposite directions and carrying the two conjugate color indices, c and d.
- the general rule:



’t-Hooft:

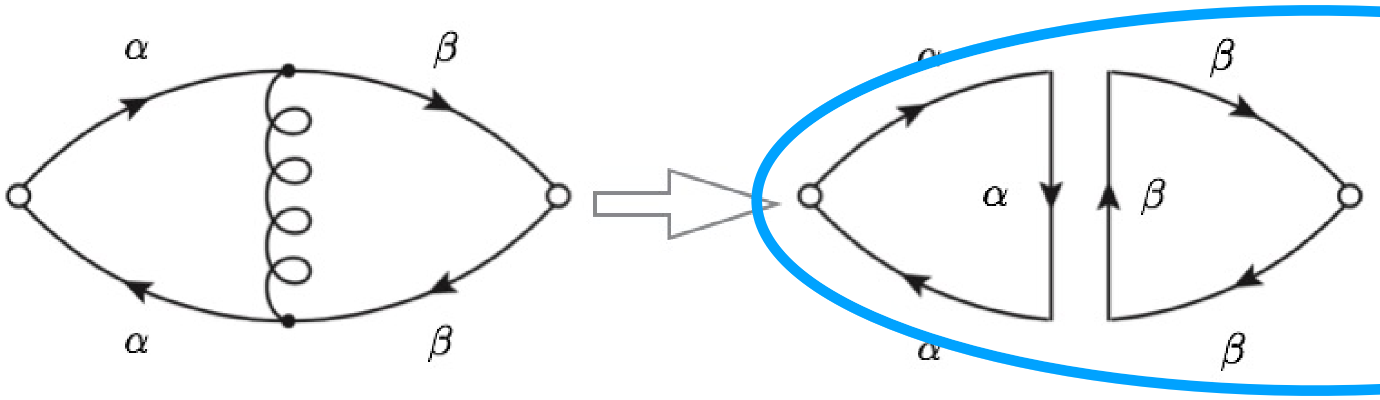
Order of diagram:

$N^k, \; k = 2 - L - 2H$

$L =$ n. of fermion loops

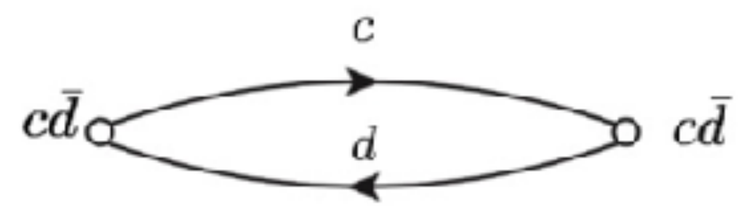
$H =$ n. of handles

- To check (open dots= insertion of color siglet operators) L=1, H=0 ⇒ k=1:

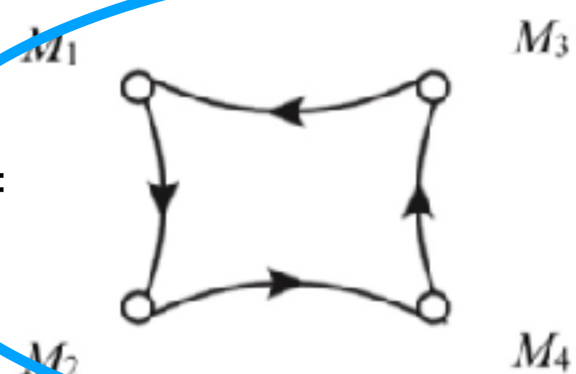


$$\times \frac{\lambda}{N} = \lambda N$$

Mesons ($q\bar{q}$): interactions and scattering

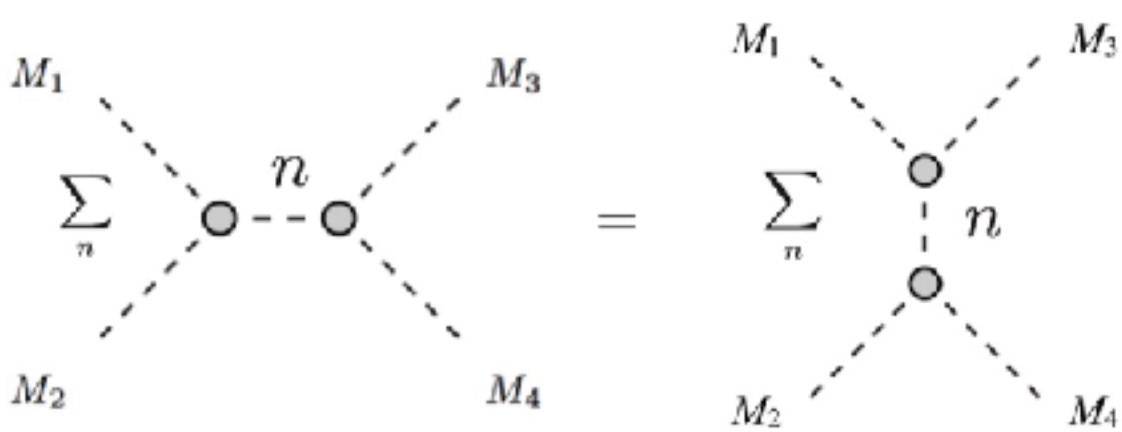
- External mesons are represented by the insertion color singlet ($q\bar{q}$) operators in one fermion loop.
  - Meson propagator: 

$\propto N$
  - To normalize the amplitude, we introduce one factor $1/\sqrt{N}$ for each insertion
  - With the same convention, the meson-meson scattering amplitude is

$\mathscr{A}(M_1 + M_2 \to M_3 + M_4) =$ 

$\times \left(\frac{1}{\sqrt{N}}\right)^4 \propto \frac{1}{N}$ Quark picture

  - cuts in the s-channel or in the t-channel give poles corresponding to stable mesons

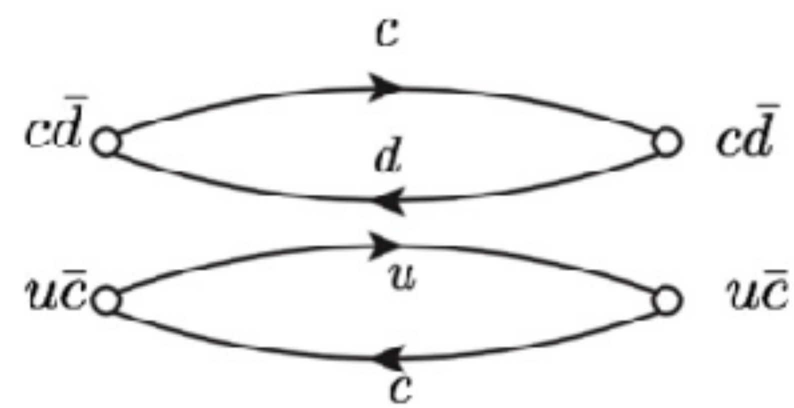


Meson picture
- Equality corresponds to the Dolen-Horn Schmidt duality: sum of resonances in s-ch = exchange of poles in t-ch
- $N \to \infty$ in planar approximation = Veneziano Model

2. Tetraquarks in the large N expansion

- Respectability of tetraquarks was somehow tarnished by a theorem of S. Coleman:

S. Coleman, Aspects of Symmetry (Cambridge University Press, Cambridge, England, 1985), pp. 377–378.



- by Fierz rearrangements, tetraquark operators can be reduced to products of color singlet bilinears;
- tetraquarks correlators for N→∞ reduce to disconnected meson-meson propagators *i.e. tetraquark operators, to leading order create out of vacuum only pairs of mesons.*
- The argument was reexamined by S. Weinberg who argued that if the connected tetraquark correlator develops a pole, it will be irrelevant that the residue of the pole is of order 1/N with respect to the disconnected part: *at the pole the connected part will dominate anyhow.*

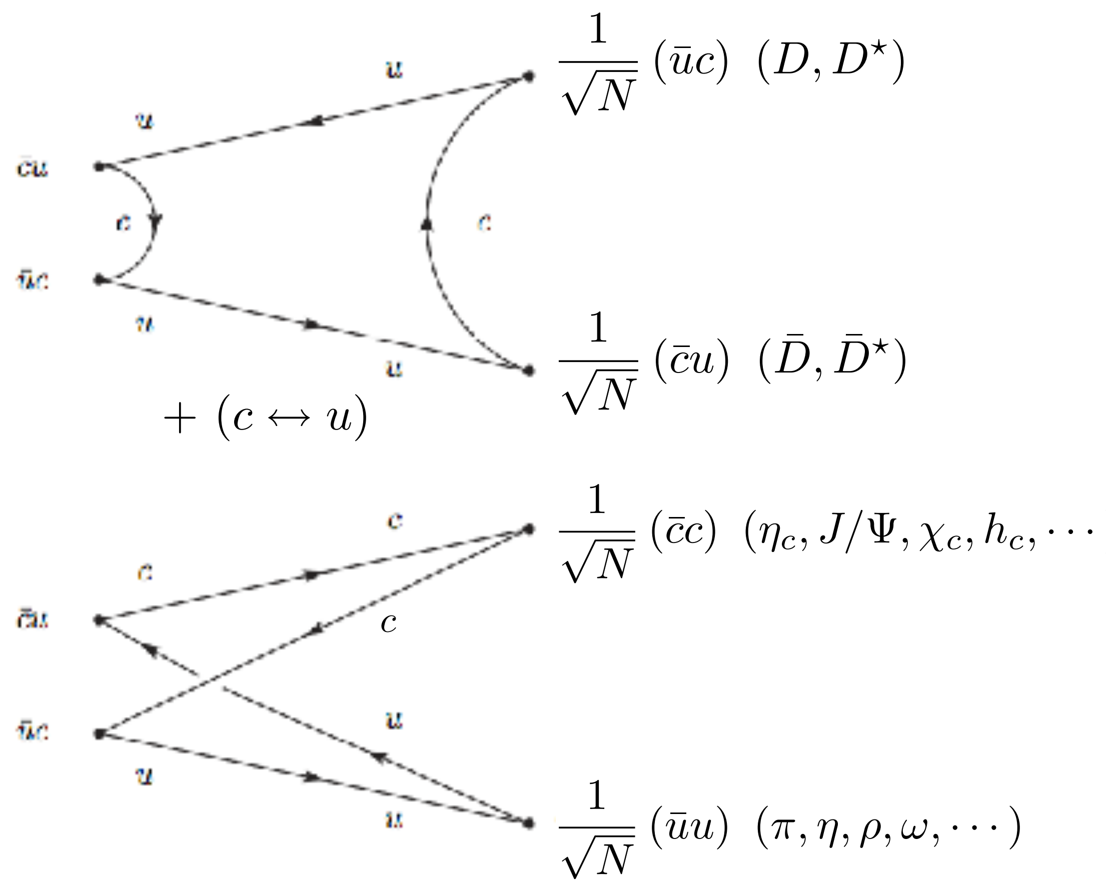
S. Weinberg, PRL 110, 261601 (2013),

- The real issue is the width of the tetraquark state: it may increase for large N, to the point of making the state unobservable;
- Weinberg's conclusions is the the decay rate goes like 1/N, making tetraquarks a respectable possibility.
- Weinberg's discussion has been enlarged by M. Knecht and S. Peris (arXiv:1307.1273) and further considered by T. Cohen and R. Lebed et al. (arXiv: 1401.1815, arXiv:1403.8090).

Weinberg: Decay amplitudes of tetraquarks in 1/N expansion

- Interpolating field operators have to be multiplied by powers of N, such as to make the connected two-point correlators to be normalized to unity;
- one loop amplitude with insertions of quark color singlet operators gives a factor N.

$$\mathcal{Q} = \frac{1}{\sqrt{N}}[cu][\bar{c}\bar{u}]$$

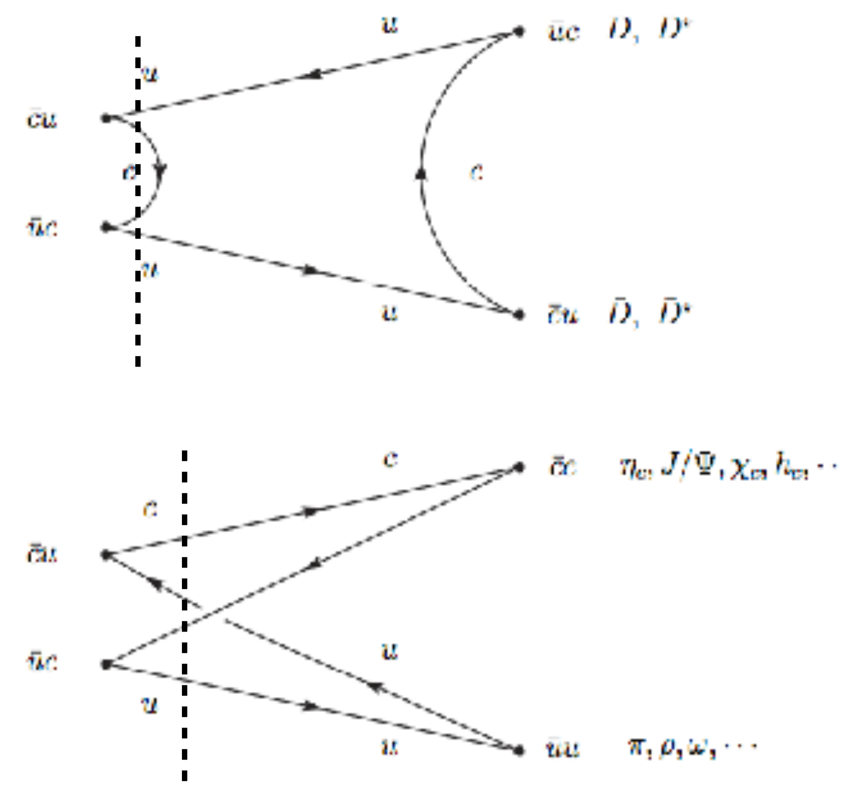


- two independent amplitudes, same order in N

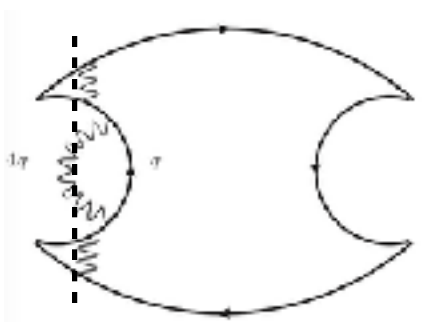
- The result is that *decay amplitudes into two mesons are of order* $\frac{1}{N^{3/2}}\,N = \frac{1}{\sqrt{N}}$
- No divergence with N of the rates, tetraquarks are observable!

3. Beyond planar diagrams

T. D. Cohen and R. F. Lebed, Phys. Rev. **D 90**, 016001 (2014)
L. Maiani, A. D. Polosa and V. Riquer, JHEP 1606 (2016) 160

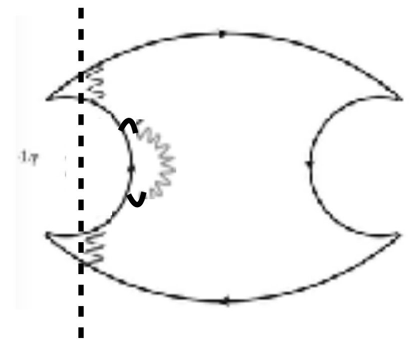


- The typical diagrams of order N show a 4 quarks cut
- but: are these ***free or interacting quarks***?
- Note: The same order applies if we fill the quark loop by a multigluon, ***planar*** diagram('t-Hooft)
- Do ***planar*** interactions resolve the problem?

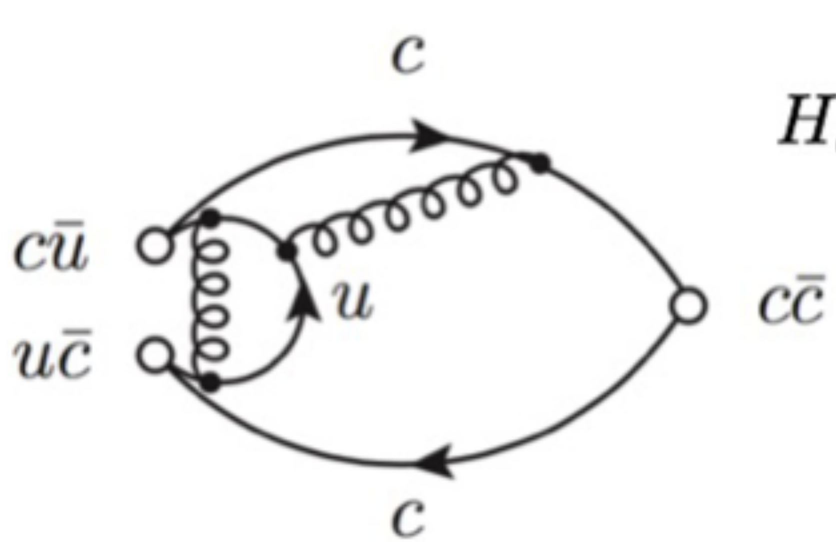


in a planar diagram all lines must be inside the quark loop

=



- for planar gluons, the cut corresponds to a pair of ***free mesons***
- effect of non planar gluon interaction: the two mesons do interact !



$H_a = 1$

$\propto \frac{1}{N}$

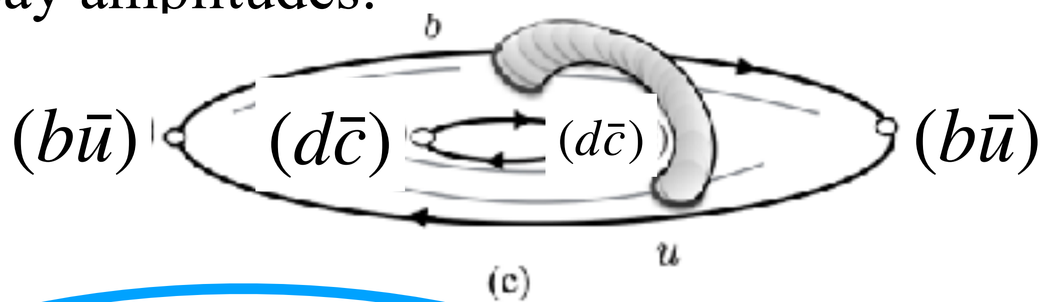
contributions by
R. F. Lebed, Phys. Rev. **D 88** (2013) 057901.
W. Lucha, D. Melikhov and H. Sazdjian, Phys. Rev. **D 96** (2017) 014022; Eur. Phys. J. C 77 (2017) 866.

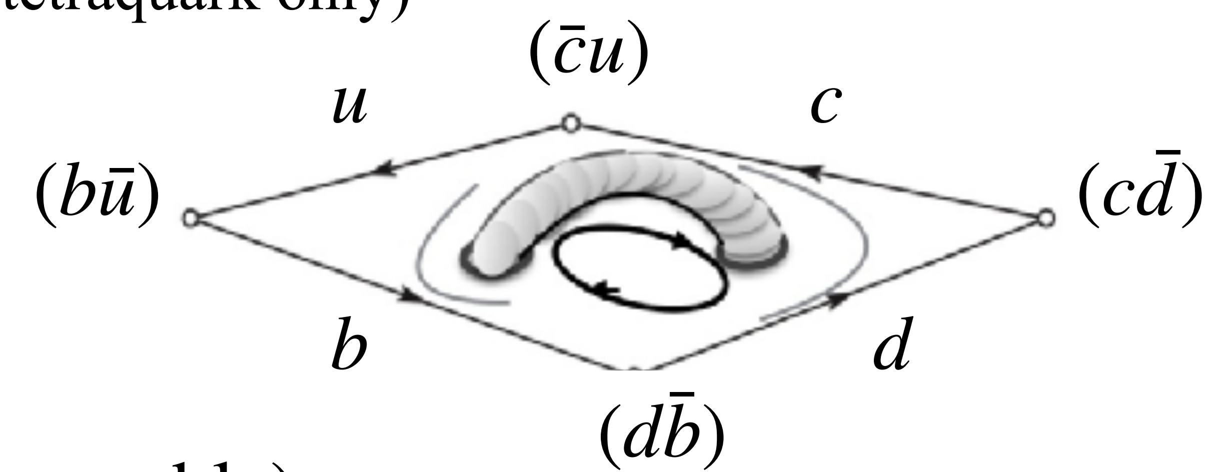
Our conclusion: tetraquarks diagrams are next-to next-to leading

L. Maiani, A.D. Polosa, V. Riquer, Phys. Rev. **D 98** (2018) 054023

Only one diagram is relevant, with meson insertions distributed on the two fermion loops

1. for given flavours, ***one tetraquark suffices*** to obtain a consistent solution (qq are attractive in color 3-bar and repulsive in 6, one tetraquark only)
2. decay amplitudes:





$$A(T \to M_1 + M_2) \propto \frac{1}{N^2}$$ (Tetraquarks are observable)

$$A(T \to T' + M) \propto \frac{1}{\sqrt{N}}$$

$$A(T \to T' + \gamma) \propto eN^0$$

3. The decays: Y→ Z+π and Y→ X+γ are ***allowed transitions***

4. ***neutral hidden-charm tetraquarks mix with charmonia*** to order: $\frac{1}{N\sqrt{N}}$
   they may have ***charmonia-like decays***
   Y states ***may annihilate into $e^+ e^-$*** via mixing.

4. An alternative approach: Witten's baryons

- Tetraquarks of composition $[qq'][\bar{q}''\bar{q}''']$ can be generalized to N colours

- Pentaquarks, however, are related to baryons, which in QCD with general N require a completely different toolkit, since the three quark configuration in the baryon generalizes to the color antisymmetric combination $[q_1 q_2 \ldots q_N]$, first studied by Witten.

- the baryonium scheme, a formulation *a'la Witten* to generalize tetraquarks to any N, has been explored by Rossi and Veneziano (1977, 2016):

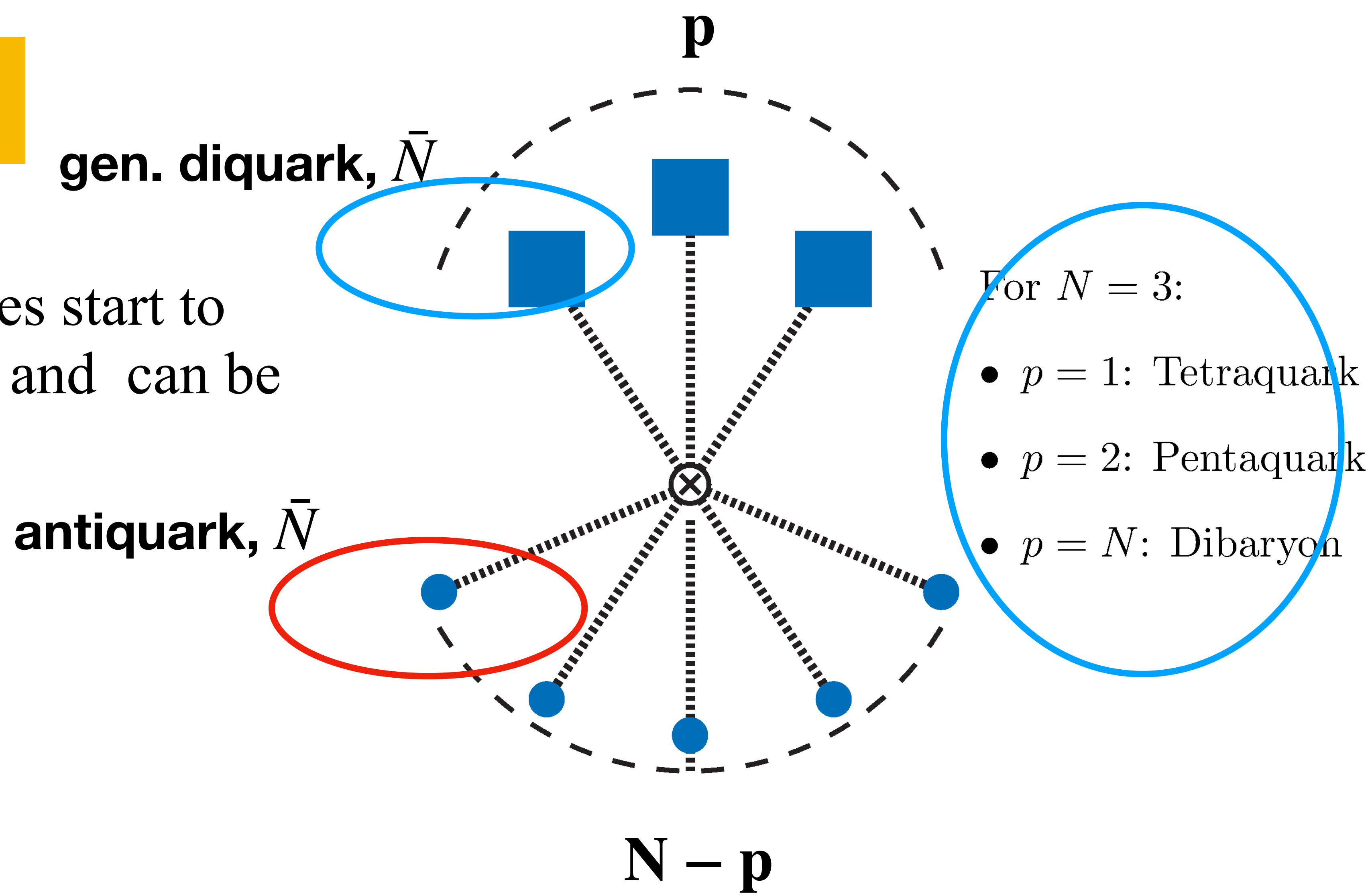
$$T = \epsilon_{aa_1\ldots a_{N-1}}\left(q^{[a_1}\ldots q^{a_{N-1}]}\right)\epsilon^{ab_1\ldots b_{N-1}}\left(\bar{q}_{[b_1}\ldots\bar{q}_{b_{N-1}]}\right)$$

- the generalised diquark $[q_1 q_2 \ldots q_{N-1}]$ transform as an antiquark and can generalise the construction used in N=3 for pentaquarks and dibaryons

- In a ***world of two colours***, the new structures disappear:

- N=2 QCD is made only of mesons, $\bar{q}q$, ``baryons", qq, and molecules thereof.

L. Maiani, V. Riquer and W. Wang, Eur. Phys. J. C **78**} (2018) 1011



- The new spectroscopic series start to appear at N=3, (our world!) and can be extended to N colours.

generalised pentaquark :

$$P = (q^{[1}q^2 \ldots q^{A]}) \ldots (q^{[1}q^2 \ldots q^{B]})\ \bar{q}_{[1} \ldots \bar{q}_{C]}\ \epsilon_{aA} \ldots\ \epsilon_{bB}\ \epsilon^{a \ldots bC}\ (C = N - p)$$

p gen. diquarks.

generalised dibaryon;

$$P = (q^{[1}q^2 \ldots q^{A]}) \ldots (q^{[1}q^2 \ldots q^{B]})\ \ \epsilon_{aA} \ldots \epsilon_{bB}\ \epsilon^{a \ldots b}$$

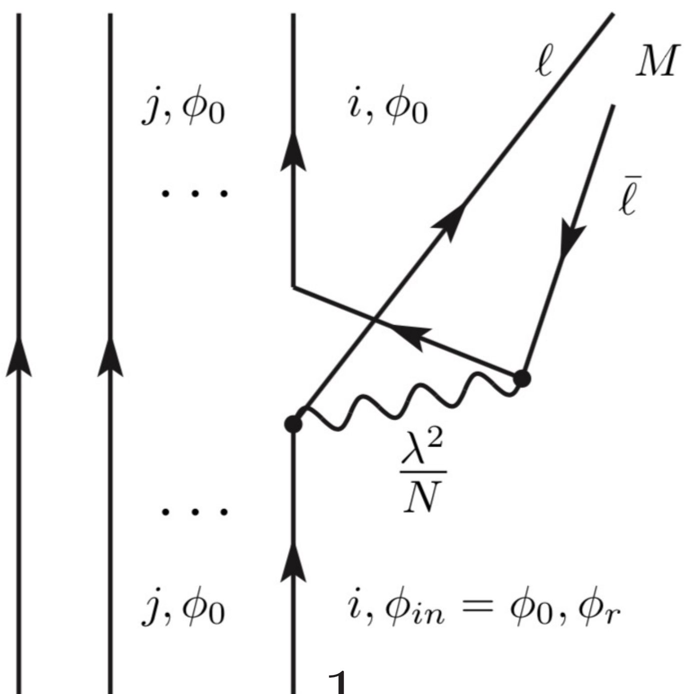
N gen. diquarks.

Baryons: coupling to meson and decay

$$M^* = NM_0 + \sum_r n_r \epsilon_r, \quad \sum_r n_r = N$$

- quarks live in N-independent Harthree-Fock wavefunctions
- space wave functions of N bosons

- Ground and excited states: mass



$$M = \frac{1}{\sqrt{N}} \sum_\ell \bar{q}_\ell q^\ell$$

ground state :

$$\Psi_0^{1\ 2\ \ldots N}(x_1, x_2, \ldots N) = \epsilon^{12\ldots N} \phi_0(x_1) \times \times \phi_0(x_2) \ldots \phi_0(x_N)$$

excited state :,

$$\Psi_r^{1\ 2\ \ldots N}(x_1, x_2, \ldots N) = \epsilon^{12\ldots N} \times \frac{1}{\sqrt{N}} \times [\phi_{\mathbf{r}}(x_1)\phi_0(x_2)\ldots\phi_0(x_N) + \phi_0(x_1)\phi_{\mathbf{r}}(x_2)\ldots\phi_0(x_N) + \cdots + \phi_0(x_1)\phi_0(x_2)\ldots\phi_{\mathbf{r}}(x_N)].$$

- Transition operator

$$O = \frac{\lambda^2}{N}\sqrt{N}\ \bar{q}_i \mathcal{O}(x) q^i.$$

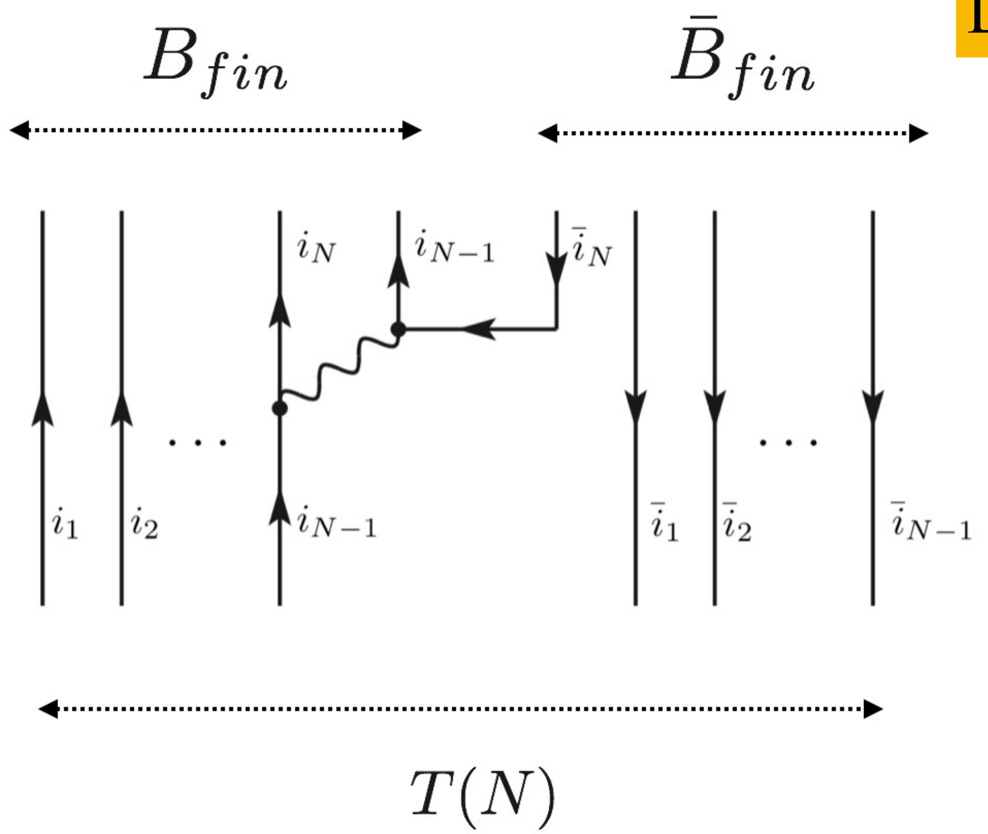
- ground state coupling
- excited state decay amplitude

$$g_{B\bar{B}M} \sim \frac{\lambda^2}{N}\sqrt{N}\ N \propto \sqrt{N}$$

$$A(B^* \to B + M) \sim \frac{\lambda^2}{N}\sqrt{N}\ \frac{N}{\sqrt{N}} \propto N^0$$

tetraquark to $B\bar{B}$

L. Maiani, V. Riquer and W. Wang, Eur. Phys. J. C **78** (2018) 1011



$$T = \frac{1}{\sqrt{N}} B_a \bar{B}^a;\ \ B_a = \frac{\partial}{\partial q^a} B;$$

$$q^a B_a = q^a \frac{\partial}{\partial q^a} B = B, \text{for any } a$$

$$A(T \to B + \bar{B}) = \frac{1}{\sqrt{N}} \frac{\lambda^2}{N} (N-1)\ N \propto \sqrt{N}$$

$$A(T^* \to B + \bar{B}) = \frac{1}{\sqrt{N}} \frac{\lambda^2}{N} (N-1)\ \sqrt{N} \propto N^0$$

RESULTS 1

- decay amplitudes from the ***ground states*** may diverge at large N.
- However such decays are generally forbidden by phase space and the divergent amplitudes do not affect the observability for such particles.
- At $N = \infty$, ground states of multiquark hadrons are narrow or stable, particularly in the case of the dibaryon.

RESULTS 2

- Decay amplitudes of excited states are summarised in the Table

Table 1:

$T^* \to$	$B + \bar{B}$		$T+$ Meson	Mesons
$A^* \propto$	N^0		N^0	$< e^{-\frac{N}{2}}$
$P^* \to$	$B + T$	$B + B + \bar{B}$	$P+$Meson	$B+$Mesons
$A^* \propto$	N^0	$N^{1/2}$	N^0	$< e^{-\frac{N}{2}}$
$D^* \to$		$NB + \bar{B}$	$D+$ Meson	$(N-1)B$
$A^* \propto$		$> e^{+\frac{N}{2}\log N}$	N^0	$> e^{+\frac{N}{2}\log N}$

- Third column: decay of an excited into the ground state by meson emission.
- Last column refers to decays obtained by reorganising the quark-antiquark pairs of the initial state into a multimeson state or redistribuiting the quarks of one diquark to the other diquarks, to form a set of N-1 baryons.

Excited tetraquarks:

- the amplitudes for the decay of the excited states vanish or remain constant for $N \to \infty$: observable states in this limit;

- tetraquark de-excitation amplitudes same order of $B\bar{B}$ decay amplitudes

- For N=3 and flavour composition $[cu][\bar{c}\bar{u}]$ the threshold for two-baryon decay is $2M(\Lambda_c)$~ 4570MeV; Cotugno et al (2009) argued that X(4660) is a P-wave tetraquark decaying predominantly into $\Lambda_c\bar{\Lambda}_c$ in addition to the mode into $\psi(2S)\pi\pi$.

- tetraquark-charmonium mixing is exponentially suppressed;

•The divergence at large N is not relevant for the width and the observability of the ground state, which is below threshold for the decay,
•Tetraquark ground state is dominant as intermediate state in elastic $B\bar{B}$ scattering. The 1/N behaviour we find for the latter amplitude is in agreement with the result given in Rossi(2016).

Excited pentaquarks and dibaryons:

- de-excitation amplitudes into the ground state and a meson remain limited for large N;
- at $N = \infty$ there are modes which give divergent amplitudes, namely $P* \to B + B + \bar{B}$ and $D* \to NB + \bar{B}$ or $(N-1)B$
- literally, these results, imply ***sharp thresholds at $2B + \bar{B}$ and (N-1)B respectively, below which we expect observable pentaquarks and dibaryons, and above which we expect large, unobservable widths***
- a situation similar to charmonia above and below the open charm-anticharm meson threshold.
- For N=3 and pentaquark with flavour composition:$\sim[cu][ud]\bar{c}$, corresponding to the states observed by LHCb, the threshold for ``non-observabilty" would be $2M(\Lambda_c) + M(P) \sim 5510$ MeV,
- for a double charmed dibaryon with flavour [cu][cd][ud] the threshold would be at: $2M(\Lambda_c)$.

- Finally, it is interesting to compare the results for tetraquarks with the analysis based on the large N generalisation of diquarks following Weinberg.
- The results by the Roma group feature
- a narrow ground state, with a suppressed decay amplitude into two mesons, of order N^{-2};
- for large N this is larger that the exponentially suppressed amplitude in the Table, but it takes N> 6 for the power suppression to win over the exponential suppression
- amplitude of order $N^{-1/2}$ for the de-excitation into the ground state by meson emission;
- tetraquark-charmonium mixing occurs to order $N^{-3/2}$;
- the decay of an excited tetraquark into $B\bar{B}$ cannot be computed..

The similarities of two very different multiquark generalisations are very intriguing!