Nucleon partonic structure: concepts and measurements

Part 3: factorisation

M. Diehl

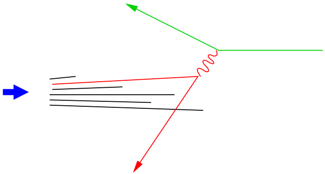
Deutsches Elektronen-Synchroton DESY

Frontiers in Nuclear and Hadronic Physics 2022
GGI, Firenze, 21 Feb – 4 March

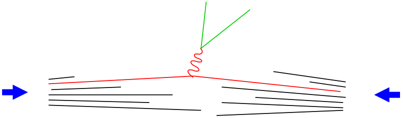
The parton model

description for deep inelastic scattering, Drell-Yan process, etc.

- fast-moving hadron
  $\approx$ set of free partons $(q, \bar{q}, g)$ with low transverse momenta
- physical cross section
  $=$ cross section for partonic process $(\gamma^* q \to q,\ q\bar{q} \to \gamma^*)$
  $\times$ parton densities



Deep inelastic scattering (DIS): $\ell p \to \ell X$



Drell-Yan: $pp \to \ell^+ \ell^- X$

Nobel prize 1990 for
Friedman, Kendall, Taylor

The parton model

description for deep inelastic scattering, Drell-Yan process, etc.

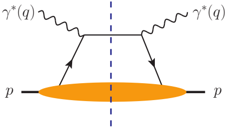
- fast-moving hadron
  $\approx$ set of free partons $(q, \bar{q}, g)$ with low transverse momenta
- physical cross section
  $=$ cross section for partonic process $(\gamma^* q \to q,\ q\bar{q} \to \gamma^*)$
  $\times$ parton densities

Factorisation

- implement and correct parton-model ideas in QCD
  - conditions and limitations of validity
    kinematics, processes, observables
  - corrections: partons interact
    α_s is small at large scales $\rightsquigarrow$ perturbation theory
  - definition of parton densities in QCD
    derive their general properties
    make contact with non-perturbative methods

Example: inclusive DIS (deep inelastic scattering)

- $\sigma_{\text{tot}}(\gamma^* p \to X)$
  $\xrightarrow{\text{opt. theorem}}$ $\operatorname{Im} \mathcal{A}(\gamma^* p \to \gamma^* p)$
  forward amplitude



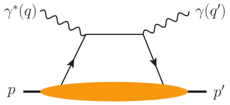
- measure in $ep \to eX$
- Bjorken limit: $Q^2 = -q^2 \to \infty$ at fixed $x_B = Q^2/(2p \cdot q)$
- $\operatorname{Im} \mathcal{A}(\gamma^* p \to \gamma^* p) =$
  hard-scattering coefficient $\otimes$ parton distribution
  - hard-scattering coefficient $\sim$ $\operatorname{Im} \mathcal{A}(\gamma^* q \to \gamma^* q)$ small print $\to$ later
  - parton densities (PDFs): process independent
    also appear in $pp \to \ell^+ \ell^- X$, $\gamma^* p \to \text{jet} + X$, ...

Example: DVCS (deeply virtual Compton scattering)

- exclusive cross section

  $\propto \left|\mathcal{A}(\gamma^* p \to \gamma p)\right|^2$

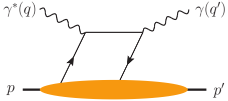
  square of amplitude



- measure in $ep \to ep\gamma$
- Bjorken limit: $Q^2 = -q^2 \to \infty$ at fixed x_B and $t = (p - p')^2$
- $\mathcal{A}(\gamma^* p \to \gamma p) =$

  hard-scattering coefficient $\otimes$ generalized parton distribution
  - GPD depends on momentum fractions x, ξ and on t
  - hard-scattering coefficient $\sim \mathcal{A}(\gamma^* q \to \gamma q)$

Example: DVCS (deeply virtual Compton scattering)



- exclusive cross section

  $\propto \left|\mathcal{A}(\gamma^* p \to \gamma p)\right|^2$

  square of amplitude

- measure in $ep \to ep\gamma$
- Bjorken limit: $Q^2 = -q^2 \to \infty$ at fixed x_B and $t = (p - p')^2$
- $\mathcal{A}(\gamma^* p \to \gamma p) =$

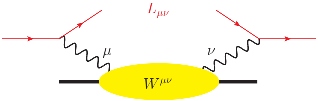
  hard-scattering coefficient $\otimes$ generalized parton distribution

  - GPD depends on momentum fractions x, ξ and on t
  - hard-scattering coefficient $\sim \mathcal{A}(\gamma^* q \to \gamma q)$ or $\mathcal{A}(\gamma^* q\bar{q} \to \gamma)$

    both cases included in "$\otimes$" of factorisation formula

Interlude: DIS structure functions

- aim: separate QED/electroweak from QCD part



  - leptonic tensor $L_{\mu\nu} \propto \mathcal{A}_{\ell\to\ell+V(\mu)} \left[\mathcal{A}_{\ell\to\ell+V(\nu)}\right]^*$, $V = \gamma^*, Z^*$
  - hadronic tensor $W^{\mu\nu} \propto \mathrm{Im} \int d^4x \, e^{iqx} \langle p|J^\mu(x) J^\nu(0)|p\rangle$
  - $\sigma_{\ell+p\to\ell+X} \propto L_{\mu\nu} W^{\mu\nu}$

- using symmetries (parity, time reversal, current conservation) get

$$W^{\mu\nu}(p,q) = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1(x_B,Q^2) + \frac{1}{pq}\left(p^\mu - \frac{pq}{q^2}\, q^\mu\right)\left(p^\nu - \frac{pq}{q^2}\, q^\nu\right) F_2(x_B,Q^2)$$
$$- i\epsilon^{\mu\nu\alpha\beta} \frac{q_\alpha p_\beta}{2pq} F_3(x_B,Q^2) + \text{ proton spin dependent terms}$$

F_3 only with Z exchange

- $\sigma_{\ell+p\to\ell+X}$ expressed through structure functions F_1, F_2, F_3
- analogs for SIDIS $\ell + p \to \ell + h + X$, Drell-Yan, etc.

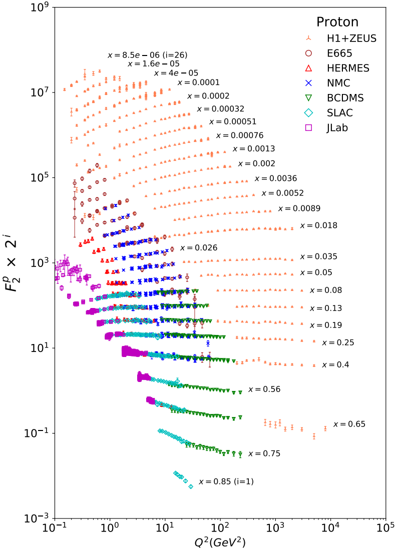
Interlude: DIS structure functions

- $\sigma_{\ell+p\to\ell+X}$ expressed through structure functions F_1, F_2, F_3
- valid in any kinematics
  no reference to factorisation
- do not confuse structure functions with parton distributions
- parton model:

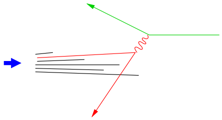
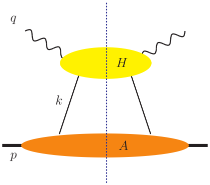
$$F_2(x,Q^2) = \sum_q e_q^2\, x\big[q(x) + \bar{q}(x)\big]$$
$$+\, Z \text{ exchange}$$

- Bjorken scaling: F_2 independent of Q^2 holds approx. at $x \sim$ a few 0.1 and large enough Q^2

figure: Review of Particle Properties 2020



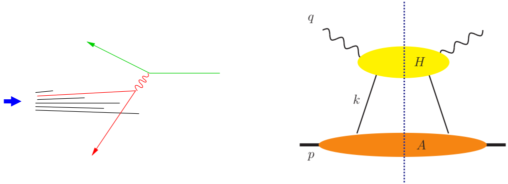
Factorisation: physics idea and technical implementation

- idea: separation of physics at different scales
  - high scales: quark-gluon interactions
    $\rightsquigarrow$ compute in perturbation theory
  - low scale: proton $\to$ quarks, antiquarks, gluons
    $\rightsquigarrow$ parton densities

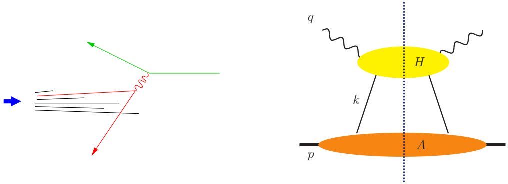
requires hard momentum scale in process
large photon virtuality $Q^2 = -q^2$ in DIS

Factorisation: physics idea and technical implementation



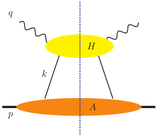
- implementation: separate process into
  - “hard” subgraph H with particles far off-shell
    compute in perturbation theory
  - “collinear” subgraph A with particles moving along proton
    turn into definition of parton density

Factorisation: physics idea and technical implementation



- note difference with high-energy/small x factorisation
  - separate dynamics according to rapidity (not virtuality) of particles
  - overlap of the factorisation schemes if have strong ordering in rapidity and virtuality

Collinear expansion



- graph gives $\int d^4k\, H(k) A(k)$; simplify further
- light-cone coordinates: $v^{\pm} = \frac{1}{\sqrt{2}}(v^0 \pm v^3)$, $\boldsymbol{v} = (v^1, v^2)$

$$v \cdot w = v^\mu w_\mu = v^+ w^- + v^- w^+ - \vec{v}\,\vec{w}$$

$$v^2 = 2 v^+ v^- - \vec{v}_T^{\,2}$$

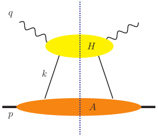
boost along z : $v^+ \to \alpha v^+$, $\vec{v} \to \vec{v}$, $v^- \to \frac{1}{\alpha} v^-$

FAST RIGHT-MOVING PARTICLE $m, p_T \ll p^3$

$$p^0 = \sqrt{m^2 + \vec{p}_T^{\,2} + (p^3)^2} = p^3 + \frac{p_T^2 + m^2}{2p^3} + \ldots$$

$$p^+ = \frac{1}{\sqrt{2}}(p^0 + p^3) \approx \sqrt{2}\, p^3, \quad p^- \approx \frac{p_T^2 + m^2}{\sqrt{2}\, 2p^3} \ll p_T, m$$

Collinear expansion



- graph gives $\int d^4k\, H(k) A(k)$; simplify further
- in hard graph neglect small components of external lines
  $\rightsquigarrow$ Taylor expansion

$$H(k^+, k^-, k_T) = H(k^+, 0, 0) + \text{corrections}$$

  $\rightsquigarrow$ loop integration greatly simplifies:

$$\int d^4k\; H(k)\, A(k) \approx \int dk^+\, H(k^+, 0, 0) \int dk^- d^2k_T\, A(k^+, k^-, k_T)$$

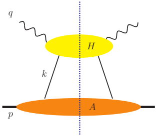
- in hard scattering treat incoming/outgoing partons as exactly collinear $(k_T = 0)$ and on-shell $(k^- = 0)$

$k^2 = 2k^+k^- - k_T^2$

- in collin. matrix element integrate over k_T and virtuality
  $\rightsquigarrow$ collinear (or k_T integrated) parton densities
  only depend on $k^+ = xp^+$

further subtleties related with spin of partons, not discussed here

Definition of parton distributions



- matrix elements of quark/gluon operators

$$f(x) = \int \frac{dz^-}{2\pi}\, e^{ixp^+z^-}\, \langle p \,|\, \bar{q}(0)\, \tfrac{1}{2}\gamma^+ W[0,z]\, q(z) \,|\, p\rangle \Big|_{z^+=0,\, z_T=0}$$

$q(z)$ = quark field operator: annihilates quark

$\bar{q}(0)$ = conjugate field operator: creates quark

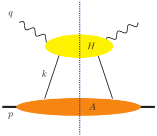
$\frac{1}{2}\gamma^+$ sums over quark spin

$\int \frac{dz^-}{2\pi}\, e^{ixp^+z^-}$ projects on quarks with $k^+ = xp^+$

$W[0,z]$ = Wilson line, makes product of fields gauge invariant $\rightsquigarrow$ later

- analogous definitions for polarised quarks, antiquarks, gluons
- analysis of factorisation used Feynman graphs
  but here provide non-perturbative definition

Definition of parton distributions



- matrix elements of quark/gluon operators

$$f(x) = \int \frac{dz^-}{2\pi} \, e^{ixp^+z^-} \, \langle p \big| \bar{q}(0) \, \tfrac{1}{2}\gamma^+ W[0,z] \, q(z) \big| p \rangle \Big|_{z^+=0,\, z_T=0}$$

$q(z) =$ quark field operator: annihilates quark and creates antiquark

$\bar{q}(0) =$ conjugate field operator: creates quark and annihilates antiquark

- matrix element generates both quark and antiquark distributions:

$$f(x) = \begin{cases} q(x) & \text{for } x > 0, \\ -\bar{q}(-x) & \text{for } x < 0 \end{cases}$$

minus sign from $dd^\dagger = -d^\dagger d$

Lowest order results for DIS and DVCS

- hard-scattering part of handbag graphs: kinematics

FRAME WITH $p_T = q_T = 0$

$$Q^2 = -q^2 = -2q^+q^- \qquad m^2 = p^2 = 2p^+p^-$$

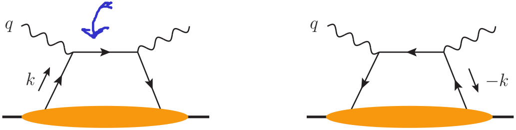
$$\frac{Q^2}{x_B} = 2p.q = 2p^+q^- + 2p^-q^+ = -\frac{p^+}{q^+}Q^2 + \frac{q^+}{p^+}m^2 \overset{Bj}{\approx} -\frac{p^+}{q^+}Q^2$$

$$\Rightarrow \quad q^+ \overset{Bj}{=} -x_B p^+ \, , \quad q^- = Q^2/2xp^+$$

BREIT FRAME: $q^0 = 0 \Rightarrow q^- = -q^+ = \frac{Q}{\sqrt{2}}$

$$p^+ = \frac{Q}{\sqrt{2}\, x_B} \qquad p^- = \frac{x_B m^2}{\sqrt{2}\, Q}$$

Lowest order results for DIS and DVCS



- hard-scattering part of handbag graphs: kinematics

$$x = k^+/p^+ \neq k^3/p^3$$

$$(q+k)^2 + i\epsilon = 2(q^- + k^-)(q^+ + k^+) - k_T^2 + i\epsilon$$

$$\approx 2q^- (-x_B p^+ + x p^+) + i\epsilon \approx \frac{Q^2}{x_B} (x - x_B + i\epsilon)$$

Lowest order results for DIS and DVCS

- hard-scattering part of handbag graphs:

$$\frac{1}{x - x_B + i\varepsilon} + \{\text{crossed graph}\} = \mathrm{PV}\,\frac{1}{x - x_B} - i\pi\delta(x - x_B) + \{\text{crossed graph}\}$$

- for DIS:

$$\sigma_{\text{tot}} \propto \mathrm{Im}\,\mathcal{A}(\gamma_T^* p \to \gamma_T^* p) = \sum_q (e e_q)^2 \big[q(x_B) + \bar{q}(x_B) \big]$$

$$\mathcal{A}(\gamma_L^* p \to \gamma_L^* p) = 0$$

$$2x_B F_1 = F_2 = x_B \sum_q e_q^2 \big[q(x_B) + \bar{q}(x_B) \big]$$

Lowest order results for DIS and DVCS

- hard-scattering part of handbag graphs:

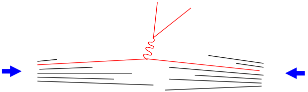
$$\frac{1}{x - x_B + i\varepsilon} + \{\text{crossed graph}\} = \mathrm{PV}\,\frac{1}{x - x_B} - i\pi\delta(x - x_B) + \{\text{crossed graph}\}$$

- for DVCS:

$$\mathcal{A}(\gamma_T^* p \to \gamma_T p) = \sum_q (e e_q)^2 \left[\mathrm{PV}\int dx\, \frac{\mathsf{GPD}(x, x_B, t)}{x_B - x} + i\pi\,\mathsf{GPD}(x_B, x_B, t) \right] + \{\text{crossed graph}\}$$

Factorisation for pp collisions

- example: Drell-Yan process $pp \to \gamma^* + X \to \mu^+\mu^- + X$
  where $X =$ any number of hadrons
- one parton distribution for each proton $\times$ hard scattering
  $\rightsquigarrow$ deceptively simple physical picture



Factorisation for pp collisions

- example: Drell-Yan process $pp \to \gamma^* + X \to \mu^+\mu^- + X$
  where $X =$ any number of hadrons
- one parton distribution for each proton $\times$ hard scattering
  $\rightsquigarrow$ deceptively simple physical picture

- "spectator" interactions produce additional particles
  which are also part of unobserved system X ("underlying event")
- need not calculate this thanks to unitarity
  as long as cross section/observable sufficiently inclusive
- but must calculate/model if want more detail on the final state
  $\rightsquigarrow$ factorisation does not work for all observables

More complicated final states

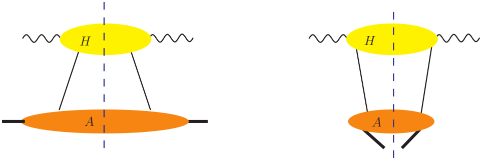
- production of W, Z or other colourless particle (Higgs, etc) same treatment as Drell-Yan
- jet production in ep or pp: hard scale provided by p_T
- heavy quark production: hard scale is m_c, m_b, m_t

Importance of factorisation concept

- describe high-energy processes: study electroweak physics, search for new particles, e.g.
  - discovery of top quark at Tevatron ($p + \bar{p}$ at $\sqrt{s} = 1.8\,\mathrm{TeV}$)
  - measurement of W mass at Tevatron and LHC
  - determination of Higgs boson properties at LHC
- determine parton densities as a tool to make predictions and to learn about proton structure
  $\rightsquigarrow$ require many processes to disentangle quark flavours and gluons

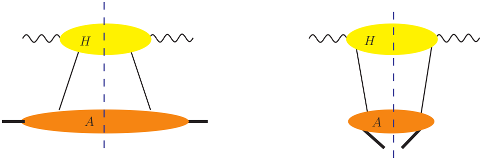
Fragmentation

- cross DIS $eh \to e + X$ to $e^+e^- \to \bar{h} + X$
  i.e., $\gamma^* h \to X$ to $\gamma^* \to \bar{h} + X$

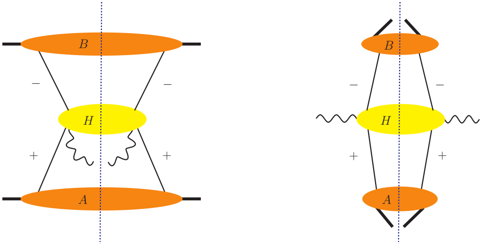


Fragmentation

- cross DIS $eh \to e + X$ to $e^+e^- \to \bar{h} + X$
  i.e., $\gamma^* h \to X$ to $\gamma^* \to \bar{h} + X$

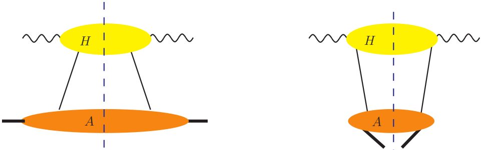


- or Drell-Yan $h_1 h_2 \to \gamma^* + X$ to $\gamma^* \to \bar{h}_1 \bar{h}_2 + X$

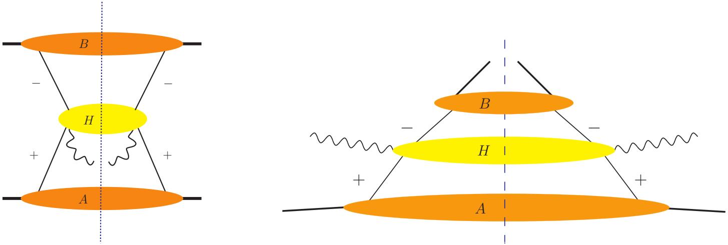


Fragmentation

- cross DIS $eh \to e + X$ to $e^+e^- \to \bar{h} + X$
  i.e., $\gamma^* h \to X$ to $\gamma^* \to \bar{h} + X$



- or Drell-Yan $h_1 h_2 \to \gamma^* + X$ to SIDIS $\gamma^* h_1 \to \bar{h}_2 + X$



Fragmentation functions

- replace parton density $\quad k^+ = xp^+$

$$f(x) = \int \frac{d\xi^-}{4\pi}\, e^{i\xi^- p^+ x}\, \langle h|\bar{q}(0)\gamma^+ W(0,\xi^-)\, q(\xi^-)|h\rangle$$

$$= \sum_X \int \frac{d\xi^-}{4\pi}\, e^{i\xi^- p^+ x}$$

$$\times \sum_X \big\langle h\big|(\bar{q}(0)\gamma^+)_\alpha W(0,\infty)\big|X\big\rangle \big\langle X\big|W(\infty,\xi^-)\, q_\alpha(\xi^-)\big|h\big\rangle$$

by fragmentation function $\quad p^+ = zk^+$

$$D(z) = \frac{1}{2N_c z} \int \frac{d\xi^-}{4\pi}\, e^{i\xi^- p^+/z}$$

$$\times \sum_X \big\langle 0\big|W(\infty,\xi^-)q_\alpha(\xi^-)\big|\bar{h}X\big\rangle \big\langle \bar{h}X\big|(\bar{q}(0)\gamma^+)_\alpha W(0,\infty)\big|0\big\rangle$$

$N_c = 3$ number of colours

A closer look at one-loop corrections

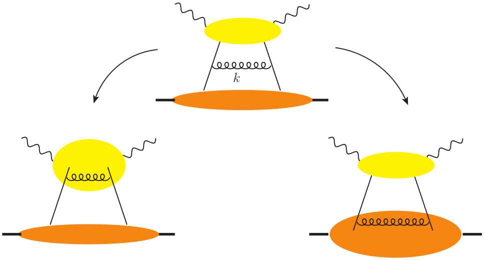
- example: DIS

- UV divergences removed by standard renormalisation
- soft divergences cancel in sum over graphs
- collinear div. do not cancel, have integrals

$$\int_0 \frac{dk_T^2}{k_T^2}$$

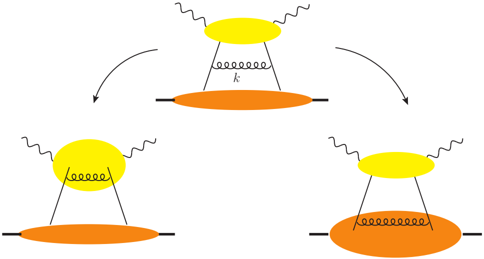
what went wrong?

- hard graph should not contain internal collinear lines
  collinear graph should not contain hard lines
- must not double count $\rightsquigarrow$ factorisation scale μ



- with cutoff: take $k_T > \mu$ take $k_T < \mu$
  $1/\mu \sim$ transverse resolution

- hard graph should not contain internal collinear lines
  collinear graph should not contain hard lines
- must not double count $\rightsquigarrow$ factorisation scale μ

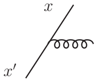


- with cutoff: take $k_T > \mu$ — take $k_T < \mu$
  $1/\mu \sim$ transverse resolution
- in dim. reg.:
  subtract collinear divergence — subtract ultraviolet div.

The evolution equations

- DGLAP equations

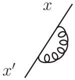
$$\frac{d}{d\log\mu^2}\, f(x,\mu) = \int_x^1 \frac{dx'}{x'}\, P\Big(\frac{x}{x'}\Big)\, f(x',\mu) \;=\; \big(P\otimes f(\mu)\big)(x)$$



- $P =$ splitting functions
  - have perturbative expansion

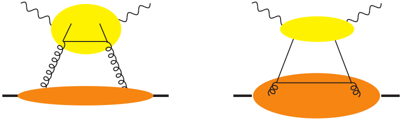
$$P(x) = \alpha_s(\mu)\, P^{(0)}(x) + \alpha_s^2(\mu)\, P^{(1)}(x) + \alpha_s^3(\mu)\, P^{(2)}(x)\ldots$$

    fully known to 3 loops, partially to 4 loops
    Moch et al. 2004, 2017



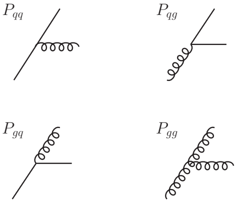
  - contains terms $\propto \delta(1-x)$ from virtual corrections

- quark and gluon densities mix under evolution:



- matrix evolution equation

$$\frac{d}{d\log\mu^2} f_i(x,\mu) = \sum_{j=q,\bar{q},g} \big(P_{ij}\otimes f_j(\mu)\big)(x) \qquad (i,j=q,\bar{q},g)$$



- parton content of proton depends on resolution scale μ

Mellin moments

- Mellin moments: $M(j) = \int_0^1 dx\, x^{j-1}\, f(x)$
- anomalous dimensions: $\gamma(j) = \int_0^1 dx\, x^{j-1}\, P(x)$

$$\frac{d\,f(x)}{d\log\mu^2} = (P \otimes f)(x) \quad \Rightarrow \quad \frac{dM(j)}{d\log\mu^2} = \gamma(j)\, M(j)$$

Exercise

- flavour non-singlet combinations (e.g. $f_u - f_d$) do not mix with f_g: same solution as RGE for running quark mass

Mellin moments

- Mellin moments: $M(j) = \int_0^1 dx\, x^{j-1}\, f(x)$
- matrix element definition

$$f(x) = \int \frac{dz^-}{2\pi}\, e^{ixp^+z^-}\, \langle p \,|\, \bar{q}(-\tfrac{1}{2}z)\, \tfrac{1}{2}\gamma^+ W[-\tfrac{1}{2}z, \tfrac{1}{2}z]\, q(\tfrac{1}{2}z) \,|\, p \rangle \Big|_{z^+=0,\, z_T=0}$$

operator identity (Exercise: verify for $n=1$ and $n=2$)

$$\int dx\, x^{n-1} \int dz^-\, e^{ixp^+z^-}\, \bar{q}(-\tfrac{1}{2}z)\gamma^+ W[-\tfrac{1}{2}z, \tfrac{1}{2}z]\, q(\tfrac{1}{2}z) \Big|_{z^+=0,\, z_T=0}$$
$$\propto \bar{q}(z)\gamma^+ \big[\overleftrightarrow{D}^+(z)\big]^{n-1}\, q(z)\, \Big|_{z=0}$$

with $\overleftrightarrow{D} = \frac{1}{2}(\overrightarrow{D} - \overleftarrow{D})\,,\qquad \overrightarrow{D}(z) = \partial_z + igA(z)\,,\qquad \overleftarrow{D}(z) = \overleftarrow{\partial}_z - igA(z)$

$\Rightarrow\qquad \int_{-1}^{1} dx\, x^{n-1}\, f(x) =$ matrix element of a local operator

$$= \begin{cases} \int_0^1 dx\, x^{n-1}\, [q(x) - \bar{q}(x)] & \text{for odd } n \\ \int_0^1 dx\, x^{n-1}\, [q(x) + \bar{q}(x)] & \text{for even } n \end{cases}$$

Factorisation formulae

- example: $p + p \to H + X$

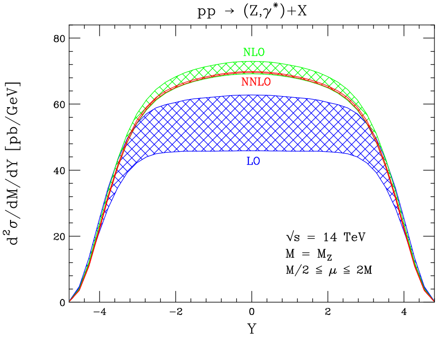
$$\sigma(p + p \to H + X) = \sum_{i,j=q,\bar{q},g} \int dx_i \, dx_j \; f_i(x_i, \mu_F) \, f_j(x_j, \mu_F) \times \hat{\sigma}_{ij}\big(x_i, x_j, \alpha_s(\mu_R), \mu_R, \mu_F, m_H\big) + \mathcal{O}\left(\frac{\Lambda^2}{m_H^4}\right)$$

  - $\hat{\sigma}_{ij}$ = cross section for hard scattering $i + j \to H + X$
    m_H provides hard scale
  - μ_R = renormalisation scale, μ_F = factorisation scale
    may take different or equal
  - μ_F dependence in C and in f cancels up to higher orders in α_s
    similar discussion as for μ_R dependence
  - accuracy: α_s expansion and power corrections $\mathcal{O}(\Lambda^2/m_H^2)$

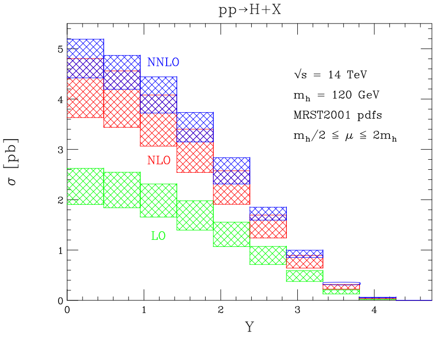
- can make σ and $\hat{\sigma}$ differential in kinematic variables, e.g. p_T of H

Scale dependence

examples: rapidity distributions in Z/γ^* and in Higgs production



Anastasiou, Dixon, Melnikov, Petriello, hep-ph/0312266



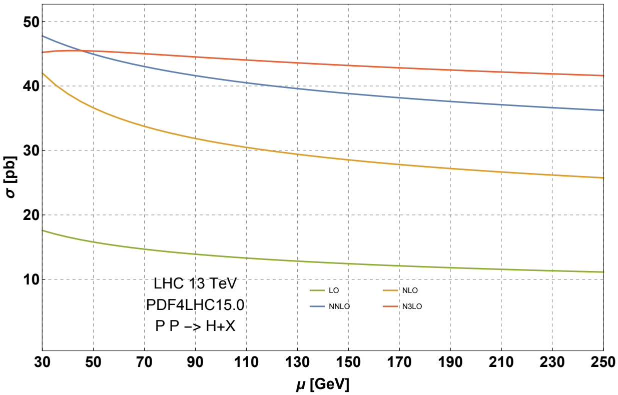
Anastasiou, Melnikov, Petriello, hep-ph/0501130

$\mu_F = \mu_R = \mu$ varied within factor 1/2 to 2

Scale dependence

example: inclusive Higgs production

$pp \to H + X$



Mistlberger, arXiv:1802.00833

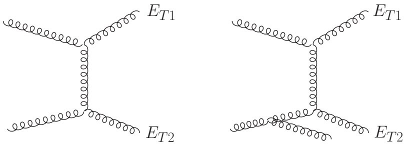
LO, NLO, and higher

- instead of varying scale(s) may estimate higher orders by comparing N^{n}LO result with N^{n-1}LO
- caveat: comparison NLO vs. LO may not be representative for situation at higher orders

often have especially large step from LO to NLO

  - certain types of contribution may first appear at NLO e.g. terms with gluon density $g(x)$ in DIS, $pp \to Z + X$, etc.
  - final state at LO may be too restrictive

    e.g. in $\dfrac{d\sigma}{dE_{T1}\, dE_{T2}}$ for dijet production



Summary of part 3

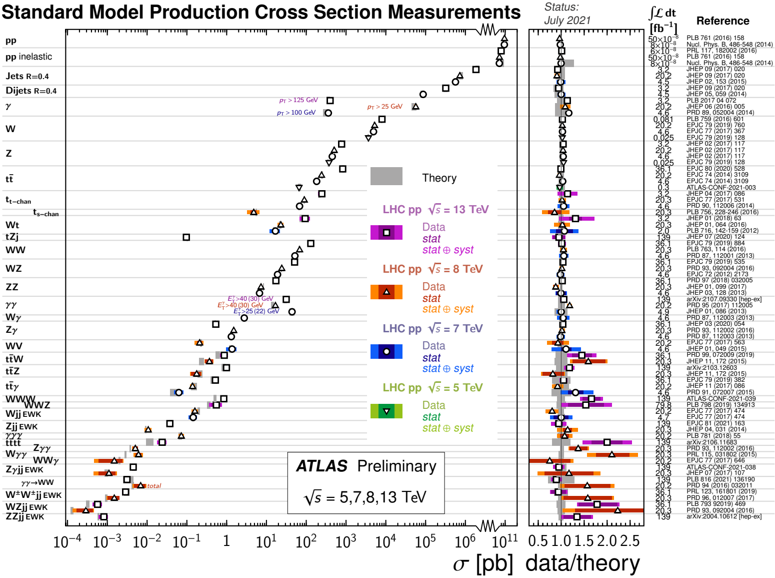
Factorisation

- implements ideas of parton model in QCD
  - perturbative corrections (NLO, NNLO, N^{3}LO, ...)
  - field theoretical def. of parton densities
    $\rightsquigarrow$ bridge to non-perturbative QCD
- valid for sufficiently inclusive observables
  and up to power corrections in Λ/Q or $(\Lambda/Q)^2$
  which are most often not calculable
- must in a consistent way
  - remove collinear kinematic region in hard scattering
  - remove hard kinematic region in parton densities
    $\leftrightarrow$ UV renormalisation

this introduces factorisation scale μ_F

- separates "collinear" from "hard", "object" from "probe"

Factorisation at work



Exercises

1. Light-cone coordinates
   - Derive the relations
   $$d^4v = dv^+ \, dv^- \, d^2\boldsymbol{v}\,, \qquad \partial_v^+ = \partial/(\partial v^-)\,, \qquad \partial_v^- = \partial/(\partial v^+)\,,$$
   where $\partial_v^\mu = \partial/(\partial v_\mu)$ and $\boldsymbol{v}$ is the transverse part of the vector.
   - Show that under a boost along the z axis, the light-cone plus-component transforms as $v^+ \to \alpha v^+$, where α characterises the boost.

2. Kinematics of DIS
   Consider a frame in which the proton and photon momenta, p and q, have zero transverse components.
   - Derive the explicit forms of p^+, p^-, q^+, q^- in the Bjorken limit for (i) the proton rest frame and (ii) the $\gamma^* p$ centre-of-mass. You can neglect $m^2 \ll Q^2$ and use $q^+ = -x_B \, p^+$.
   - Express p^+/q^+ in terms of x_B, Q^2 and m^2, without neglecting the proton mass.

Exercises

3. Mellin moments (slides 32–33)

- Derive the evolution equation of the Mellin moment $M(j)$ from the one for $f(x)$.
- Compute $M(j)$ for $f(x) = x^{-\alpha}(1-x)^{\beta}$ and discuss its analytic structure in the complex j plane.
- Show that the inverse of the Mellin transform

$$M(j) = \int_0^1 dx\, x^{j-1} f(x) \tag{1}$$

is

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dj\, x^{-j}\, M(j)\,, \tag{2}$$

where c is a real constant. The path of integration thus is a vertical line in the complex j plane, and it must be to the right of all singularities of $M(j)$. Hint: insert (1) into (2) and rewrite the integration over j as a Fourier transform.

Exercises

3. Mellin moments (continued) FOR WEDNESDAY

   - Verify the operator relation on slide 33 for $n = 1$ and $n = 2$.
     The Wilson line is defined as

     $$W[z_0, z_1] = \mathrm{P}\exp\left[ig \int_{z_0^-}^{z_1^-} dy^- A^+(z_0^+, y^-, \boldsymbol{z}_0)\right]$$

     for the light-like path with $z_0^+ = z_1^+$ and $\boldsymbol{z}_0 = \boldsymbol{z}_1$.
     The path ordering P acts such that, when one Taylor expands the exponential, the fields $A^+(y_1) \cdots A^+(y_n)$ are ordered along the integration path, starting with z_0 and ending with z_1. (Recall that A^+ is a matrix in colour space.)
     To simplify the calculation, you could first consider the case with light-cone gauge, $A^+(z) = 0$.

Exercises

4. Step functions and Fourier transform

- Show that

$$\int \frac{d\ell}{2\pi}\, e^{i\ell y}\, \frac{i}{\ell - i\epsilon} = -\theta(y)\,,$$
$$\int \frac{d\ell}{2\pi}\, e^{i\ell y}\, \frac{i}{\ell + i\epsilon} = \theta(-y)\,.$$

Hint: the integrals can be done in the complex ℓ plane using the residue theorem.

Note: we will use these relations in the lecture on TMDs. They are useful when manipulating Wilson lines $W[z_0, z_1]$ (see previous page) with $z_1^- \to \infty$ or $z_0^- \to -\infty$.

Notes

Notes

Notes